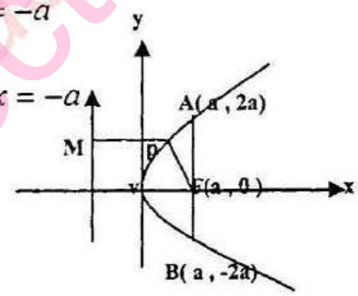
Unit – 9 : Conics – II .

Conic: The set of all points in the plane in such away that its distances from a fixed point bears a constant ratio to its distance from a fixed straight line is called conic .
The fixed point is called focus point and fixed straight line is called directrix .
The constant ratio is called eccentricity .
(i) If e = 1 , the conic is called is parabola .
(ii) If e < 1 , the conic is called ellipse .
(iii) If e > 1 , the conic is called hyperbola .

Parabola:
It is a conic in which the distance of each of point from the fixed point is equal to its distance from the fixed straight line. In parabola, e =1,

Standard equation of parabola: (a) $y^2 = 4ax$

Proof : Let $F(a , 0)$ is a fixed point.
Then equation of directrix is $x = -a$
=> $x + a = 0$ .....(i)



Let $P(x , y)$ is any point on the parabola . Let M be any point on the directrix .
Then by definition of parabola

$$\frac{|PF|}{|PM|} = 1 \quad => |PF| = |PM| \text{ ...(ii)}$$

Now $|PF| = \sqrt{(x-a)^2 + (y-0)^2}$ ...(iii)

And $|PM|$ is the perpendicular distance of the point $P(x , y)$ from the directrix (i) . Then

$$|PM| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \quad \text{put } x_1 = x,\ y_1 = y\ ,\ a = 1\ , b = 0\ , c = a$$

$$=> |PM| = \frac{|x + 0 + a|}{\sqrt{(1)^2 + 0}} = \frac{|x + a|}{\sqrt{1}} = |x + a| \text{ ....(iv)}$$

Put (iii) and (iv) in (ii).

$\sqrt{(x-a)^2+y^2}=|x+a|$ **squaring both sides**

$\Rightarrow (x-a)^2+y^2=(x+a)^2$

$\Rightarrow x^2-2ax+a^2+y^2=x^2+2ax+a^2$

$\Rightarrow y^2=x^2-x^2+2ax+2ax+a^2-a^2$

$\Rightarrow y^2=4ax$ *is the required equation of parabola*

<u>Some important points</u>:

(i) Fixed point is called Focus point in this case Focus point is $x=a, y=0 \Rightarrow F(a,0)$

(ii) The fixed straight line is called dirctrix.
In this case its equation is $x=-a$ *or* $x+a=0$

(iii) The line through the focus and perpendicular to directrix is called symmetry axis of parabola.
In this case x – axis is the symmetry axis of parabola and its equation is $y=0$

(iv) <u>vertex</u>: The mid point of the line from the focus point to directrix is called vertex. In this case origin is the vertex i-e x = 0 , y = 0 => v (0 ,0)

(v) The double ordinates through the focus point and parallel to directrix is called focal length.
The coordinates of the end points of focal chord (latus rectum) are $A(a\ ,2a)$, $B(a\ ,-2a)$.
Its equation is x = - a or x + a = 0
its length = 4a

(b) $y^2=-4ax$

(i) **<u>Focus point</u>** : $x=-a, y=0 \Rightarrow F(-a,0)$

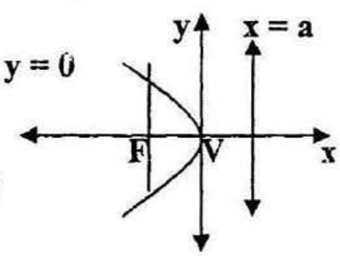
(ii) Equation of Directrix is x = a

(iii) Equation of axis of parabola is y = 0

(iv) <u>Vertex</u>: $x=0, y=0 \Rightarrow V(0,0)$

(v) Length of Focal length = 4a

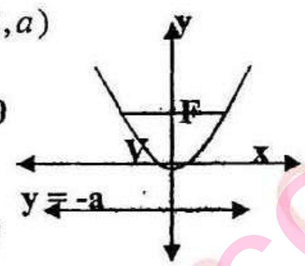
The coordinates of the end points of focal chord (latus rectum) are
$A(-a\ ,2a)$, $B(-a\ ,-2a)$.



Its equation is x = a or x - a = 0

(c) $x^2 = 4ay$

(i) **Focuse point** : $y = a, x = 0 \Rightarrow F(0, a)$



(ii) **Equation of Directrix** : $y = -a$

(iii) Equation of axis of parabola, x = 0

(iv) **Vertex**: $x = 0, y = 0 \Rightarrow v(0,0)$

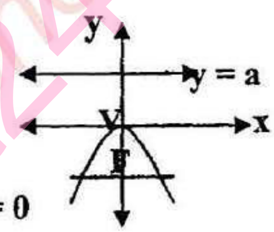
(v) Length of Focal length = 4a

The coordinates of the end points of focal chord (latus rectum) are $A(2a, a)$, $B(-2a, a)$.

Its equation is $y = -a$ or $y + a = 0$

(D) $x^2 = -4ay$

(i) **Focuse**: $x = 0, y = -a \Rightarrow F(0, -a)$



(ii) Equation of Directrix: y = a

(iii) **Vertex**: $x = 0, y = 0 \Rightarrow v(0,0)$

(iv) Equation of axis of parabola is x = 0

(v) Length of Focal length= 4a.

The coordinates of the end points of focal chord (latus rectum) are $A(2a, -a)$, $B(2a, -a)$.

Its equation is $y = a$ or $y - a = 0$

Translation of parabola: $y^2 = 4ax$: It is the standard form with vertex is $V(0, 0)$ and focus is $F(a, 0)$. If we translate the vertex to $V(h, k)$, then focus is $F(a + h, k)$

Then equation of parabola becomes

$(y - k)^2 = 4a(x - h)$, with directrix is $x = h - a$.

The curve is toward right if $a > 0$ and the curve is toward left if $a < 0$.

The equation of symmetry of axis is $y - k = 0$

Translation of parabola: $x^2 = 4ay$: It is the standard form with vertex is $V(0, 0)$ and focus is $F(0, a)$. If we translate the vertex to $V(h, k)$, then focus is $F(h, a+k)$

Then equation of parabola becomes

$(x-h)^2 = 4a(y-k)$, with directrix is $y = k-a$.

The curve is toward up if $a > 0$ and
the curve is toward down if $a < 0$.

The equation of symmetry of axis is $x-h=0$

Equation of tangent line in slope form :

(a) If m is the slope of tangent line to the parabola
$y^2 = 4ax$ ....(i)

Then tangent line of the form $y = mx + c$ ......(ii)

The line (ii) touch the parabola (i) at only one point if

$c = \frac{a}{m}$ ......(iii) put (iii) in (ii)

$y = mx + \frac{a}{m}$ .....(iv).

Then eq(iv) is tangent equation to the parabola $y^2 = 4ax$.

Eq(iii) is the condition of tangency.

(b) If parabola of the form $x^2 = 4ay$, then the line (ii)
is tangent to the parabola if $c = -am^2$ .....(iv)
then $y = mx - am^2$ is tangent line to the parabola.
Eq (iv) is the condition of tangency.

(c) The equation of tangent line to the parabola $y^2 = 4ax$
at point $P(x_1, y_1)$ is

$$y_1 y = 2a(x + x_1) \quad ....(a)$$

The contact point is given by $P(x_1, y_1) = P\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

(d) The equation of tangent line to the parabola $x^2 = 4ay$
at point $P(x_1, y_1)$ is

$$x_1 x = 2a(y + y_1)$$

The contact point is given by $P(x_1, y_1) = P(2am, am^2)$.

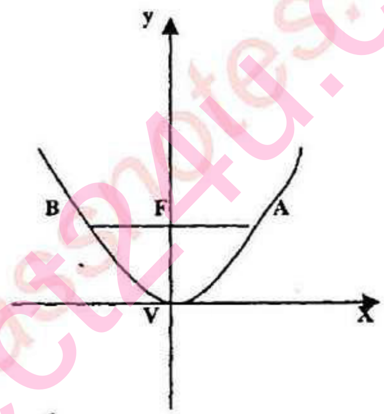
Equation of normal line to parabola at a point :

The equation of normal line to parabola $y^2 = 4ax$ at a point $P(x_1 , y_1)$ is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

Exercise 9.1:

Q.1: In each case, sketch the parabola represented by the equation, indicate the vertex, the focus, the end points of the focal chord (latus rectum) and axis of symmetry.

(a): $x^2 = 2y$



Solution: $x^2 = 2y$ ....(i)

The standard equation of parabola is $x^2 = 4ay$ ....(ii)

comparing (i) and (ii)

Then $4a = 2 \Rightarrow a = \frac{1}{2}$

(i) vertex : $x = 0$, $y = 0$

$\Rightarrow V(0 , 0)$.

(ii) Focus : As $x = 0$ *and* $y = \frac{1}{2}$

Then $F\left(0 , \frac{1}{2}\right)$

(iii) The end points of latus rectum:

Put $y = \frac{1}{2}$ in (i) $\Rightarrow x^2 = 2\left(\frac{1}{2}\right) = 1$

$\Rightarrow x = \pm 1$

$A\left(1 , \frac{1}{2}\right)$ and $B\left(-1 , \frac{1}{2}\right)$ are the end points of latus rectum.

(iv) Axis of symmetry is y – axis . Then its equation is $x = 0$.

(b) $y^2 = -3(x + 1)$

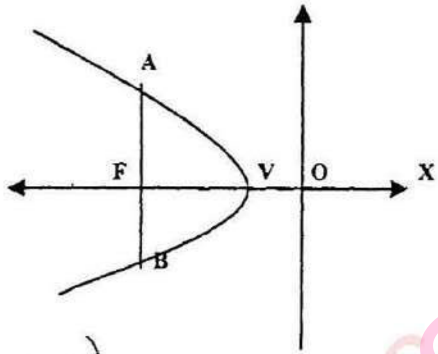
Solution: $y^2 = -3(x+1)$....(i)

Comparing with parabola

$(y-k)^2 = 4a(x-h)$ ...(ii)

comparing (i) and (ii)

$4a = -3 \Rightarrow a = -\frac{3}{4}$



And $k = 0$, $-h = 1 \Rightarrow h = -1$

(i) Vertex: $V(h, k) = V(-1, 0)$

(ii) Focus: $F(a+h, k) = F\left(-\frac{3}{4}-1, 0\right)$

$= F\left(-\frac{7}{4}, 0\right)$

(iii) The end points of focal chard: Put $x = -\frac{7}{4}$ in (i)

$$y^2 = -3\left(-\frac{7}{4}+1\right) = -3\left(\frac{-7+4}{4}\right)$$

$$\Rightarrow y^2 = -3\left(-\frac{3}{4}\right) = \frac{9}{4} \quad \Rightarrow y = \pm\frac{3}{2}$$

Thus the end points are $A\left(-\frac{7}{4}, \frac{3}{2}\right)$, $B\left(\frac{7}{4}, -\frac{3}{2}\right)$.

(iv) The equation of symmetry of axis is $y - k = 0$

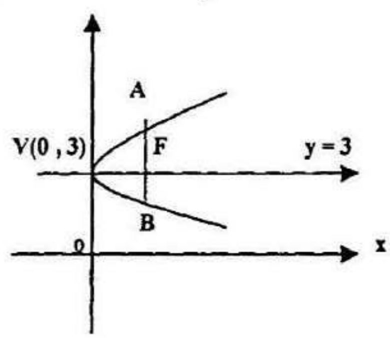
$\Rightarrow y = 0$

(c) $(y-3)^2 = x$

Solution: $(y-3)^2 = x$ ...(i)

Comparing with parabola

$(y-k)^2 = 4a(x-h)$....(ii)



$4a = 1 \Rightarrow a = \frac{1}{4}$

$k = 3$ and $h = 0$

(i) vertex: $V(h, k) = V(0, 3)$

(ii) **Focus :**

$$F(a+h\ ,\ k)=F\left(\frac{1}{4}+0\ ,\ 3\right)=F\left(\frac{1}{4}\ ,\ 3\right)$$

(iii) **End points of focal chord :** Put $x=\frac{1}{4}$ in (i)

$$(y-3)^2=\frac{1}{4} \quad => \quad y-3=\pm\frac{1}{2}$$

$$=> \quad y=3\pm\frac{1}{2}$$

$$=> \quad y=3+\frac{1}{2} \quad and \quad y=3-\frac{1}{2}$$

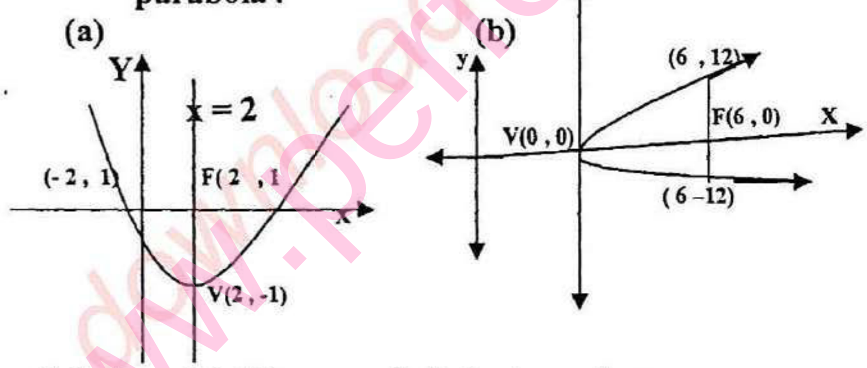
$$=> \quad y=\frac{7}{2} \quad and \quad y=\frac{5}{2}$$

Thus end points are $A\left(\frac{1}{4}\ ,\ \frac{7}{2}\right)$, $B\left(\frac{1}{4}\ ,\ \frac{5}{2}\right)$ **Ans.**

(iv) **Equation of axis of symmetry is** $y-k=0$

$$y-3=0 \quad => \quad y=3$$

Q.2: In each case , determined the equation of graphed parabola .



Solution: (a) **Since parabola is toward up .**

Then equation of parabola is $(x-h)^2=4a(y-k)^2$ ....(i)

vertex : $V(2\ ,-1\ )=V(h,\ k) \quad => \quad h=2 \quad and \quad k=-1$

Focus : $F(2\ ,\ 1)=F(h\ \ ,a+k)$

$$=> h=2 \quad and \quad a+k=1 \qquad put\ \ k=-1$$

$=> \ a-1=1 \ => \ a=1+1=2$

Put values in (i)

$(x-2)^2=4(2)(y-(-1)) \ => \ (x-2)^2=8(y+1)$ **Ans.**

(b) Since parabola is toward up .

Then equation of parabola is $(x-h)^2=4a(y-k)^2$ **....(i)**

vertex : $V(-1\ ,\ 0)=V(h,\ k) \ => \ h=-1 \ \ and \ \ k=0$

Focus : $F\left(-1\ ,\ \frac{3}{4}\right)=F(h\ \ ,a+k)$

$=> h=-1 \ \ and \ \ a+k=\frac{3}{4} \qquad put \ \ k=0$

$=> \ a+0=\frac{3}{4} \ => \ a=\frac{3}{4}$

Put values in (i)

$(x-(-1))^2=4\left(\frac{3}{4}\right)(y-0) \ => \ (x+1)^2=3y$

$=> x^2+2x-3y+1=0$ **Ans.**

Q.3: In each case , write the equation of parabola through the given information .

