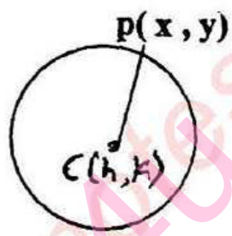
Unit – 8: Conics – I:

Circle: The set of points in which the distance of each point from a fixed point is same is called circle.

The fixed point is called centre of the circle and the distance between the centre and the point of circle is called radius of the circle.



Let $C(h\ ,\ k)$ be the centre of a circle.

Let $P(x\ ,y)$ is point on the circle,

Then $|CP| = r$

$$\Rightarrow \sqrt{(x-h)^2+(y-k)^2} = r$$

$$\Rightarrow (x-h)^2+(y-k)^2 = r^2 \quad \text{....(i)}$$

Eq(i) is the equation of the circle with centre $(h\ ,\ k)$ and radius r.

Note: If centre at $(0\ ,\ 0)$, put $h=0$, $k=0$ in (i)

$$x^2+y^2=r^2 \quad \text{.....(ii)}$$

Eq(ii) is the circle equation centre at origin.

Example .1: Determine the equation of a circle with centre at $(-2\ ,1)$ and radius $r=3$.

Solution: The equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \quad \text{....(i)}$$

Put $h=-2$, $k=1$ and $r=3$ in (i)

$$(x-2)^2+(y-1)^2=(3)^2$$

$$\Rightarrow x^2-6x+4+y^2-2y+1=9$$

$$\Rightarrow x^2+y^2-6x-2y-5=0 \quad Ans.$$

General equation of the circle:

General equation of the circle is

$x^2+y^2+2gx+2fy+c=0$, with centre is $C(-g\ ,\ -f)$

and radius is $r=\sqrt{g^2+f^2-c}$.

Note: (i) If $g^2+f^2-c>0$, then the circle is real and different from zero.

(ii) If $g^2+f^2-c=0$, then the circle is reduced to a single point is called point circle.

(iii) If $g^2+f^2-c<0$, then the circle is imaginary or virtual.

(iv) The coefficients of x^2 and y^2 are same and there is no term containing xy.

Example .2: Find the centre and radius of a circle

$$45x^2+45y^2-60x+36y+19=0$$

Solution: $45x^2+45y^2-60x+36y+19=0$

Both sides divided by 45.

$$x^2+y^2-\frac{60}{45}x+\frac{36}{45}y+\frac{19}{45}=0$$

$$\Rightarrow x^2+y^2-\frac{4}{3}x+\frac{4}{5}y+\frac{19}{45}=0 \quad \text{....(i)}$$

General equation of the circle is

$$x^2+y^2+2gx+2fy+c=0 \text{..........(ii)}$$

Comparing (i) and (ii)

$$2g=-\frac{4}{3} \Rightarrow g=-\frac{2}{3}$$

$$2f=\frac{4}{5} \Rightarrow f=\frac{2}{5} \text{ and } c=\frac{19}{45}$$

The centre is $C(-g,\ -f)=C\left(\frac{2}{3},-\frac{2}{5}\right)$

and radius is $r=\sqrt{g^2+f^2-c}=\sqrt{\left(-\frac{2}{3}\right)^2+\left(\frac{2}{5}\right)^2-\frac{19}{45}}$

$$\Rightarrow r=\sqrt{\frac{4}{9}+\frac{4}{25}-\frac{19}{45}}=\sqrt{\frac{100+36-95}{225}}=\sqrt{\frac{41}{225}}$$

$$\Rightarrow r = \frac{\sqrt{41}}{15} \quad \textbf{Ans.}$$

Example 3: Find the equation of the circle which passes through the points $A(1\ ,0)$, $B(0\ ,-6)$ and $C(3\ ,4)$.

Solution: Let the equation of the required circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad .....(a)$$

Since (a) passes through the points $A(1\ ,0)$, $B(0\ ,-6)$ and $C(3\ ,4)$. Put $x = 1$, $y = 0$ $in(a)$

$$(1)^2 + 0 + 2g(1) + 0 + c = 0$$

$$\Rightarrow 2g + c = -1 \quad .....(i) ,$$

Take $B(0\ ,-6)$: put $x = 0$, $y = -6$ in (a)

$$0 + (-6)^2 + 0 + 2f(-6) + c = 0$$

$$\Rightarrow -12f + c = -36 \quad .....(ii)$$

Take $C(3\ ,4)$: Put $x = 3$, $y = 4$ in (a)

$$(3)^2 + (4)^2 + (2g)(3) + (2f)(4) + c = 0$$

$$\Rightarrow 9 + 16 + 6g + 12f + c = 0$$

$$\Rightarrow 6g + 8f + c = -25 \quad ....(iii)$$

Now we can solve the eq(i) , eq (ii) and eq(iii) by simple addition or subtraction or by matrix method to find the values of g , f and c.

The system of linear equations (i) , (ii), (iii) in the matrix form is $AX = b$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -12 & 1 \\ 6 & 8 & 1 \end{bmatrix} \begin{bmatrix} g \\ f \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -36 \\ -25 \end{bmatrix}$$

Now the augmented matrix is

$$A/b = \begin{bmatrix} 2 & 0 & 1 & -1 \\ 0 & -12 & 1 & -36 \\ 6 & 8 & 1 & -25 \end{bmatrix}$$ we change it into echelon form

$$\underset{\sim}{R}\begin{bmatrix} 2 & 0 & 1 & -1 \\ 0 & -12 & 1 & -36 \\ 0 & 8 & -2 & -22 \end{bmatrix} R_3 - 3R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 2 & 0 & 1 & -1 \\ 0 & -12 & 1 & -36 \\ 0 & 0 & -\frac{4}{3} & -46 \end{bmatrix} R_3 + \frac{2}{3}R_2 \quad \textbf{Which is in echelon form}$$

From $R_3 \Rightarrow -\frac{4}{3}c = -46 \Rightarrow c = \frac{46\times 3}{4} = \frac{69}{2}$ **...(iv)**

From $R_2 \Rightarrow -12f + c = -36$ **.....(v)**

From $R_1 \Rightarrow 2g + c = -1$ **........(vi)**

Put $c = \frac{69}{2}$ **in (v)**

$$-12f + \frac{69}{2} = -36 \Rightarrow -12f = -36 - \frac{69}{2}$$

$$\Rightarrow -12f = \frac{-72-69}{2} \Rightarrow f = \frac{141}{12\times 2} = \frac{47}{8}$$

Put $c = \frac{69}{2}$ **in (vi)**

$$2g + \frac{69}{2} = -1 \Rightarrow 2g = -1 - \frac{69}{2}$$

$$\Rightarrow 2g = \frac{-71}{2} \Rightarrow g = -\frac{71}{4}$$

Put values of f , g and c in (a)

$$x^2 + y^2 + 2\left(-\frac{71}{4}\right)x + 2\left(\frac{47}{8}\right)y + \frac{69}{2} = 0$$

$$