(a) Focus at $F(0\ ,\ 3)$**, directrix** $y=-3$

Solution : As $F(0\ ,\ a)=F(0\ ,\ 3)$ **, then** $a=3$

The parabola towards up , then its equation is

$x^2=4ay \ => \ x^2=4(2)y$

$=> \ x^2=8y$ *Ans.*

(b) Focus at $F(4\ ,\ 0)$ **and directrix** $x=-4$

Solution: As $F(4\ ,\ 0)=F(a\ ,\ 0)$ **, then** $a=4$

The parabola is open toward right ,then its equation is

$y^2=4ax \ => \ y^2=4(4)x$

$=> \ y^2=16x$ *Ans.*

(c) Vertex at $V(0\ ,\ 0)$ **, x –axis is the line of symmetry , passes through** $(3\ ,\ 6)$ **.**

Solution: Since x – axis is the line of symmetry , then the focus point is on the x –axis .

Let $F(a, 0)$ is the focus point .

Then the required equation of parabola is

$y^2 = 4ax$ ….(i) , Since (i) passes through $(3, 6)$.

Put $x = 3$ and $y = 6$ in (i)

=> $(6)^2 = 4a(3)$ => $36 = 12a$

=> $a = \frac{36}{12} = 3$

Put $a = 3$ in (i)

=> $y^2 = 4(3)x$ => $y^2 = 12x$ Ans.

(d) Vertex at $V(0, 0)$, y –axis is the line of symmetry , passes through $(-12, -3)$.

Solution: Since y – axis is the line of symmetry , then the focus point is on the y –axis .

Let $F(0, a)$ is the focus point .

Then the required equation of parabola is

$x^2 = 4ay$ ….(i) , Since (i) passes through $(-12, -3)$.

Put $x = -12$ and $y = -3$ in (i)

=> $(-12)^2 = 4a(-3)$ => $144 = -12a$

=> $a = -\frac{144}{12} = -12$

Put $a = -12$ in (i)

=> $y^2 = 4(-12)x$ => $y^2 = -48x$ Ans.

(e) Line of symmetry is vertical , passes through $(-3, 4)$, vertex at $V(5, 1)$.

Solution: As vertex is $V(5, 1) = V(h, k)$

=> $h = 5$ and $k = 1$

and the line of symmetry is vertical then equation of parabola is $(x-h)^2 = 4a(y-k)$ ….(i)

put $h = 5$, $k = 1$ in (i)

$$(x-5)^2 = 4a(y-1) \quad .....(ii)$$

Since (ii) passes through the point $(-3, 4)$.

Put $x=-3$ *and* $y=4$ in (ii)

$$\Rightarrow (-3-5)^2 = 4a(4-1) \qquad \Rightarrow (-8)^2 = 4a(-3)$$

$$\Rightarrow 12a = 64 \qquad \Rightarrow a = \frac{64}{12} = \frac{16}{3}$$

Put $a = \frac{16}{3}$ in (ii)

$$(x-5)^2 = \frac{16}{3}(y-1) \quad \textbf{Ans.}$$

(f) Line of symmetry is horizontal, passes through $(7, 9)$, vertex at $(3, -7)$.

Solution: As $V(3, -7) = V(h, k)$

$$\Rightarrow h = 3 \quad and \quad k = -7$$

and the line of symmetry is horizontal then equation of parabola is $(y-k)^2 = 4a(x-h)$ ....(i)

put $h=3$ *and* $k=-7$ in (i)

$$(y-(-7))^2 = 4a(x-3)$$

$$\Rightarrow (y+7)^2 = 4a(x-3) \quad ....(ii)$$

Since (ii) is passes through the point $(7, 9)$.

Put $x=7$, $y=9$ in (ii)

$$(9+7)^2 = 4a(7-3) \qquad \Rightarrow (16)^2 = 4a(4)$$

$$\Rightarrow 16a = 256 \qquad \Rightarrow a = 16$$

Put $a=16$ in (ii) $\Rightarrow (y+7)^2 = 4(16)(x-3)$

$$\Rightarrow (y+7)^2 = 64(x-3) \quad \textbf{Ans.}$$

Q.4: Find the equation of the set of all points with distances from $(4, 3)$ that equal their distances from $(-2, 1)$.

Solution: Let A $(4, 3)$ and B$(-2, 1)$.

Let $P(x, y)$ be the set of all points such that

$$|AP| = |BP|$$

$$\Rightarrow \sqrt{(x-4)^2+(y-3)^2} = \sqrt{(x-(-2))^2+(y-1)^2} \text{ squaring}$$

$$\Rightarrow (x-4)^2+(y-3)^2 = (x+2)^2+(y-1)^2$$

$$\Rightarrow x^2-8x+16+y^2-6y+9 = x^2+4x+4+y^2-2y+1$$

$$\Rightarrow -8x-4x-6y+2y+25-5=0$$

$$\Rightarrow -12x-4y+20=0 \quad \textit{Divided by } -4$$

$$\Rightarrow 3x+4y-5=0$$

$$\Rightarrow 3x+4y=5 \quad \textit{Ans.}$$

Q.5: Find an equation for a parabola whose focal length has length 6, if it is known that parabola has focus $(4,-2)$ and its directrix is parallel to the y- axis.

Solution: Given focal length is $4a=6$

$$\Rightarrow a = \frac{6}{4} = \frac{3}{2}$$

Focus $F(4,-2)$ **and directrix is parallel to y –axis.**

Then equation of parabola is $(y-k)^2 = 4a(x-h)$ …..(i)

with focus is $F(a+h, k)$

Then $F(a+h, k) = F(4,-2)$

$$\Rightarrow a+h=4 \Rightarrow \frac{3}{2}+h=4$$

$$\Rightarrow h = 4-\frac{3}{2} = \frac{8-3}{2} = \frac{5}{2}$$

and $k=-2$ **put values in (i)**

$$(y-(-2))^2 = 4\left(\frac{3}{2}\right)\left(x-\frac{5}{2}\right)$$

$$\Rightarrow (y+2)^2 = 6\left(x-\frac{5}{2}\right) \quad \textbf{Ans.}$$

Q.6: In each case, find the points of intersection in between the line and the parabola.

(a). $y^2+3x=-8$, $x-y+2=0$

Solution: $y^2+3x=-8$ ….(i) **and** $x-y+2=0$ …….(ii)

From (ii) $\Rightarrow x = y - 2$ .....(iii)

Put (iii) in (i)

$y^2 + 3(y-2) = -8 \Rightarrow y^2 + 3y - 6 + 8 = 0$

$\Rightarrow y^2 + 3y + 2 = 0$

$\Rightarrow y^2 + y + 2y + 2 = 0$

$\Rightarrow y(y+1) + 2(y+1) = 0$

$\Rightarrow (y+1)(y+2) = 0$

$\Rightarrow y + 1 = 0$ *and* $y + 2 = 0$

$\Rightarrow y = -1$ *and* $y = -2$

Put $y = -1$ in (iii) $\Rightarrow x = -1 - 2 = -3$

Put $y = -2$ in (iii) $\Rightarrow x = -2 - 2 = -4$

Thus intersection points are $A(-3, -1)$ *and* $B(-4, -2)$

(b) $x^2 = 2y$, $x - y - 2 = 0$

Solution: $x^2 = 2y$ ....(i) and $x - y - 2 = 0$ .....(ii)

From (ii) $\Rightarrow x - 2 = y$ or $y = x - 2$ ......(iii)

Put (iii) in (i) $\Rightarrow x^2 = 2(x-2)$

$\Rightarrow x^2 = 2x - 4 \Rightarrow x^2 - 2x + 4 = 0$

Now by quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{put } a = 1, b = -2, c = 4$$

$$\Rightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)} = \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$\Rightarrow x = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{3}\,i}{2} = 1 \pm i\sqrt{3}$$

Which is imaginary . Then line (ii) does not intersect (i).

Q.7: For what value of c ,

(a) The line $x - y + c = 0$ will touch the parabola $y^2 = 9x$

Solution: $y^2 = 9x$ .....(i) and $x - y + c = 0$ ......(ii)

As we know that line (ii) touch the parabola (i) if $c = \frac{a}{m}$ ……(A)

From (i) => $4a = 9$ => $a = \frac{9}{4}$

From (ii) => $y = x - c$ then $m = 1$

Put value of m and a in (A)

$$c = \frac{9/4}{1} = \frac{9}{4} \quad \text{Ans.}$$

(b) The line $x - y + c = 0$ will touch the parabola $x^2 = \frac{2}{3}y$

Solution: $x^2 = \frac{2}{3}y$…..(i) and $x - y + c = 0$ ……(ii)

As we know that line (ii) touch the parabola (i)

if $c = am$ ……(A)

From (i) => $4a = \frac{2}{3}$ => $a = \frac{2}{4 \times 3} = \frac{1}{6}$

From (ii) => $y = x + c$ then $m = 1$

Put value of m and a in (A)

$$c = \left(\frac{1}{6}\right)(1) = \frac{1}{6} \quad \text{Ans.}$$

Q.8: In each case, find the tangent equation and normal equation.

(a) At a point $(3, 6)$ to parabola $y^2 = 12x$

Solution: $y^2 = 12x$ ….(i) and $(3, 6)$

The equation of tangent line to the parabola $y^2 = 4ax$ at point $P(x_1, y_1)$ is

$$y_1 y = 2a(x + x_1) \quad ….(a)$$

Here $4a = 12$ => $a = 3$ and $x_1 = 3$, $y_1 = 6$ put in (a)

$$6y = 2(3)(x+3) \quad => \quad y = x + 3……..(A) \quad Ans.$$

Eq(A) is the equation of tangent to parabola (i)

The equation of normal is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$….(b)

Put values in (b)

$$y-6=-\frac{6}{2(3)}(x-3) \Rightarrow y-6=-(x-3)$$

$$\Rightarrow x+y-6-3=0 \Rightarrow x+y-9=0 \quad ......(B)$$

Eq(B) is the equation of normal to parabola (i).

(b) At a point $\left(\frac{1}{2}, -\frac{1}{3}\right)$ **to parabola** $x^2=-\frac{3}{4}y$

Solution: $x^2=-\frac{3}{4}y$ **....(i) and** $\left(\frac{1}{2}, -\frac{1}{3}\right)$

The equation of tangent line to the parabola $x^2=4ay$ **at point** $P(x_1, y_1)$ **is**

$$x_1x=2a(y+y_1) \quad ....(a)$$

Here $4a=-\frac{3}{4} \Rightarrow a=-\frac{3}{16}$ **and** $x_1=\frac{1}{2},\ y_1=-\frac{1}{3}$

put in (a)

$$\frac{1}{2}x=2\left(-\frac{3}{16}\right)\left(y-\frac{1}{3}\right) \Rightarrow \frac{1}{2}x=-\frac{3}{8}\left(\frac{3y-1}{3}\right)$$

$$\Rightarrow \frac{1}{2}x=-\frac{1}{8}(3y-1) \Rightarrow \frac{8}{2}x=-(3y-1)$$

$$\Rightarrow 4x+3y-1=0 \quad ....(A)$$

Eq(A) is the equation of tangent to parabola (i)

The equation of normal is $y-y_1=-\frac{y_1}{2a}(x-x_1)$**....(b)**

Put values in (b)

$$y-\left(-\frac{1}{3}\right)=-\frac{-\frac{1}{3}}{2\left(-\frac{3}{16}\right)}\left(x-\frac{1}{2}\right) \Rightarrow y+\frac{1}{3}=-\frac{1}{3}\times\frac{8}{-3}\left(x-\frac{1}{2}\right)$$

$$\Rightarrow \frac{3y+1}{3}=\frac{8}{9}\left(\frac{2x-1}{2}\right) \Rightarrow \frac{9(3y+1)}{3}=4(2x-1)$$

$$\Rightarrow 3(3y+1)=8x-4 \Rightarrow 9y+3=8x-4$$

$\Rightarrow\ 0 = 8x - 9y - 4 - 3$

$\Rightarrow\ 8x - 9y - 7 = 0$ ......(B)

Eq(B) is the equation of normal to parabola (i).

Q.9: Find the tangent equation :

(a) To a parabola $y^2 = x$, which make an angle 135^o with x-axis.

Solution: $y^2 = x$ ....(i) and $\theta = 135^o$

As equation of tangent line is $y = mx + \dfrac{a}{m}$ .....(ii)

Comparing (i) with $y^2 = 4ax$, then

$$4a = 1 \quad \Rightarrow \quad a = \frac{1}{4}$$

Also $m = \tan 135^o = -1$, put values in (ii)

$$y = -1.x + \frac{\frac{1}{4}}{-1} \quad \Rightarrow \quad y = -x - \frac{1}{4} \quad \text{Ans.}$$

(b) To a parabola $y^2 = x$, which make an angle 60^o with x-axis.

Solution: $y^2 = x$ ....(i) and $\theta = 60^o$

As equation of tangent line is $y = mx + \dfrac{a}{m}$ .....(ii)

Comparing (i) with $y^2 = 4ax$, then

$$4a = 1 \quad \Rightarrow \quad a = \frac{1}{4}$$

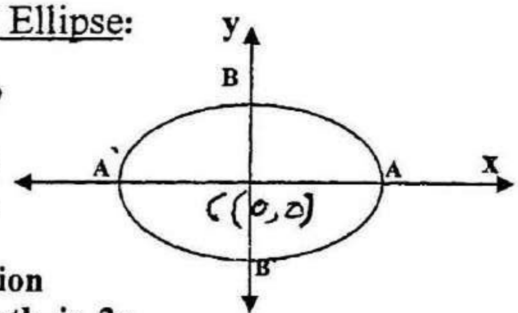
Also $m = \tan 60^o = \sqrt{3}$, put values in (ii)

$$y = \sqrt{3}x + \frac{\frac{1}{4}}{\sqrt{3}} \quad \Rightarrow \quad y = \sqrt{3}x + \frac{1}{4\sqrt{3}} \quad \text{Ans.}$$

Ellipse: It is a conic in which the distance of each point from the focus point bears a constant ratio less then "1" to its distance from the fixed straight line.

In ellipse "$e < 1$"

Standard Equation of Ellipse:



(a) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \quad , a>b$

The Following points Should be remember.

(i) **x-axis is called major axis. Its equation is y = 0 and its length is 2a.**

(ii) **y-axis is called minor axis. Its equation is x = 0, its Length = 2b**

(iii) **Center:** $x=0, y=0 \Rightarrow c(0,0)$

(iv) **Vertices: The extremities of major axis is called vertices. i.e.** $x=\pm a,\ y=0$

$\Rightarrow A(a,\ 0)\ ,A'(-a\ ,0)$

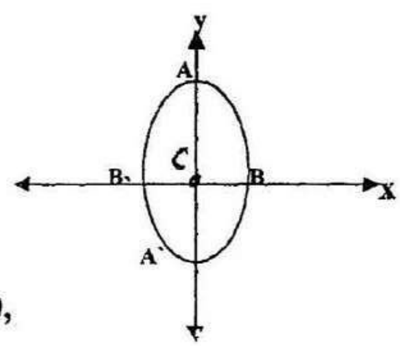
(v) **Extremities or end points of minor axis is**

$x=0, y=\pm b \qquad \Rightarrow B(0\ ,b)\ ,B'(0,\ -b)$

(vi) **Foci points:** $x=\pm c\ ,\ y=0$

$\Rightarrow F(c\ ,0),\ F'(-c\ ,0)$. **Where** $c=\sqrt{a^2-b^2}$

(vii) **Equation of Directries are** $x=\pm\dfrac{a}{e}$



(b) $\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1,\ a>b$

(i) **x-axis is minor axis. its equation is x = 0, its length = 2a**

(ii) **y-axis is called major axis. Its equation is x = 0, its length = 2b**

(iii) **Center:** $x=0,\ y=0 \Rightarrow c(0,0)$

(iv) **Vertices:** $x=0\ ,\ y=\pm a$

$\Rightarrow A(0,a),\ \ A'(0,-a)$

(v) **extremities or end points of minor axis are**
$x = \pm b$, $y = 0$ => $B(b\ ,0)$, $B'(-b\ ,\ 0)$

(vi) **Foci point**: $x = 0$, $y = \pm c$
=> $F(0,\ c)$, $F'(0,\ -c)$

(vii) **Equation of Directrices**: $y = \pm\dfrac{a}{e}$

Translation of Ellipse horizontal and vertical:

(a) $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ ......(i)

Let $X = x - h$ **and** $Y = y - k$

with centre $C(h\ ,\ k)$

(b) $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$ ......(i)

Let $X = x - h$ **and** $Y = y - k$

with centre $C(h\ ,\ k)$

Equation of tangent line to the ellipse:

(a) **Equation of tangent line to the ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **is**

$y = mx \pm \sqrt{a^2m^2 + b^2}$ ......(a)

Note: The tangent line $y = mx + c$ **touch the ellipse**

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **if** $c = \pm\sqrt{a^2m^2 + b^2}$ **Ans.**

(b) **Equation of tangent line at point** $P(x_1\ ,\ y_1)$ **to the ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **is** $\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$

(c) **Equation of normal at point** $P(x_1\ ,\ y_1)$ **to the ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **is** $\dfrac{a^2(x - x_1)}{x_1} = \dfrac{b^2(y - y_1)}{y_1}$

Orientation of ellipse to centre $C(h\ ,\ k)$:

(a) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ (i) centre is $C(h,\ k)$

(ii) equation of major axis is $y=k$

(iii) equation of minor axis is $x=h$

(iv) Foci points $F(h+c\ ,\ k)$, $F'(h-c\ ,\ k)$

Where $c=\sqrt{a^2-b^2}$

(v) vertices are $A(h+a\ ,\ k)$, $A'(h-a\ ,\ k)$

(vi) End points of minor axis are

$B(h\ ,\ b+k)$, $B'(h\ ,\ b-k)$

(b) $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$,(i) centre is $C(h,\ k)$

(ii) equation of major axis is $x=h$

(iii) equation of minor axis is $y=k$

(iv) Foci points $F(h\ ,\ k+c)$, $F'(h\ ,\ k-c)$

Where $c=\sqrt{a^2-b^2}$

(v) vertices are $A(h\ ,\ k+a)$, $A'(h\ ,\ k-a)$

(vi) End points of minor axis are

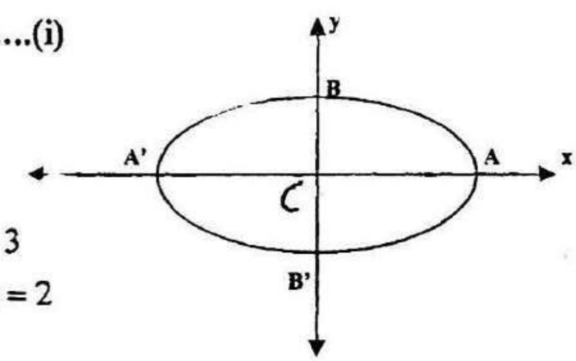
$B(h+b,\ k)$, $B'(h-b\ ,\ k)$

Exercise 9.2:

Q.1: In each case, sketch the ellipse represented by the equation. Indicate the centre, foci, endpoints of the major axis and end points of minor axis.

(a) $\frac{x^2}{9}+\frac{y^2}{4}=1$

Solution: $\frac{x^2}{9}+\frac{y^2}{4}=1$ .....(i)



Comparing (i) with

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

Then $a^2=9$ => $a=3$

And $b^2=4$ => $b=2$

(i) **centre :** $x=0$, $y=0$

$\Rightarrow C(0\ ,\ 0)$

(ii) **Foci points:** As $c=\sqrt{a^2-b^2}=\sqrt{9-4}=\sqrt{5}$

Then $F(c,\ 0)=F(\sqrt{5}\ ,\ 0)$

and $F'(-c\ ,\ 0)=F'(-\sqrt{5}\ ,0)$

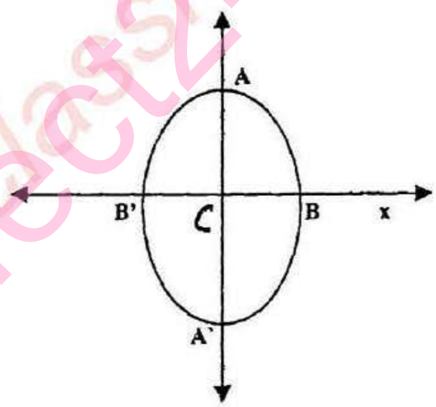
Thus foci points are $F(\sqrt{5},\ 0)$ *and* $F'(-\sqrt{5}\ ,0)$

(iii) **End points of major axis: The end points of major axis are** $A(a\ ,0)=A(3\ ,\ 0)$ **and** $A'(-a\ ,0)=A'(-3\ ,\ 0)$

(iv) **End points of minor axis : The end points of minor axis are** $B(0\ ,\ b)=B(0\ ,\ 2)$ **and** $B'(0\ ,-b)=B'(0\ ,-2)$.

(b) $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1$

<u>**Solution:**</u> $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1$ .....(i)



Comparing (i) with

$\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$,

Then $a^2=25 \Rightarrow a=5$

And $b^2=16 \Rightarrow b=4$

(i) **centre :** $x=0$, $y=0$

$\Rightarrow C(0\ ,\ 0)$

(ii) **Foci points:** As $c=\sqrt{a^2-b^2}=\sqrt{25-16}=\sqrt{9}=3$

Then $F(c,\ 0)=F(3\ ,\ 0)$

and $F'(-c\ ,\ 0)=F'(-3\ ,0)$

Thus foci points are $F(3,\ 0)$ *and* $F'(-3,0)$

(iii) **End points of major axis: The end points of major axis are** $A(0\ ,a)=A(0\ ,\ 5)$ **and**

$A'(-a\ ,0)=A'(0\ ,\ 5)$

(iv)**End points of minor axis : The end points of minor axis**

are $B(b\ ,\ 0) = B(4\ ,\ 0)$ and
$B'(-b\ ,0) = B'(-4\ ,\ 0)$.

(c) $\dfrac{(x+1)^2}{16} + \dfrac{(y-2)^2}{9} = 1$

Solution: $\dfrac{(x+1)^2}{16} + \dfrac{(y-2)^2}{9} = 1$ ....(i)

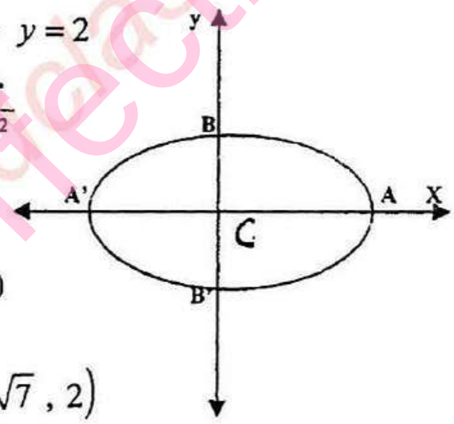
Let $x+1 = X$ and $y-2 = Y$ put in (i)

$\dfrac{X^2}{16} + \dfrac{Y^2}{9} = 1$, Then

$a^2 = 16 \Rightarrow a = 4$

$b^2 = 9 \Rightarrow b = 3$

(i) Centre: As $X = 0 \Rightarrow x+1 = 0$
$\Rightarrow x = -1$
and $Y = 0 \Rightarrow y-2 = 0 \Rightarrow y = 2$
Thus centre is $C(-1\ ,\ 2)$.



(ii) Foci point: As $c = \sqrt{a^2 - b^2}$
$\Rightarrow c = \sqrt{16-9} = \sqrt{7}$
$X = \pm c$ and $Y = 0$
$\Rightarrow x+1 = \pm\sqrt{7}$ and $y-2 = 0$
$\Rightarrow x = -1 \pm \sqrt{7}$ and $y = 2$
Thus foci points are $F\left(-1+\sqrt{7}\ ,\ 2\right)$
and $F'\left(-1-\sqrt{7}\ ,\ 0\right)$

(iii) The end points of major axis are
$X = \pm a$ and $Y = 0$
$\Rightarrow x+1 = \pm 4$ and $y-2 = 0$
$\Rightarrow x = 1 \pm 4$ and $y = 2$
$\Rightarrow x = 1+4$ and $x = 1-4$
$\Rightarrow x = 5$ and $x = -3$

Thus the end points of major axis are $A(5\ ,\ 2)$ and
$A'(-3\ ,\ 2)$.

(iv) The end points of minor axis are

$X = 0 \quad and \quad Y = \pm b$

$\Rightarrow \quad x + 1 = 0 \quad and \quad y - 2 = \pm 3$

$\Rightarrow \quad x = -1 \quad and \quad y = 2 \pm 3$

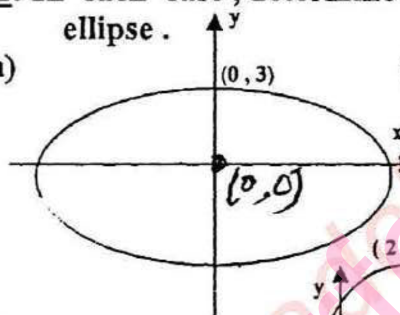
$\Rightarrow \quad y = 2 + 3 \quad and \quad y = 2 - 3$

$\Rightarrow \quad y = 5 \quad and \quad y = -1$

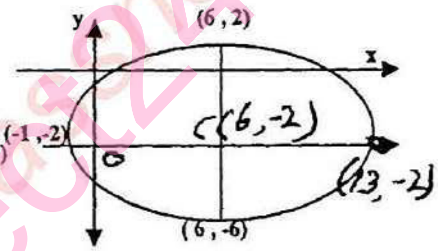
Thus end points of minor axis are $B(-1\ ,\ -1)$ and
$B'(-1\ ,\ 5)$.

Q.2: In each case, determine the equation of graphed ellipse.

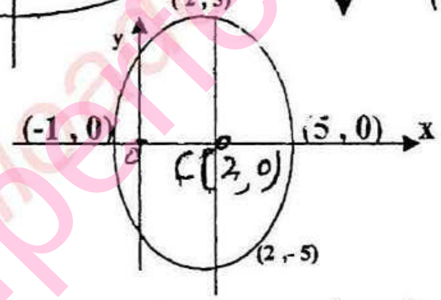
(a)



(b)



(c)



Solution: (i) From figure (i) Centre $C(0\ ,\ 0)$
and major axis is along x – axis .

Extremities of major axis $A(a\ ,\ 0) = A(6\ ,\ 0)$

Then $a = 6$

Extremities of minor axis is $B(0\ ,\ b) = B(0\ ,\ 3)$

$\Rightarrow \quad b = 3$

Now equation of required ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Put values

$$\frac{x^2}{(6)^2}+\frac{y^2}{(3)^2}=1$$

$$\Rightarrow \frac{x^2}{36}+\frac{y^2}{9}=1 \quad \textbf{Ans.}$$

(b) From figure (ii) the centre is $C(h\ ,\ k)=C(6\ ,\ -2)$.

$$\Rightarrow h=6 \quad and \quad k=-2$$

and major axis is along x – axis .

From figure $a=13-6=7$

and $b=2-(-2)=2+2=4$

Now equation of ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

Put values $\Rightarrow \frac{(x-6)^2}{49}+\frac{(y-(-2))^2}{4}=1$

$$\Rightarrow \frac{(x-6)^2}{49}+\frac{(y+2)^2}{4}=1 \quad \textbf{Ans.}$$

(c) From figure (iii) $\Rightarrow C(h\ ,\ k)=C(2\ ,\ 0)$

$$\Rightarrow h=2 \quad and \quad k=0$$

and major axis is along y – axis .

Here $a=5-0=5$ and $b=5-2=3$

Now equation of ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

Put values $\Rightarrow \frac{(x-2)^2}{(3)^2}+\frac{(y-0)^2}{(5)^2}=1$

$$\Rightarrow \frac{(x-2)^2}{9}+\frac{y^2}{25}=1 \quad \textbf{Ans.}$$

Q.3: In each case, write the equation of ellipse through the given information .

(a) Centre is at $C(-3\ ,\ 2)$ and $a=2\ ,\ b=1$, major axis is horizontal.

Solution: Since the major axis on the x – axis. Then

the equation of ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

put $a=2$, $b=1$, $h=-3$ and $k=2$

$$\frac{(x-(-3))^2}{(2)^2}+\frac{(y-2)^2}{(1)^2}=1$$

$$\Rightarrow \frac{(x+3)^2}{4}+\frac{(y-2)^2}{1}=1 \quad \textbf{Ans.}$$

(b) Vertices are $A'(4, 2)$ and $A(12, 2)$, $b=2$

Solution: $A'(4, 2)$ and $A(12, 2)$

Since centre is the mid point of the vertices. Then

$$C(h, k)=C\left(\frac{4+12}{2}, \frac{2+2}{2}\right)$$

$$=C\left(\frac{16}{2}, \frac{4}{2}\right)=C(8, 2)$$

Since vertex is $A(h+a, k)=A(12, 2)$

$h+a=12 \Rightarrow 8+a=12$

$\Rightarrow a=12-8=4$

The equation of ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

put $a=4$, $b=2$, $h=8$ and $k=2$

$$\frac{(x-8)^2}{(4)^2}+\frac{(y-2)^2}{(2)^2}=1$$

$$\Rightarrow \frac{(x-8)^2}{16}+\frac{(y-2)^2}{4}=1 \quad \textbf{Ans.}$$

(c) A focus is at $(-2, 3)$, a vertex is at $(6, 3)$, length of minor axis is 6.

Solution: Here focus is $F(-2, 3)$ and vertex is

$A(6,\ 3)$.

Length of minor axis is $2b=6 \quad => \; b=3$

As vertex is $A(h+a\ ,k)=A(6\ ,\ 3)$

$h+a=6$ .....(i) **and** $k=3$

Also focus point is $F(h-c\ ,k)=F(-3\ ,\ 3)$

$=> h-c=-3$ ....(ii)

Subtract (ii) from (i)

$$\begin{array}{r} h+a=6 \\ \underline{\pm h \mp c=\mp 3} \\ a+c=9 \end{array}$$

$=> \; c=9-a$ .....(iii)

But $c=\sqrt{a^2-b^2}$ **put in (iii)**

$9-a=\sqrt{a^2-b^2}$ **squaring both sides**

$=> \; (9-a)^2=a^2-b^2 \qquad put \;\; b=3$

$=> \; a^2-18a+81=a^2-9$

$=> \; -18a=-9-81 \quad => \; -18a=-90$

$=> \; a=5 \qquad put \;\; a=5 \;\; in(i)$

$h+5=6 \qquad => \; h=1$

As required equation of ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad \textbf{put values}$$

$$=> \; \frac{(x-1)^2}{25}+\frac{(y-3)^2}{9}=1$$

(d) Vertices are at $(0\ ,\ 8)$ *and* $(0\ ,\ 2)$, $c=\sqrt{5}$

<u>Solution</u>: Here $A(0\ ,\ 8)$ **and** $A'(0\ ,\ 2)$

Let $C(h\ ,\ k)$ **be the centre . Then** $C(h\ ,\ k)$ **be the mid point of the vertices .**

Then $C(h\ ,\ k)=C\left(\frac{0+0}{2}\ ,\ \frac{8+2}{2}\right)=(0\ ,\ 5)$.

$=> h=0 \quad and \quad k=5$

As the vertex is $A(h\ ,\ a+k)=A(0\ ,\ 8)$

$$\Rightarrow a+k=8 \quad \Rightarrow a+5=8$$

$$\Rightarrow a=8-5=3$$

Since $c^2=a^2-b^2$

$$\Rightarrow b^2=a^2-c^2=(3)^2-(\sqrt{5})^2=9-5$$

$$\Rightarrow b^2=4 \quad \Rightarrow b=2$$

The equation of ellipse is $\dfrac{(x-h)^2}{b^2}+\dfrac{(y-k)^2}{a^2}=1$

put $a=3$, $b=2$, $h=0$ and $k=5$

$$\Rightarrow \frac{(x-0)^2}{(2)^2}+\frac{(y-5)^2}{(3)^2}=1$$

$$\Rightarrow \frac{x^2}{4}+\frac{(y-5)^2}{9}=1 \quad \textbf{Ans.}$$

Q.4: The shape of an ellipse depends on the eccentricity of the ellipse $e=\dfrac{c}{a}$. Determine

(a) The eccentricity of the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$

Solution: $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$ .....(i)

Comparing (i) with $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, then

$$a^2=25 \quad \Rightarrow \quad a=5$$

And $b^2=16 \quad \Rightarrow \quad b=4$

Since $c=\sqrt{a^2-b^2}=\sqrt{25-16}=\sqrt{9}=3$

Now $e=\dfrac{c}{a}=\dfrac{3}{5}$ **Ans.**

(b) The equation of the ellipse with vertices are at $A(-5\ ,\ 0)$ and $A'(5\ ,\ 0)$ and eccentricity

is $e = \frac{3}{5}$

Solution: Let $C(h, k)$ be the centre of the ellipse. Then $C(h, k)$ is the mid point of $A(-5, 0)$ and $A'(5, 0)$.

$$C(h, k) = C\left(\frac{-5+5}{2}, \frac{0+0}{2}\right) = C(0, 0)$$

Also vertices $V(\pm a, 0) = V(\pm 5, 0)$

$\Rightarrow a = 5$

Now $e = \frac{c}{a} = \frac{3}{5} \Rightarrow c = \frac{3}{5}a$ put $a = 5$

$\Rightarrow c = \frac{3}{5} \times 5 = 3$

Also $b^2 = a^2 - c^2 = (5)^2 - (3)^2$

$\Rightarrow b^2 = 25 - 9 = 16 \Rightarrow b = 4$

Now equation of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ put values

$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$ Ans.

(c) The eccentricity of the ellipse, if the length of the semi major axis is $a = 4$ and the length of semi major axis is $b = 2$

Solution: As $c^2 = a^2 - b^2 = (4)^2 - (2)^2$

$\Rightarrow c^2 = 16 - 4 = 12$

$\Rightarrow c = \sqrt{12}$

Now $e = \frac{c}{a} = \frac{\sqrt{12}}{4}$ Ans.

Q.5: For what value of c,

(a) The line $x - y + c = 0$ will touch the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

Solution: $\frac{x^2}{4}+\frac{y^2}{1}=1$ ....(i) and $x-y+c=0$ .....(ii)

Now we know that the line (ii) touch the ellipse (i)

if $c=\pm\sqrt{a^2m^2+b^2}$ .....(iii)

From (ii) => $y=x+c$

Then the slope of line (ii) is $m=1$

Also from (i) => $a^2=4$ => $a=2$

and $b^2=1$ => $b=1$

put values in (iii)

$$c=\pm\sqrt{(4)(1)^2+1}=\sqrt{5} \quad \textbf{Ans.}$$

(b)The line $2x-y+c=0$ will touch the ellipse $\frac{x^2}{3}+\frac{y^2}{4}=1$

Solution: $\frac{x^2}{3}+\frac{y^2}{4}=1$ ....(i) and $2x-y+c=0$ .....(ii)

Now we know that the line (ii) touch the ellipse (i)

if $c=\pm\sqrt{a^2m^2+b^2}$ .....(iii)

From (ii) => $y=2x+c$

Then the slope of line (ii) is $m=2$

Also from (i) => $a^2=4$ => $a=2$

and $b^2=3$ => $b=\sqrt{3}$

put values in (iii)

$$c=\pm\sqrt{(4)(2)^2+3}=\pm\sqrt{16+3}=\pm\sqrt{19} \quad \textbf{Ans.}$$

(c) The line $x+y+c=0$ will touch the ellipse $\frac{x^2}{25}+\frac{y^2}{11}=1$

Solution: $\frac{x^2}{25}+\frac{y^2}{11}=1$ ....(i) and $x+y+c=0$ .....(ii)

Now we know that the line (ii) touch the ellipse (i)

if $c=\pm\sqrt{a^2m^2+b^2}$ .....(iii)

From (ii) => $y=-x-c$

Then the slope of line (ii) is $m=-1$

Also from (i) => $a^2 = 25$ => $a = 5$

and $b^2 = 11$ => $b = \sqrt{11}$

put values in (iii)

$$c = \pm\sqrt{(25)(-1)^2 + 11} = \pm\sqrt{25+11} = \pm\sqrt{36} = \pm 6 \quad \textbf{Ans.}$$

Q.6: In each case, find the tangent equation and normal equation

(a) at a point $(1, 2)$ to ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Solution: $\frac{x^2}{4} + \frac{y^2}{9} = 1$ .....(i) and point $(1, 2)$.

Comparing (i) with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

Then $a^2 = 9$ => $a = 3$ and

$b^2 = 4$ => $b = 2$

Equation of tangent line at point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ is $\frac{x_1 x}{b^2} + \frac{y_1 y}{a^2} = 1$.....(ii)

put $x_1 = 1$, $y_1 = 2$, $a = 3$, $b = 2$ in (ii)

$$\frac{1.x}{(2)^2} + \frac{2y}{(3)^2} = 1 \Rightarrow \frac{x}{4} + \frac{2y}{9} = 1 \quad \text{multiplied by } 36$$

=> $9x + 8y = 36$.

Equation of normal at point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ is $\frac{b^2(x - x_1)}{x_1} = \frac{a^2(y - y_1)}{y_1}$ ...(iii)

put values in (iii)

$$\frac{4(x-1)}{1} = \frac{9(y-2)}{2}$$

=> $8x - 8 = 9y - 18$

=> $8x - 9y + 10 = 0$ *Ans.*

(b) at a point $(3\ ,\ 6)$ to ellipse $\frac{x^2}{7}+\frac{y^2}{4}=1$

Solution: $\frac{x^2}{7}+\frac{y^2}{4}=1$ .....(i) and point $(3\ ,\ 6)$.

Comparing (i) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Then $a^2=7$ **and** $b^2=4$

Equation of tangent line at point $P(x_1\ ,\ y_1)$ **to the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **is** $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$ .....(ii)

put $x_1=3\ ,\ y_1=6\ ,\ a^2=7\ ,\ b^2=4$ **in (ii)**

$\frac{3x}{7}+\frac{6y}{4}=1$ **multiplied by 28**

=> $12x+42y=28$ **, Divided by 2**

=> $6x+21y=14$ **Ans.**

Equation of normal at point $P(x_1\ ,\ y_1)$ **to the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **is** $\frac{a^2(x-x_1)}{x_1}=\frac{b^2(y-y_1)}{y_1}$ ...(iii)

put values in (iii)

$\frac{7(x-3)}{3}=\frac{4(y-6)}{6}$ => $\frac{7(x-3)}{3}=\frac{2(y-6)}{3}$

=> $7x-21=2y-12$

=> $7x-2y-21+12=0$

=> $7x-2y-9=0$ *Ans.*

(c) at a point $(1\ ,1)$ to ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

Solution: $\frac{x^2}{4}+\frac{y^2}{1}=1$ .....(i) and point $(1\ ,\ 1)$.

Comparing (i) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Then $a^2 = 4$ and $b^2 = 1$

Equation of tangent line at point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$ .....(ii)

put $x_1 = 1$, $y_1 = 1$, $a^2 = 4$, $b^2 = 1$ in (ii)

$\frac{1.x}{4} + \frac{1.y}{1} = 1$ multiplied by 4

$\Rightarrow x + 4y = 4$ Ans.

Equation of normal at point $P(x_1, y_1)$ to the

ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2(x - x_1)}{x_1} = \frac{b^2(y - y_1)}{y_1}$ ...(iii)

put values in (iii)

$$\frac{4(x-1)}{1} = \frac{1(y-1)}{1}$$

$\Rightarrow 4x - 4 = y - 1$

$\Rightarrow 4x - y - 3 = 0$ *Ans.*

Q.7: Find the tangent equation

(a) at a point $(1, 2)$ to ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ which is perpendicular to the line $9x + 8y - 36 = 0$

Solution: $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ....(i) and $9x + 8y - 36 = 0$....(ii)

From (ii) $\Rightarrow 8y = -9x + 36$

$\Rightarrow y = -\frac{9}{8}x + \frac{9}{2}$

Then the slope of line (ii) is $m_1 = -\frac{9}{8}$

Let $y - y_1 = m(x - x_1)$ .....(iii) is the equation of required tangent. As (ii) is perpendicular to (iii) then $m_1.m = -1 \Rightarrow m = -\frac{1}{m_1} = -\frac{1}{-\frac{9}{8}} = \frac{8}{9}$

put $m = \frac{8}{9}$, $x_1 = 1$, $y_1 = 2$ in (iii)

$$y - 2 = \frac{8}{9}(x-1) \Rightarrow 9y - 18 = 8x - 8$$

$\Rightarrow 0 = 8x - 9y - 8 + 18$

$\Rightarrow 8x - 9y + 10 = 0$ **Ans.**

(b) at a point $(3, 6)$ to ellipse $\frac{x^2}{7} + \frac{y^2}{4} = 1$ which is perpendicular to the line $6x + 21y - 14 = 0$

Solution: $\frac{x^2}{7} + \frac{y^2}{4} = 1$....(i) and $6x + 21y - 14 = 0$....(ii)

From (ii) $\Rightarrow 21y = -6x + 14$, Divided by 7

$$\Rightarrow y = -\frac{6}{21}x + \frac{14}{21}$$

Then the slope of line (ii) is $m_1 = -\frac{6}{21}$

Let $y - y_1 = m(x - x_1)$ .....(iii) is the equation of required tangent. As (ii) is perpendicular to (iii)

then $m_1.m = -1 \Rightarrow m = -\frac{1}{m_1} = -\frac{1}{-\frac{6}{21}} = \frac{21}{6}$

put $m = \frac{21}{6}$, $x_1 = 3$, $y_1 = 6$ in (iii)

$$y - 6 = \frac{21}{6}(x-3) \Rightarrow 6y - 36 = 21x - 63$$

$\Rightarrow 0 = 21x - 6y - 63 + 36$

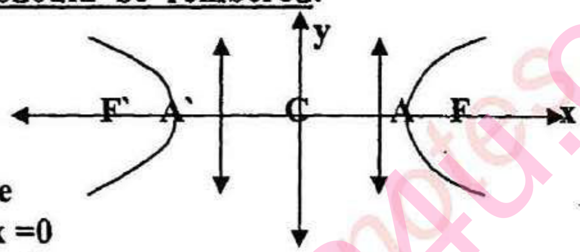
$\Rightarrow 21x - 6y - 27 = 0$ **Ans.**

Hyperbola: It is a conic in which the distance of each point from the fixed point bears a constant ratio greater than "1" to its distance from the fixed straight line.

Standerd Equation of Hyperbola:

(a) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The following point should be rembered:



(i) x-axis is tranverse axis its equation $y = 0$ its length $=2a$

(ii) y-axis is conjugate axis its equation, $x = 0$

(iii) **Centre:** $x = 0$, $y = 0$
$\Rightarrow C(0, 0)$

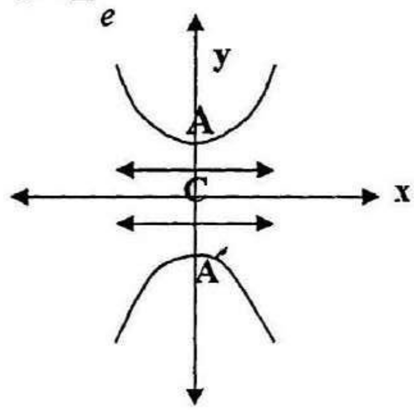
(iv) **Foci points:** $x = \pm c$, $y = 0$

$\Rightarrow F(c, 0), F'(-c, 0)$ Where $c = \sqrt{a^2 + b^2}$

(v) **Vertices:** The extremities of Transverse axis is called vertices. i.e. $x = \pm a$, $y = 0$

$\Rightarrow V_1(a, 0)$, $V_2(-a, 0)$

(vi) **Equation of Directrices:** $x = \pm \frac{a}{e}$



(b) $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

(i) y-axis is Transverse axis, its equation is $x=0$, its length $= 2a$

(ii) x-axis is conjugate axis, its equation is $y = 0$

(iii) **centre:**
$x = 0$, $y = 0 \Rightarrow c(0,0)$

(iv) **Foci points:** $x = 0$, $y = \pm c$

$\Rightarrow F(0,\ c),\ F'(-c\ ,\ 0)$, where $c=\sqrt{a^2+b^2}$

(v) **Vertices:** $V_1(0\ ,\ a)\ ,V_2(0,-a)$

(vi) **Equation of Directrices:** $y=\pm\frac{a}{e}$.

Orientation of hyperbola: Let $C(h\ ,\ k)$ be the centre of hyperbola. Then

(a) $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

(i) Equation of transverse axis is $y=k$

(ii) Equation of conjugate axis is $x=h$

(iii) Vertices are $V_1(h-a\ ,\ k)$ and $V_2(h+a\ ,\ k)$

(iv) Foci points $F'(h-c\ ,\ k)$ and $F(h+c\ ,\ k)$

(b) $\frac{(y-h)^2}{a^2}-\frac{(x-k)^2}{b^2}=1$

(i) Equation of transverse axis is $x=h$

(ii) Equation of conjugate axis is $y=k$

(iii) Vertices are $V_1(h\ ,\ k-a)$ and $V_2(h\ ,\ k+a)$

(iv) Foci points $F'(h\ ,\ k-c)$ and $F(h\ ,\ k+c)$

(a) **Equation of tangent line to the hyperbola** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

in slope form is $y=mx\pm\sqrt{a^2m^2-b^2}$ ......(a)

Note: The tangent line $y=mx+c$ touch the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $c=\pm\sqrt{a^2m^2-b^2}$ Ans.

(b) **Equation of tangent line at point** $P(x_1\ ,\ y_1)$ to the

Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1$

(c) **Equation of normal at point** $P(x_1\ ,\ y_1)$ to the

hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{b^2(y-y_1)}{y_1}=-\frac{a^2(x-x_1)}{x_1}$.

Definition: If a straight line cuts a hyperbola in two points at an infinite distance from the origin and is itself at a finite distance from the origin is then called the asymptotes.

(a) For hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ the asymptotes equation is

$$y=\pm\frac{b}{a}x$$

(b) For hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ the asymptotes equation is

$$y=\pm\frac{a}{b}x$$

(c) For translating hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ the asymptotes equation is $y-k=\pm\frac{b}{a}(x-h)$

(d) For translating hyperbola $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ the asymptotes equation is $y-k=\pm\frac{a}{b}(x-h)$

Exercise 9.3:

Q.1: In each case, the hyperbola represented by the equation indicate the centre, vertices, foci and the equation of the asymptotes.

(a) $\frac{x^2}{4}-\frac{y^2}{9}=1$

Solution: $\frac{x^2}{4}-\frac{y^2}{9}=1$ ......(i) Compare (i) with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$$a^2=4 \Rightarrow a=2 \quad and \quad b^2=9 \Rightarrow b=3$$

(i) As centre $x=0$ and $y=0$

Thus the centre is $C(0\,,\,0)$

(ii) **Vertices :** As $V_1(-a\ ,\ 0)=V_1(-2\ ,0)$

And $V_2(a\ ,0)=V_2(2\ ,\ 0)$

(iii) **Foci :** As $c=\sqrt{a^2+b^2}=\sqrt{4+9}=\sqrt{13}$

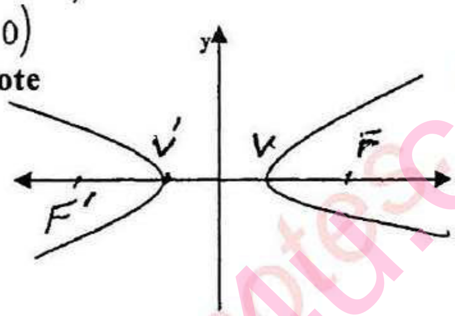
Then $F_1(-c\ ,0)=F_1(-\sqrt{13}\ ,\ 0)$

$F_2(c\ ,\ 0)=F_2(\sqrt{13}\ ,\ 0)$

(iv) **As equation of asymptote**

is $y=\pm\dfrac{b}{a}x$

=> $y=\pm\dfrac{3}{2}x$



(b) $\dfrac{y^2}{25}-\dfrac{x^2}{4}=1$

Solution: $\dfrac{y^2}{25}-\dfrac{x^2}{4}=1$ ....(i) **Comparing (i) with**

$\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=1$ **, then** $a^2=25$ => $a=5$

And $b^2=4$ => $b=2$

(i) **As centre** $x=0$ *and* $y=0$

Thus the centre is $C(0\ ,0)$

(ii) **Vertices :** As $V_1(0\ ,\ -a)=V_1(0\ ,-5)$

And $V_2(0\ ,a)=V_2(0\ ,\ 5)$

(iii) **Foci :** As $c=\sqrt{a^2+b^2}=\sqrt{25+4}=\sqrt{29}$

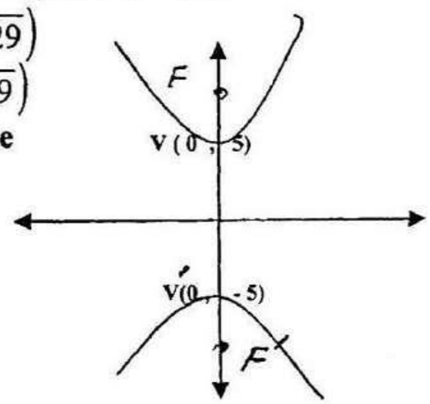
Then $F_1(0\ ,-c)=F_1(0\ ,\ \sqrt{29})$

$F_2(0\ ,\ c)=F_2(0\ ,-\sqrt{29})$

(iv) **As equation of asymptote**

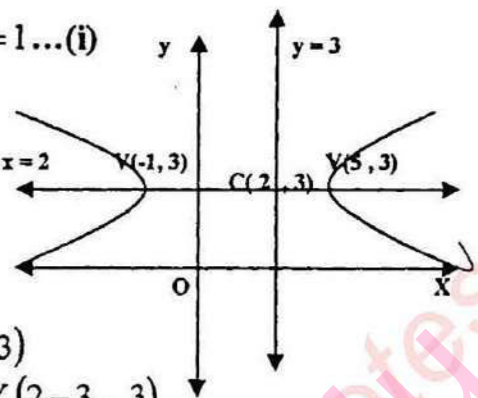
is $y=\pm\dfrac{a}{b}x$

=> $y=\pm\dfrac{5}{2}x$



(c) $\dfrac{(x-2)^2}{9}-\dfrac{(y-3)^2}{16}=1$

Solution: $\dfrac{(x-2)^2}{9}-\dfrac{(y-3)^2}{16}=1\ldots$(i)



Comparing with

$\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$

Then $a^2=9 \Rightarrow a=3$

And $b^2=16 \Rightarrow b=4$

(i) **Centre** $C(h\ ,\ k)=C(2\ ,\ 3)$

(ii) **Vertices :** $V_1(h-a,\ k)=V_1(2-3\ ,\ 3)$

$V_1(-1\ ,\ 3)$

And $V_2(h+a,\ k)=V_2(2+3\ ,3)=V_2(5\ ,\ 3)$

(iii) **Foci :** **As** $c=\sqrt{a^2+b^2}=\sqrt{9+16}=\sqrt{25}=5$

$F_1(h-c\ ,\ k)=F_1(2-5\ ,\ 3)=F_1(-3\ ,\ 3)$

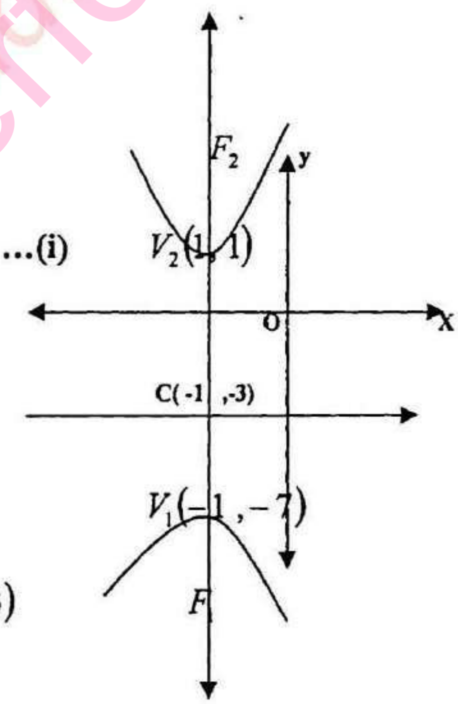
$F_2(h+c\ ,k)=F_2(2+5\ ,\ 3)=F_2(7\ ,\ 3)$

(iv) **As equation of asymptote**

is $y=\pm\dfrac{b}{a}x \Rightarrow y=\pm\dfrac{4}{3}x$

(d) $\dfrac{(y+1)^2}{16}-\dfrac{(x+3)^2}{25}=1$

Solution: $\dfrac{(y+1)^2}{16}-\dfrac{(x+3)^2}{25}=1\ \ldots$(i)



Comparing with

$\dfrac{(y-h)^2}{a^2}-\dfrac{(x-k)^2}{b^2}=1$

Then $a^2=16 \Rightarrow a=4$

And $b^2=25 \Rightarrow b=5$

$h=-1$ *and* $k=-3$

(i) **Centre** $C(h\ ,\ k)=C(-1\ ,-3)$

(ii) Vertices : $V_1(h,\ k-a) = V_1(-1\ ,\ -3-4)$

$V_1(-1\ ,\ -7)$

And $V_2(h,\ k+a) = V_2(-1\ ,-3+4) = V_2(-1\ ,\ 1)$

(iii) Foci : **As** $c = \sqrt{a^2+b^2} = \sqrt{16+25} = \sqrt{41}$

$F_1(h\ ,\ k-c) = F_1\left(-1\ ,-3-\sqrt{41}\right)$

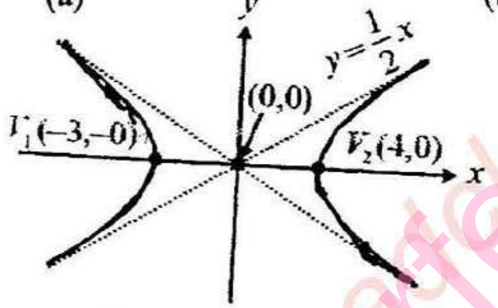
$F_2(h\ ,\ k+c) = F_2\left(-1\ ,\ -3+\sqrt{41}\right)$

(iv) As equation of asymptote

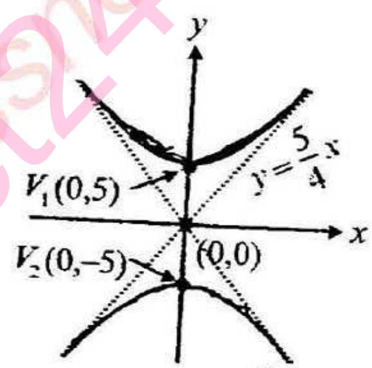
is $y = \pm\dfrac{a}{b}x$ => $y = \pm\dfrac{4}{5}x$

<u>Q.2</u>: In each case, determine the equation of graphed hyperbola

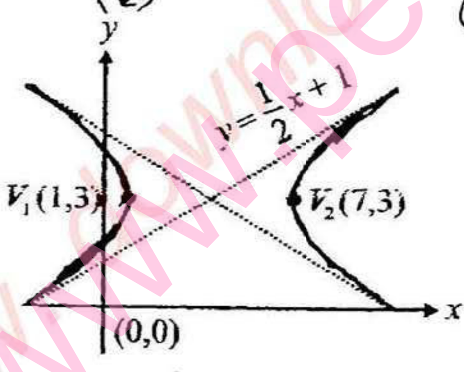
(a)



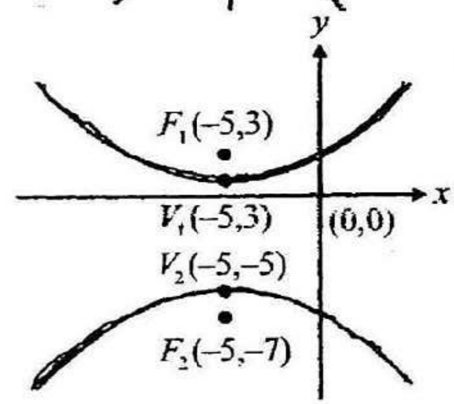
(b)



(c)



(d)



<u>Solution</u>: From figure (a) , The transverse axis along x- axis . Centre is $C(0\ ,\ 0)$.

Vertex is $V(a\ ,\ 0) = V(4\ ,\ 0)$

$\Rightarrow\ a = 4$

As asymptotes equation is $y = \dfrac{b}{a}x = \dfrac{b}{4}x$ .....(i)

From figure asymptotes equation is $y = \dfrac{1}{2}x = \dfrac{2}{4}x$ ...(ii)

From (i) and (ii) $b = 2$

Now equation of hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$\Rightarrow\ \dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$ Ans.

(b) From figure (b) the transverse axis along y – axis and centre is $C(0\ ,\ 0)$ and vertex is $V(0\ ,\ a) = V(0\ ,\ 5)$ then $a = 5$

Also we know the asymptotes equation is

$y = \dfrac{a}{b}x = \dfrac{5}{b}x$ .....(i)

But from figure the asymptotes is $y = \dfrac{5}{4}x$ .....(ii)

From (i) and (ii) $\Rightarrow\ b = 4$

Now equation of hyperbola is $\dfrac{x^2}{(5)^2} - \dfrac{y^2}{(4)^2} = 1$

$\Rightarrow\ \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$ Ans.

(c) From figure (c) the transverse axis is parallel to x – axis. And vertices are $V_1(1\ ,\ 3)$ *and* $V_2(7\ ,\ 3)$.

Let $C(h\ ,\ k)$ be the centre then $C(h\ ,\ k)$ is the mid point of $V_1(1\ ,\ 3)$ *and* $V_2(7\ ,\ 3)$.

$$C(h\ ,\ k) = C\left(\frac{1+7}{2}\ ,\ \frac{3+3}{2}\right) = C(4\ ,\ 3)$$

$\Rightarrow\ h=4 \quad and \quad k=3$

Now vertex is $V(h+a\ ,\ k)=V(7\ ,\ 3)$

$\Rightarrow\ h+a=7$ **, put** $h=4$

$\Rightarrow\ 4+a=7$

$\Rightarrow\ a=7-4=3$

Also we know the asymptotes equation is

$$y-k=\frac{b}{a}(x-h)=\frac{b}{3}x \text{ .....(i)}$$

But from figure the asymptotes is $y=\frac{1}{2}x+1$ **.....(ii)**

From (i) and (ii) $\Rightarrow\ \frac{b}{3}=\frac{1}{2}$

$\Rightarrow\ b=\frac{3}{2}$

Now equation of hyperbola is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

$$\Rightarrow\ \frac{(x-4)^2}{(3)^2}-\frac{(y-3)^2}{\left(\frac{3}{2}\right)^2}=1$$

$$\Rightarrow\ \frac{(x-4)^2}{9}-\frac{4(y-3)^2}{9}=1 \quad \textbf{Ans.}$$

(d) From figure (d) the transverse axis is parallel to y – axis.

Vertices are $V_1(-5\ ,\ -5)$ *and* $V_2(-5\ ,\ 1)$

Let $C(h\ ,\ k)$ **be the centre then** $C(h\ ,\ k)$ **is the mid point of** $V_1(-5\ ,\ -5)$ *and* $V_2(-5\ ,\ 1)$**, Then**

$$C(h\ ,\ k)=C\left(\frac{-5-5}{2}\ ,\frac{-5+1}{2}\right)=C\left(-\frac{10}{2}\ ,\ -\frac{4}{2}\right)$$

$\Rightarrow\ C(h,\ k)=C(-5\ ,\ -2)$

$\Rightarrow\ h=-5 \quad and \quad k=-2$

Now vertex is $V(h\ ,\ k+a)=V(-5\ ,\ 1)$

$k+a=1 \quad \Rightarrow\ -2+a=1 \quad \Rightarrow\ a=3$

As from figure focus $F(-5\ ,3)=F(h\ ,\ k+c)$

$\Rightarrow k+c=3 \quad \Rightarrow -2+c=3$

$\Rightarrow c=2+3=5$

Now we know that $c^2=a^2+b^2$

Or $b^2=c^2-a^2=(5)^2-(3)^2$

$\Rightarrow b^2=25-9=16 \quad \Rightarrow b=4$

Now equation of hyperbola is $\dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1$

Put values

$$\frac{(y-(-2))^2}{(3)^2}-\frac{(x-(-5))^2}{(4)^2}=1$$

$$\Rightarrow \frac{(y+2)^2}{9}-\frac{(x+5)^2}{16}=1 \quad \text{Ans.}$$

Q.3: In each case, write the equation of hyperbola through the given information .

(a) Foci are at $(0,\ 3)$ and $(0\ ,-3)$, one vertex at $V_1(0\ ,-2)$

Solution: Since foci and vertices on y – axis and the

centre is origin $O(0\ ,\ 0)$. Then the equation of hyperbola is $\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=1$ ......(i)

As one vertex is $V_1(0\ ,-2)$ then the other vertex is $V_2(0\ ,2)$

Also $V(0\ ,\ a)=V(0\ ,2) \quad \Rightarrow a=2$

As $F_1(0\ ,\ 3)=F(0\ ,\ c) \quad \Rightarrow c=3$

Now we know that $c^2=a^2+b^2$

$\Rightarrow b^2=c^2-a^2=(3)^2-(2)^2=9-4=5$

$\Rightarrow b^2=5$

Put values of a^2 and b^2 in (i)

$$\frac{y^2}{4}-\frac{x^2}{5}=1 \quad \text{Ans.}$$

(b) Vertices are at $(5\ ,\ 0)$ *and* $(-5\ ,\ 0)$, one focus is at $F(-7\ ,\ 0)$.

Solution: As $V(a,\ 0)=V(5\ \ 0)$

=> $a=5$. Also centre is $C(0\ ,\ 0)$

And $F(c\ ,0)=F(-7\ ,\ 0)$ => $c=-7$

Now we know that $c^2=a^2+b^2$

=> $b^2=c^2-a^2=(-7)^2-(5)^2=49-25$

=> $b^2=24$

As equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ .....(i)

Put values of a^2 *and* b^2 in (i)

$$=> \quad \frac{x^2}{25}-\frac{y^2}{24}=1 \quad \text{Ans.}$$

(c) Transverse axis on the x- axis, asymptotes are the line $y=3x$ *and* $y=-3x$

Solution: As transverse axis on the x- axis then the

equation of required hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \ \text{......(i)}$$

As we know that the asymptote equation is

$$y=\pm\frac{b}{a}x \ \text{....(i)}$$

But given asymptote equation is $y=\pm 3x$ ....(ii)

From (i) and (ii) => $\frac{b}{a}=\frac{3}{1}$

=> $a=3$ *and* $b=1$, put values in (i)

$$\frac{x^2}{1}-\frac{y^2}{(3)^2}=1 \quad => \quad x^2-\frac{y^2}{9}=1$$

$=> \ 9x^2-y^2=9$

(d) **Foci are at** $(5\ ,\ 0)$ *and* $(-5\ ,\ 0)$ **, eccentricity is** $\frac{5}{3}$.

<u>Solution</u>: Here Foci lie on x –axis then transverse axis is x- axis. As $F(5\ ,\ 0)=F(c\ ,\ 0)$ **and** $e=\frac{5}{3}$

$=> \ c=5$ **and** $e=\frac{c}{a} \ => \ \frac{c}{a}=\frac{5}{3}$ **put value of e and c**

$$=> \ \frac{5}{a}=\frac{5}{3} \quad => \quad a=3$$

Also $b^2=c^2-a^2=(5)^2-(3)^2=25-9$

$=> \ b^2=16$

As required equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Put values $=> \ \frac{x^2}{9}-\frac{y^2}{16}=1$

$=> 16x^2-9y^2=144$ **Ans.**

(e) **Vertices are** $(3\ ,\ -1)$ **and** $(-1\ ,\ -1)$ **and asymptotes are the lines** $y=\frac{9}{4}x-\frac{13}{4}$ **and** $y=-\frac{9}{4}x+\frac{5}{4}$.

<u>Solution</u>: Centre is the mid point of vertices . Then

$$C(h\ ,\ k)=C\left(\frac{3-1}{2}\ ,\ \frac{-1-1}{2}\right)=C(1\ ,\ -1)$$

Also vertex is $V(h+a\ ,\ k)=V(3\ ,\ -1)$

$=> \ h+c=3$ **.....(i) and** $k=-1$

And $V(h-a,\ k)=V(-1,\ -1)$

$\Rightarrow h-a=-1$ .....(ii) adding (i) and (ii)

$$h+a=3$$
$$h-a=-1$$
$$2h=2$$

$\Rightarrow h=1$, put $h=1$ in (i)

$1+a=3 \Rightarrow a=2$

As we know that the asymptote equation is

$$y-k=\pm\frac{b}{a}(x-h)$$

$$\Rightarrow y-(-1)=\pm\frac{b}{2}(x-1)$$

$$\Rightarrow y+1=\pm\frac{b}{2}(x-1)$$

$$\Rightarrow y=\pm\frac{b}{2}(x-1)-1$$

$$\Rightarrow y=\pm\frac{b}{2}x\pm\frac{b}{2}-1 \text{......}(iii)$$

But given asymptote equation is

$y=\frac{9}{4}x-\frac{13}{4}$ ......(iv) and $y=-\frac{9}{4}x+\frac{5}{4}$......(v)

From (iii) and (iv) $\Rightarrow \frac{b}{2}=\frac{9}{4} \Rightarrow b=\frac{9}{2}$

Now equation of hyperbola is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Put values $\Rightarrow \frac{(x-1)^2}{4}-\frac{(y+1)^2}{\frac{81}{4}}=1$

$$\Rightarrow \frac{(x-1)^2}{4}-\frac{4(y+1)^2}{81}=1 \text{ Ans.}$$

Q.4: Determine the path of a point that moves so that the difference of its distances from

(a) The points $(-5, 0)$ and $(5, 0)$ is 8.

Solution: Here foci are $F_1(-5, 0)$ and $F_2(5, 0)$

And difference is 8. i.e. distance between vertices are 8

Then $2a = 8 \Rightarrow a = 4$

And $F(c, 0) = F(5, 0) \Rightarrow c = 5$

Also $b^2 = c^2 - a^2 = (5)^2 - (4)^2$

$\Rightarrow b^2 = 25 - 16 = 9 \Rightarrow b = 3$

Now equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{x^2}{(4)^2} - \frac{y^2}{(3)^2} = 1 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$ Ans.

(b) The points $(0, -13)$ and $(0, 13)$ is 10.

Solution: Here foci are $F_1(0, -13)$ and $F_2(0, 13)$

And difference is 10. i.e. distance between vertices are 10

Then $2a = 10 \Rightarrow a = 5$

And $F(0, 13) = F(0, c) \Rightarrow c = 13$

Also $b^2 = c^2 - a^2 = (13)^2 - (5)^2$

$\Rightarrow b^2 = 169 - 25 = 144 \Rightarrow b = 12$

Now equation of hyperbola is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

$\Rightarrow \frac{y^2}{(5)^2} - \frac{x^2}{(12)^2} = 1 \Rightarrow \frac{y^2}{25} - \frac{x^2}{144} = 1$ Ans.

Q.5: Write the equation of hyperbola

(a) With vertices at $(2, -2)$, $(-4, -2)$ and that passes through the point with coordinates $(5, 1)$.

Solution: As vertices are $V_1(2, -2)$, *and* $V_2(-4, -2)$

Here we see from vertices that x – coordinates are changes and y –coordinates is same. Thus the transverse axis is parallel to x – axis.

Let $C(h, k)$ be the centre then $C(h, k)$ is the mid point of vertices. Then

$$C(h, k) = C\left(\frac{2-4}{2}, \frac{-2-2}{2}\right) = C\left(-\frac{2}{2}, -\frac{4}{2}\right)$$

$\Rightarrow C(h, k) = C(-1, -2) \quad \Rightarrow h = -1 \quad and \quad k = -2$

Also $V(h+a, k) = V(2, -2) \quad \Rightarrow h + a = 2$

$\Rightarrow -1 + a = 2 \quad \Rightarrow a = 3$ Now equation of hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{put values}$$

$$\Rightarrow \frac{(x-(-1))^2}{(3)^2} - \frac{(y-(-2))^2}{b^2} = 1$$

$$\Rightarrow \frac{(x+1)^2}{9} - \frac{(y+2)^2}{b^2} = 1 \quad ....(i)$$

Since (i) passes through the point $(5, 1)$.

Put $x = 5$ and $y = 1$ in (i)

$$\frac{(5+1)^2}{9} - \frac{(1+2)^2}{b^2} = 1 \Rightarrow \frac{36}{9} - \frac{9}{b^2} = 1$$

$$\Rightarrow 4 - \frac{9}{b^2} = 1 \qquad \Rightarrow 4 - 1 = \frac{9}{b^2}$$

$\Rightarrow \frac{9}{b^2} = 3 \qquad \Rightarrow b^2 = \frac{9}{3} = 3$ Put value of $b^2 = 3$ in (ii)

$$\Rightarrow \frac{(x+1)^2}{9} - \frac{(y+2)^2}{3} = 1 \qquad \text{Ans.}$$

(b) With vertices at $(-3, 1)$, $(-3, 3)$ and that passes through the point with coordinates $(0, 4)$.

Solution: As vertices are $V_1(-3, 1)$, and $V_2(-3, 3)$

Here we see from vertices that y– coordinates are changes and x–coordinates is same. Thus the transverse axis is parallel to y axis.

Let $C(h, k)$ be the centre then $C(h, k)$ is the mid point of vertices. Then

$$C(h\,,\,k)=C\left(\frac{-3-3}{2}\,,\,\frac{1+3}{2}\right)=C\left(-\frac{6}{2}\,,\,\frac{4}{2}\right)$$

$\Rightarrow\ C(h\,,\,k)=C(-3\,,\,2)\quad \Rightarrow\ h=-3\quad and\quad k=2$

Also $V(h\,,\,k+a)=V(-3\,,\,3)\quad \Rightarrow\ k+a=3$

$\Rightarrow\ 2+a=3\quad \Rightarrow\ a=1$ **Now equation of hyperbola is**

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \quad \textbf{put values}$$

$$\Rightarrow \frac{(y-2)^2}{(1)^2}-\frac{(x-(-3))^2}{b^2}=1 \Rightarrow (y-2)^2-\frac{(x+3)^2}{b^2}=1 \ ....(i)$$

since (i) passes through the point $(0\,,\,4)$.

Put $x=0\quad and\quad y=4$ **in (i)**

$$(4-2)^2-\frac{(0+3)^2}{b^2}=1 \Rightarrow (2)^2-\frac{9}{b^2}=1$$

$$\Rightarrow 4-\frac{9}{b^2}=1 \qquad \Rightarrow 4-1=\frac{9}{b^2}$$

$\Rightarrow \dfrac{9}{b^2}=3 \qquad \Rightarrow b^2=\dfrac{9}{3}=3$ **Put value of** $b^2=3$ **in (ii)**

$$\Rightarrow \frac{(x+1)^2}{9}-\frac{(y+2)^2}{3}=1 \qquad \textbf{Ans.}$$

<u>Q.6</u>: In each case, sketch the rectangular hyperbola and identify the vertices, the foci and the asymptotes.

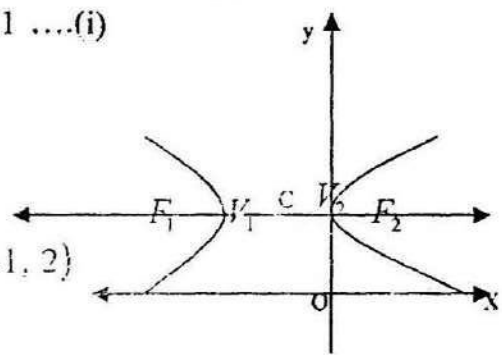
(a) $(x+1)^2-(y-2)^2=1$

<u>Note</u>: A hyperbola is called rectangular hyperbola if $a=b$

<u>Solution</u>: $(x+1)^2-(y-2)^2=1\ \ldots.(i)$

Comparing (i) with

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$



Then $h=-1\quad and\quad k=2$

The centre is $C(h,\,k)=C(-1,\,2)$

And $a^2=1=b^2$

(i) Vertices:

As $V_1(h-a\ ,\ k)=V_1(-1-1,\ 2)$

$=> V_1(h-a\ ,\ k)=V_1(-2\ ,\ 2)$

And $V_2(h+a,\ k)=V_2(-1+1\ ,2)$

$=V_2(0\ ,\ 2)$

Thus vertices are $V_1(-2\ ,\ 2)$ *and* $V_2(0\ ,\ 2)$.

(ii) Foci : As $c^2=a^2+b^2=1+1=2$

$=> c=\sqrt{2}$

Now $F_1(h-c\ ,\ k)=F_1(-1-\sqrt{2}\ ,\ 2)$

$F_2(h+c\ ,\ k)=F_2(-1+\sqrt{2}\ ,\ 2)$

(iii) The asymptotes equation is $y-k=\pm\frac{b}{a}(x-h)$

$=> y-2=\pm\frac{1}{1}(x-(-1)) \quad => y-2=\pm(x+1)$

$=> y=x+1+2 \quad$ *and* $\quad y=-x-1+2$

$=> y=x+3 \quad$ *and* $\quad y=-x+1$

(b) $\frac{(x-3)^2}{4}-\frac{(y+1)^2}{4}=1$

Solution: $\frac{(x-3)^2}{4}-\frac{(y+1)^2}{4}=1$ ....(i)

Comparing (i) with

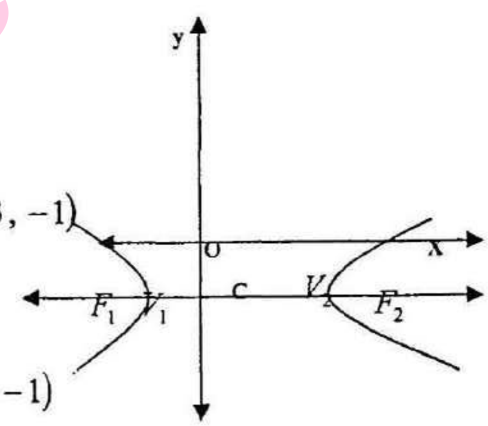
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Then $h=3$ *and* $k=-1$

The centre is $C(h,\ k)=C(3,\ -1)$

And $a^2=4=b^2$

$=> a=2=b$



(i) Vertices:

As $V_1(h-a\ ,\ k)=V_1(3-2,\ -1)$

$=> V_1(h-a\ ,\ k)=V_1(1\ ,\ -1)$

And $V_2(h+a,\ k)=V_2(3+2\ ,-1)$

$=V_2(5\ ,\ -1)$

Thus vertices are $V_1(1\ ,-1)$ *and* $V_2(5\ ,\ -1)$.

(ii) Foci : As $c^2=a^2+b^2=4+4=8$

$\Rightarrow\ c=2\sqrt{2}$

Now $F_1(h-c\ ,\ k)=F_1\left(3-2\sqrt{2}\ ,\ -1\right)$

$F_2(h+c\ ,\ k)=F_2\left(3+2\sqrt{2}\ \ ,-1\right)$

(iii) The asymptotes equation is $y-k=\pm\dfrac{b}{a}(x-h)$

$\Rightarrow\ y-(-1)=\pm\dfrac{2}{2}(x-3)\quad \Rightarrow\ y+1=\pm(x-3)$

$\Rightarrow\ y=x-3-1\quad$ *and* $\quad y=-x+3-1$

$\Rightarrow\ y=x-4\quad$ *and* $\quad y=-x+2$

<u>Q.7</u>: In each case, find the point of intersection in between the line and the hyperbola :

(a) $xy=4$ **and** $y=x-3$

<u>Solution</u>: $xy=4$ **….(i) and** $y=x-3$ **……(ii)**

Put (ii) in (i) $\Rightarrow\ x(x-3)=4$

$\Rightarrow\ x^2-3x-4=0$

$\Rightarrow\ x^2+x-4x-4=0$

$\Rightarrow\ x(x+1)-4(x+1)=0\quad \Rightarrow\ (x+1)(x-4)=0$

$\Rightarrow\ x+1=0\quad$ *and* $\quad x-4=0$

$\Rightarrow\ x=-1\quad$ *and* $\quad x=4$

Put $x=-1$ **in (ii)** $y=-1-3=-4$

Put $x=4$ **in (ii)** $y=4-3=1$

Thus intersection points are $(-1\ ,\ -4)\ ,\ (4\ ,\ 1)$.

(b) $\dfrac{y^2}{1}-\dfrac{x^2}{4}=1\ ,\quad x+4y-4=0$

Solution: $\frac{y^2}{1}-\frac{x^2}{4}=1$ => $4y^2-x^2=4$ .....(i)

And $x+4y-4=0$ .....(ii)

From (ii) => $x=4-4y$ ....(iii)

Put (iii) in (i)

$4y^2-(4-4y)^2=4$

=> $4y^2-(16+16y^2-32y)-4=0$

=> $4y^2-16-16y^2+32y-4=0$

=> $-12y^2+32y-20=0$ => $-4(3y^2-8y+5)=0$

=> $3y^2-8y+5=0$

=> $3y^2-3y-5y+5=0$ => $3y(y-1)-5(y-1)=0$

=> $(y-1)(3y-5)=0$

=> $y-1=0$ *and* $3y-5=0$

=> $y=1$ *and* $y=\frac{5}{3}$

Put $y=1$ in (iii) => $x=4-4(1)=4-4=0$

Put $y=\frac{5}{3}$ in (iii) => $x=4-4\left(\frac{5}{3}\right)=\frac{12-20}{3}$

=> $x=-\frac{8}{3}$

Thus intersection points are $(0\ ,\ 1)$ *and* $\left(-\frac{8}{3},\frac{5}{3}\right)$.

Q.8: For what value of c ,

(a) The line $y=x+c$ will touch the hyperbola $\frac{x^2}{16}-\frac{y^2}{4}=1$

Solution: $y=x+c$ ....(i) and $\frac{x^2}{16}-\frac{y^2}{4}=1$ ......(ii)

As we know that the tangent line $y=mx+c$ touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

if $c = \pm\sqrt{a^2m^2 - b^2}$ ....(A)

From (i) => $m = 1$ and from (ii)

$a^2 = 16$ *and* $b^2 = 4$, Put values in (A) .

$$c = \pm\sqrt{(16)(1)^2 - 4} = \pm\sqrt{16-4} = \pm\sqrt{12}$$

=> $c = \pm 2\sqrt{3}$

(b) The line $y = -x + c$ will touch the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Solution: $y = -x + c$ ....(i) and $\frac{x^2}{16} - \frac{y^2}{9} = 1$ ......(ii)

As we know that the tangent line $y = mx + c$ touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

if $c = \pm\sqrt{a^2m^2 - b^2}$ ....(A)

From (i) => $m = -1$

and from (ii) $a^2 = 16$ *and* $b^2 = 9$, Put values in (A) .

$$c = \pm\sqrt{(16)(-1)^2 - 9} = \pm\sqrt{16-9}$$

=> $c = \pm\sqrt{7}$ *Ans.*

Q. 9: In each case , find the tangent equation and normal equation .

(a) At a point $\left(\sqrt{13}, \frac{9}{2}\right)$ to hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

Solution: $\frac{x^2}{4} - \frac{y^2}{9} = 1$ .....(i) and $\left(-\sqrt{13}, \frac{9}{2}\right)$

From (i) $a^2 = 4$ *and* $b^2 = 9$

As equation of tangent line at point $P(x_1, y_1)$ to the

Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1 \quad .....(ii)$$

Put $a^2=4$, $b^2=9$, $x_1=-\sqrt{13}$, $y_1=\frac{9}{2}$ values in (ii)

$$\frac{\left(-\sqrt{13}\right)x}{4}-\frac{9y}{2(9)}=1 \Rightarrow -\frac{\sqrt{13}}{4}x-\frac{y}{2}=1$$

$$\Rightarrow \sqrt{13}x+2y=-4 \quad multiplied\ by\ -4$$

$$\Rightarrow \sqrt{13}\,x+2y+4=0 \quad Ans.$$

Now equation of normal at point $P(x_1 , y_1)$ to the

hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{b^2(y-y_1)}{y_1}=-\frac{a^2(x-x_1)}{x_1} \quad .....(iii)$$

Put values in (iii)

$$\frac{9\left(y-\frac{9}{2}\right)}{\frac{9}{2}}=-\frac{4\left(x+\sqrt{13}\right)}{-\sqrt{13}}$$

$$\Rightarrow 2\left(y-\frac{9}{2}\right)=\frac{4\left(x+\sqrt{13}\right)}{\sqrt{13}} \Rightarrow 2y-9=\frac{4}{\sqrt{13}}x+\frac{4\sqrt{13}}{\sqrt{13}}$$

$$\Rightarrow 2y=\frac{4}{\sqrt{13}x}+4+9$$

$$\Rightarrow y=\frac{2}{\sqrt{13}}x+\frac{13}{2} \quad Ans.$$

(b) At a point $\left(\frac{16}{3} , 5\right)$ to hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$.

Solution: $\frac{x^2}{9}-\frac{y^2}{16}=1$ .....(i) and $\left(\frac{16}{3} , 5\right)$

From (i) $a^2 = 9$ *and* $b^2 = 16$

As equation of tangent line at point $P(x_1 , y_1)$ to the

Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1 \quad .....(ii)$$

Put $a^2 = 9$, $b^2 = 16$, $x_1 = \frac{16}{3}$, $y_1 = 5$ values in (ii)

$$\frac{\left(\frac{16}{3}\right)x}{9} - \frac{5y}{16} = 1 \quad => \frac{16}{27}x - \frac{5y}{16} = 1$$

$$=> \frac{16}{27}x - 1 = \frac{5}{16}y \quad \textit{multiplied by } \frac{16}{5}$$

$$=> \quad y = \frac{256}{135}x - \frac{16}{5} \quad \textit{Ans.}$$

Now equation of normal at point $P(x_1 , y_1)$ to the

hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{b^2(y - y_1)}{y_1} = -\frac{a^2(x - x_1)}{x_1} \quad .....(iii)$$

Put values in (iii)

$$\frac{16(y-5)}{5} = -\frac{9\left(x - \frac{16}{3}\right)}{\frac{16}{3}}$$

$$=> \frac{16}{5}(y-5) = -\frac{27\left(x - \frac{16}{3}\right)}{16} \quad => \frac{16}{5}y - 16 = -\frac{27}{16}x + \frac{(9)(16)}{16}$$

$$=> \frac{16}{5}y = -\frac{27}{16}x + 9 + 16$$

$$\Rightarrow \quad y=-\frac{5(27)}{256}x+\frac{25\times 5}{16} \qquad \text{multiplied by } \frac{5}{16}$$

$$\Rightarrow \quad y=-\frac{135}{256}x+\frac{125}{16} \quad Ans.$$

Note: The book answer is correct for if $\frac{y^2}{9}-\frac{x^2}{16}=1$.

As equation of tangent line at point $P(x_1\,,\,y_1)$ **to the Hyperbola** $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ **is**

$$\frac{y_1y}{a^2}-\frac{x_1x}{b^2}=1 \quad \text{.....(ii)}$$

Put $a^2=9$, $b^2=16$, $x_1=\frac{16}{3}$, $y_1=5$ **values in (ii)**

$$\frac{5y}{9}-\frac{\left(\frac{16}{3}\right)x}{16}=1 \quad \Rightarrow \frac{5}{9}y-\frac{x}{3}=1$$

$$\Rightarrow \frac{5}{9}y=\frac{x}{3}+1 \quad \Rightarrow \quad y=\frac{3}{5}x+\frac{9}{5} \quad Ans.$$

Now equation of normal at point $P(x_1\,,\,y_1)$ **to the hyperbola** $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ **is**

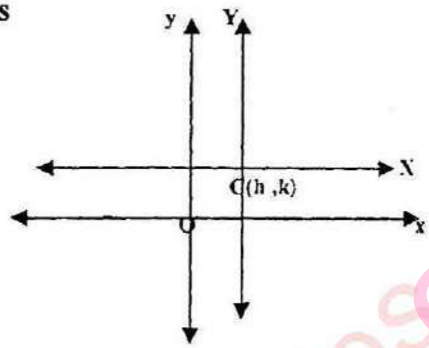
$$\frac{b^2(x-x_1)}{x_1}=-\frac{a^2(y-y_1)}{y_1} \quad \text{.....(iii) Put values in (iii)}$$

$$\frac{16\left(x-\frac{16}{3}\right)}{\frac{16}{3}}=-\frac{9(y-5)}{5} \quad \Rightarrow 3\left(x-\frac{16}{3}\right)=-\frac{9}{5}y+9$$

$$\Rightarrow \frac{9}{5}y=-3x+16+9 \quad multiiplied \;\; by \;\; \frac{5}{9}$$

$$\Rightarrow \quad y=-\frac{5}{3}x+\frac{125}{9} \quad Ans.$$

Transformation of axes : Let $O(0\ ,\ 0)$ be the origin of the rectangular coordinate axes xy - system.



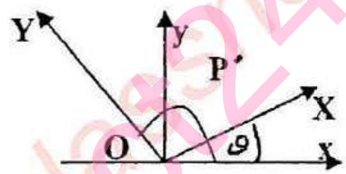
If we change the origin to a point $C(h\ ,\ k)$ we get new rectangular coordinate axes XY – system which is parallel to xy – system.

The relation between them is

$x = X + h$ and $y = Y + k$

(a) Rotation of axes about old origin :

Let X , Y coordinate system is obtained from the xy coordinate system by rotating through an angle θ with origin.



Then the relation between xy and XY – systems is given by the following equations.

$$x = X\cos\theta - Y\sin\theta \rightarrow (a)$$

and $$y = X\sin\theta + Y\cos\theta \rightarrow (b)$$

Eq (a) and eq (b) are the equation of rotations of x , y in terms of X and Y .

Now equations of X and Y in terms of x and y

$$X = x\cos\theta + y\sin\theta \rightarrow (c)$$

$$Y = y\cos\theta - x\sin\theta \rightarrow (d)$$

Eq (c) and (d) are the equation of X and Y in terms of x and y .

(b) New origin and new axes about old origin and old axes.

Let $o(0\ ,\ 0)$ be the origin of rectangular coordinate axes

ox and oy. Let $O(h, k)$ be the origin of rectangular coordinate axes OX and OY. New rectangular coordinates axes OX' and OY' are obtained by rotating of rectangular coordinate axes OX and OY through an angle θ, where $0 < \theta < 90^\circ$. Then

$x = h + X' \cos\theta - Y' \sin\theta$ ......(a) and

$y = k + X' \sin\theta + Y' \cos\theta$ ....(b)

Angle through which the axes be rotated about the origin to that the product term xy is removed from the transformed equation :

If $ax^2 + 2hxy + by^2 = 0$ .....(i), Then xy term will be remove from (i) if $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$

Exercise 9.4:

Q.1: Translate to parallel axes through the

(a) point $(0, 2)$, the equation $2x - y + 2 = 0$

Solution: $2x - y + 2 = 0$ .....(i)

Here $h = 0$ and $k = 2$

Since $x = X + h = X + 0 \Rightarrow x = X$ ...(ii)

And $y = Y + k \Rightarrow y = Y + 2$ ......(iii)

Put (ii) and (iii) in (i)

$2X - (Y + 2) + 2 = 0 \Rightarrow 2X - Y - 2 + 2 = 0$

$\Rightarrow 2X - Y = 0$ *Ans.*

(b) point $(-1, 2)$, the equation $x^2 + y^2 + 2x - 4y + 1 = 0$

Solution: $x^2 + y^2 + 2x - 4y + 1 = 0$ .....(i)

Here $h = -1$ and $k = 2$

Since $x = X + h \Rightarrow x = X - 1$ ...(ii)

And $y = Y + k \Rightarrow y = Y + 2$ ......(iii)

Put (ii) and (iii) in (i)

$$(X-1)^2+(Y+2)^2+2(X-1)-4(Y+2)+1=0$$

$\Rightarrow X^2-2X+1+Y^2+4Y+4+2X-2-4Y-8+1=0$

$\Rightarrow X^2+Y^2=4$ *Ans.*

(c) point $(3,-4)$, the equation

$$x^2+2y^2-6x+16y+39=0$$

Solution: $x^2+2y^2-6x+16y+39=0$ .....(i)

Here $h=3$ *and* $k=-4$

Since $x=X+h \Rightarrow x=X+3$ ...(ii)

And $y=Y+k \Rightarrow y=Y-4$ ......(iii)

Put (ii) and (iii) in (i)

$$(X+3)^2+2(Y-4)^2-6(X+3)+16(Y-4)+39=0$$

$\Rightarrow X^2+6X+9+2Y^2-16Y+32-6X-18+16Y-64+39=0$

$\Rightarrow X^2+2Y^2-2=0$ *Ans.*

(d) point $(-2,2)$, the equation

$$x^2+y^2-3xy+10x-10y+21=0$$

Solution: $x^2+y^2-3xy+10x-10y+21=0$ .....(i)

Here $h=-2$ *and* $k=2$

Since $x=X+h \Rightarrow x=X-2$ ...(ii)

And $y=Y+k \Rightarrow y=Y+2$ ......(iii)

Put (ii) and (iii) in (i)

$$(X-2)^2+(Y+2)^2-3(X-2)(Y+2)+10(X-2)-10(Y+2)+21=0$$

$\Rightarrow X^2-4X+4+Y^2+4Y+4-3XY-6X+6Y$

$+10X-20-10Y-20+21=0$

$\Rightarrow X^2+Y^2-3XY-11=0$ *Ans.*

Q.2: Transform to axes inclined at angle

(a) 45^o to the original axes of the conic $x^2-y^2=a^2$

Solution: $x^2-y^2=a^2$ ......(i) **and** $\theta=45^o$

As we know that equation of rotation are

$$x = X\cos\theta - Y\sin\theta \to (a)$$

and $$y = X\sin\theta + Y\cos\theta \to (b)$$

Since $\cos\theta = \cos 45^0 = \dfrac{1}{\sqrt{2}}$

And $\sin\theta = \sin 45^o = \dfrac{1}{\sqrt{2}}$

put these values in (a) and (b)

$x = \dfrac{X}{\sqrt{2}} - \dfrac{Y}{\sqrt{2}}$ => $x = \dfrac{1}{\sqrt{2}}(X - Y)$ ......**(ii)**

And $y = \dfrac{X}{\sqrt{2}} + \dfrac{Y}{\sqrt{2}}$ => $x = \dfrac{1}{\sqrt{2}}(X + Y)$.....**(iii)**

Put (iii) and (iv) in (i)

$$\left(\frac{1}{\sqrt{2}}(X-Y)\right)^2 - \left(\frac{1}{\sqrt{2}}(X+Y)\right)^2 = a^2$$

$$=> \frac{1}{2}(X^2 + Y^2 - 2XY) - \frac{1}{2}(X^2 + Y^2 + 2XY) = a^2$$

$$=> X^2 + Y^2 - 2XY - X^2 - Y^2 - 2XY = 2a^2$$

$$=> -4XY = 2a^2 \quad => \quad 2XY = -a^2$$

$$=> 2XY + a^2 = 0 \quad Ans.$$

(b) 90^o **to the original axes of the conic** $y^2 = 2px$

Solution: $y^2 = 2px$ ....**(i) and** $\theta = 45^o$

As we know that equation of rotation are

$$x = X\cos\theta - Y\sin\theta \to (a)$$

and $$y = X\sin\theta + Y\cos\theta \to (b)$$

Since $\cos\theta = \cos 90^0 = 0$

And $\sin\theta = \sin 90^o = 1$ **put these values in (a) and (b)**

$x = -Y$ .....(ii) *and* $y = X$ ......(iii)

Put (ii) and (iii) in (i)

$X^2 = -2pY$ **Ans.**

(c) 45^o **to the original axes of the conic** $x^2 + y^2 + 4xy - 1 = 0$

Solution: $x^2 + y^2 + 4xy - 1 = 0$ ......(i) and $\theta = 45^o$

As we know that equation of rotation are

$$x = X\cos\theta - Y\sin\theta \rightarrow (a)$$

and $$y = X\sin\theta + Y\cos\theta \rightarrow (b)$$

Since $\cos\theta = \cos 45^0 = \frac{1}{\sqrt{2}}$

And $\sin\theta = \sin 45^o = \frac{1}{\sqrt{2}}$ put these values in (a) and (b)

$x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}$ => $x = \frac{1}{\sqrt{2}}(X - Y)$ ......(ii)

And $y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$ => $x = \frac{1}{\sqrt{2}}(X + Y)$.....(iii)

Put (iii) and (iv) in (i)

$$\left(\frac{1}{\sqrt{2}}(X-Y)\right)^2 + \left(\frac{1}{\sqrt{2}}(X+Y)\right)^2 + 4\left(\frac{1}{\sqrt{2}}(X-Y)\right)\left(\frac{1}{\sqrt{2}}(X+Y)\right) - 1 = 0$$

=> $\frac{1}{2}(X^2 + Y^2 - 2XY) + \frac{1}{2}(X^2 + Y^2 + 2XY) + \frac{4}{2}(X^2 - Y^2) - 1 = 0$

multiiplied by 2

=> $X^2 + Y^2 - 2XY + X^2 + Y^2 + 2XY + 4X^2 - 4Y^2 - 2 = 0$

=> $6X^2 - 2Y^2 - 2 = 0$

=> $3X^2 - Y^2 - 1 = 0$ *Ans.*

(d) 45^o **to the original axes of the conic**

$$x^2 - y^2 - 2\sqrt{2}x - 10\sqrt{2}y + 2 = 0$$

Solution: $x^2 - y^2 - 2\sqrt{2}x - 10\sqrt{2}y + 2 = 0$ ......(i)

and $\theta = 45^o$

As we know that equation of rotation are

$$x = X\cos\theta - Y\sin\theta \rightarrow (a)$$

and $$y = X\sin\theta + Y\cos\theta \rightarrow (b)$$

Since $\cos\theta = \cos 45^0 = \frac{1}{\sqrt{2}}$

And $\sin\theta = \sin 45^o = \frac{1}{\sqrt{2}}$ put these values in (a) and (b)

$x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}$ => $x = \frac{1}{\sqrt{2}}(X - Y)$ ......(ii)

And $y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$ => $x = \frac{1}{\sqrt{2}}(X + Y)$.....(iii)

Put (iii) and (iv) in (i)

$$\left(\frac{1}{\sqrt{2}}(X-Y)\right)^2 - \left(\frac{1}{\sqrt{2}}(X+Y)\right)^2 - 2\sqrt{2}\left(\frac{1}{\sqrt{2}}(X-Y)\right)$$

$$-10\sqrt{2}\left(\frac{1}{\sqrt{2}}(X+Y)\right) + 2 = 0$$

$$=> \frac{1}{2}\left(X^2 + Y^2 - 2XY\right) - \frac{1}{2}\left(X^2 + Y^2 + 2XY\right) - 2(X - Y)$$

$$-10(X+Y) + 2 = 0 \quad \textit{multiiplied by } 2$$

$$=> X^2 + Y^2 - 2XY - X^2 - Y^2 - 2XY - 4X + 4Y$$

$$-20X - 20Y + 4 = 0$$

$$=> -24X - 16Y + 4 = 0 \quad \textit{diivided by } -4$$

$$=> 6X + 4Y - 1 = 0 \quad \textit{Ans.}$$

Q.3: Transform to new axes inclined at an angle

(a) $\tan^{-1}\left(\frac{1}{2}\right)$ **to the original axes of the conic**

$14x^2 + 11y^2 - 36x + 48y - 4xy + 41 = 0$ **through** $(1\ ,\ -2)$.

Solution: $14x^2 + 11y^2 - 36x + 48y - 4xy + 41 = 0$....(i)

Then equation of rotation are

$x = h + X'\cos\theta - Y'\sin\theta$ ......(a) and

$y = k + X'\sin\theta + Y'\cos\theta$ ....(b)

Let $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ => $\tan\theta = \frac{1}{2}$

As

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{1}{2}\right)^2} = \sqrt{1+\frac{1}{4}}$$

$$\Rightarrow \sec\theta = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

Now $\cos\theta = \dfrac{1}{\sec\theta} = \dfrac{2}{\sqrt{5}}$

Also $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\dfrac{2}{\sqrt{5}}\right)^2} = \sqrt{1-\dfrac{4}{5}}$

$$\Rightarrow \sin\theta = \sqrt{\frac{5-4}{5}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

Put values $\sin\theta$ *and* $\cos\theta$ **and** $h=1$, $k=-2$ **in (a) and (b)**

$$x = 1 + \frac{2}{\sqrt{5}}X' - \frac{1}{\sqrt{5}}Y' \quad \text{.......(ii)}$$

$$y = -2 + \frac{1}{\sqrt{5}}X' + \frac{2}{\sqrt{5}}Y' \quad \text{.......(iii)}$$

Put (ii) and (iii) in (i)

$$14\left(1+\frac{2}{\sqrt{5}}X' - \frac{1}{\sqrt{5}}Y'\right)^2 + 11\left(-2+\frac{1}{\sqrt{5}}X' + \frac{2}{\sqrt{5}}Y'\right)^2$$

$$-36\left(1+\frac{2}{\sqrt{5}}X' - \frac{1}{\sqrt{5}}Y'\right) + 48\left(-2+\frac{1}{\sqrt{5}}X' + \frac{2}{\sqrt{5}}Y'\right)$$

$$-4\left(1+\frac{2}{\sqrt{5}}X' - \frac{1}{\sqrt{5}}Y'\right)\left(-2+\frac{1}{\sqrt{5}}X' + \frac{2}{\sqrt{5}}Y'\right) + 41 = 0$$

$$\Rightarrow 14\left(1+\frac{4}{5}X'^2 + \frac{1}{5}Y'^2 + \frac{4}{\sqrt{5}}X' - \frac{4}{5}X'Y' - \frac{2}{\sqrt{5}}Y'\right)$$

$$+11\left(4+\frac{1}{5}X'^2 + \frac{4}{5}Y'^2 - \frac{4}{\sqrt{5}}X' + \frac{4}{5}X'Y' - \frac{8}{\sqrt{5}}Y'\right)$$

$$-36\left(1+\frac{2}{\sqrt5}X'-\frac{1}{\sqrt5}Y'\right)+48\left(-2+\frac{1}{\sqrt5}X'+\frac{2}{\sqrt5}Y'\right)$$

$$-4\left(-2-\frac{3}{\sqrt5}X'+\frac{4}{\sqrt5}Y'+\frac{2}{5}X'^2+\frac{3}{5}X'Y'-\frac{2}{5}Y'^2\right)+41=0$$

$$\Rightarrow \left(\frac{56}{5}+\frac{11}{5}-\frac{8}{5}\right)X'^2+\left(\frac{14}{5}+\frac{44}{5}+\frac{8}{5}\right)Y'^2$$

$$+\left(\frac{56}{\sqrt5}-\frac{44}{\sqrt5}-\frac{72}{\sqrt5}+\frac{48}{\sqrt5}-\frac{3}{\sqrt5}\right)X'+\left(-\frac{14}{\sqrt5}-\frac{88}{\sqrt5}+\frac{36}{\sqrt5}+\frac{96}{\sqrt5}-\frac{16}{\sqrt5}\right)Y'$$

$$+\left(-\frac{56}{5}+\frac{44}{5}-\frac{12}{5}\right)X'Y'+14+44-36-96+8+41=0$$

$$\Rightarrow \frac{59}{5}X'^2+\frac{66}{5}Y'^2-\frac{15}{\sqrt5}X'+\frac{14}{\sqrt5}Y'-\frac{24}{5}X'Y'-25=0$$

(b) $\tan^{-1}\left(-\frac{4}{3}\right)$ **to the original axes of the conic**

$11x^2+4y^2-20x-40y+24xy-5=0$ **through** $(2,-1)$.

Solution: $11x^2+4y^2-20x-40y+24xy-5=0$ **.....(i)**

Then equation of rotation are

$x=h+X'\cos\theta-Y'\sin\theta$ **......(a) and**

$y=k+X'\sin\theta+Y'\cos\theta$ **.......(b)**

Let $\theta=\tan^{-1}\left(-\frac{4}{3}\right) \Rightarrow \tan\theta=-\frac{4}{3}$

$$\Rightarrow \sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+\left(-\frac{4}{3}\right)^2}$$

$$\Rightarrow \sec\theta=\sqrt{1+\frac{16}{9}}=\sqrt{\frac{25}{9}}$$

$$\Rightarrow \sec\theta=\frac{5}{3}$$

Then $\cos\theta = \frac{1}{\sec\theta} = \frac{3}{5}$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}}$$

$$\Rightarrow \sin\theta = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Since θ lie in the fourth quadrant

Then $\sin\theta = -\frac{4}{5}$

Put $\cos\theta = \frac{3}{5}$, $\sin\theta = -\frac{4}{5}$, $h = 2$ *and* $k = -1$ **in (a) and (b)**

$$x = 2 + \frac{3}{5}X' + \frac{4}{5}Y' \quad \text{....(ii)}$$

$$y = -1 - \frac{4}{5}X' + \frac{3}{5}Y' \quad \text{.....(iii)}$$

Put (ii) and (iii) in (i)

$$11\left(2+\frac{3}{5}X'+\frac{4}{5}Y'\right)^2 + 4\left(-1-\frac{4}{5}X'+\frac{3}{5}Y'\right)^2 - 20\left(2+\frac{3}{5}X'+\frac{4}{5}Y'\right)$$
$$-40\left(-1-\frac{4}{5}X'+\frac{3}{5}Y'\right) + 24\left(2+\frac{3}{5}X'+\frac{4}{5}Y'\right)\left(-1-\frac{4}{5}X'+\frac{3}{5}Y'\right)$$
$$-5 = 0$$

$$11\left(4+\frac{9}{25}X'^2+\frac{16}{25}Y'^2+\frac{12}{5}X'+\frac{24}{25}X'Y'+\frac{16}{5}Y'\right)$$
$$+4\left(1+\frac{16}{25}X'^2+\frac{9}{25}Y'^2+\frac{8}{5}X'-\frac{6}{5}Y'-\frac{24}{25}X'Y'\right)$$
$$-20\left(2+\frac{3}{5}X'+\frac{4}{5}Y'\right)-40\left(-1-\frac{4}{5}X'+\frac{3}{5}Y'\right)$$
$$+24\left(-2-\frac{11}{5}X'-\frac{2}{5}Y'-\frac{12}{25}X'^2-\frac{7}{25}X'Y+\frac{12}{25}Y'^2\right)-5=0$$

$$\frac{1}{25}(99+64-288)X'^2+\frac{1}{25}(176+36+288)Y'^2$$

$$+\left(\frac{132}{5}+\frac{32}{5}-\frac{60}{5}+\frac{160}{5}-\frac{264}{5}\right)X'+\left(\frac{176}{5}-\frac{24}{5}+\frac{80}{5}-\frac{120}{5}-\frac{48}{5}\right)Y'$$

$$+\frac{1}{25}(264-96-168)X'Y'+44+4-40+40-48-5=0$$

$$\Rightarrow -\frac{125}{25}X'^2+\frac{500}{25}Y'^2+0.X'-\frac{64}{5}Y'-5=0$$

$$\Rightarrow -5X'^2+20Y'^2-12.8Y'-5=0 \quad Ans.$$

(c) $\tan^{-1}\left(\frac{3}{4}\right)$ **to the original axes of the conic**

$3x^2+10y^2+6x+52y-24xy=0$ **, through** $(3,1)$.

Solution : $3x^2+10y^2+6x+52y-24xy=0$ ……(i)

Then equation of rotation are

$x=h+X'\cos\theta-Y'\sin\theta$ ……(a) **and**

$y=k+X'\sin\theta+Y'\cos\theta$ …….(b)

Let $\theta=\tan^{-1}\left(\frac{3}{4}\right) \Rightarrow \tan\theta=\frac{3}{4}$

$$\sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+\left(\frac{3}{4}\right)^2}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{25}{16}}$$

$$\Rightarrow \sec\theta=\frac{5}{4} \Rightarrow \cos\theta=\frac{4}{5}$$

$$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}$$

$$\Rightarrow \sin\theta=\sqrt{\frac{25-16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

Put $h=3$, $k=1$, $\sin\theta=\frac{3}{5}$ *and* $\cos\theta=\frac{4}{5}$ **in (i)**

$$x = 3 + \frac{4}{5}X' - \frac{3}{5}Y' \quad \text{....(ii)}$$

$$y = 1 + \frac{3}{5}X' + \frac{4}{5}Y' \quad \text{......(iii)}$$

Put (ii) and (iii) in (i)

$$3\left(3 + \frac{4}{5}X' - \frac{3}{5}Y'\right)^2 + 10\left(1 + \frac{3}{5}X' + \frac{4}{5}Y'\right)^2 + 6\left(3 + \frac{4}{5}X' - \frac{3}{5}Y'\right)$$

$$+ 52\left(1 + \frac{3}{5}X' + \frac{4}{5}Y'\right) - 24\left(3 + \frac{4}{5}X' - \frac{3}{5}Y'\right)\left(3 + \frac{3}{5}X' + \frac{4}{5}Y'\right) = 0$$

$$\Rightarrow 3\left(9 + \frac{16}{25}X'^2 + \frac{9}{25}Y'^2 + \frac{24}{5}X' - \frac{24}{25}X'Y' - \frac{18}{5}Y'\right)$$

$$+ 10\left(1 + \frac{9}{25}X'^2 + \frac{16}{25}Y'^2 + \frac{6}{5}X' + \frac{24}{25}X'Y' + \frac{4}{5}Y'\right)$$

$$+ 6\left(3 + \frac{4}{5}X' - \frac{3}{5}Y'\right) + 52\left(1 + \frac{3}{5}X' + \frac{4}{5}Y'\right)$$

$$- 24\left(9 + \frac{21}{5}X' + \frac{3}{25}Y' + \frac{12}{25}X'^2 + \frac{7}{25}X'Y' - \frac{12}{25}Y'^2\right) = 0$$

$$\Rightarrow \frac{1}{25}(48 + 90 - 288)X'^2 + \frac{1}{25}(27 + 160 + 288)Y'^2$$

$$+ \frac{1}{5}(72 + 60 + 24 + 156 - 504)X' + \frac{1}{5}(-54 + 40 - 18 + 208 - 72)Y'$$

$$+ \frac{1}{25}(-72 + 240 - 168)X'Y' + 27 + 10 + 18 + 52 - 216 = 0$$

$$\Rightarrow \frac{150}{25}X'^2 + \frac{475}{25}Y'^2 - \frac{192}{5}X' + \frac{104}{5}Y' - 109 = 0$$

$$\Rightarrow 6X'^2 + 19Y'^2 - \frac{192}{5}X' + \frac{104}{5}Y' - 109 = 0 \quad \textit{Ans.}$$

Q.4: **At what angle the axes are rotated about the origin so that the transformed equation of the conic**

(a) $11x^2+4y^2-20x-40y+24xy-5=0$ **does not contain the term involving** XY .

Solution: $11x^2+4y^2-20x-40y+24xy-5=0$ .....(i)

We know that if $ax^2+2hxy+by^2=0$ .....(ii) ,

Then xy **term will be remove from (i) if**

$$\theta=\frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right) \text{ ....(iii)}$$

Comparing (i) and (ii) $a=11$, $b=4$, $2h=24$

Put these values in (iii)

$$\theta=\frac{1}{2}\tan^{-1}\left(\frac{24}{11-4}\right) \quad \Rightarrow \quad \theta=\frac{1}{2}\tan^{-1}\left(\frac{24}{7}\right)$$

$$\Rightarrow \quad \theta=\frac{1}{2}(73.74)=36.87^o$$

Or approximately $\theta=37^o$ **Ans.**

(b) $5x^2+7y^2+2\sqrt{3}xy-16=0$ **does not contain the term involving** XY .

Solution: $5x^2+7y^2+2\sqrt{3}xy-16=0$ .....(i)

We know that if $ax^2+2hxy+by^2=0$ .....(ii) ,

Then xy **term will be remove from (i) if**

$$\theta=\frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right) \text{ ....(iii)}$$

Comparing (i) and (ii) $a=5$, $b=7$, $2h=2\sqrt{3}$

Put these values in (iii)

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{3}}{5-7}\right) \quad => \quad \theta = \frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) = \frac{1}{2}\tan^{-1}\left(-\sqrt{3}\right)$$

$$=> \quad \theta = \frac{1}{2}\left(-60^o\right) = -30^o$$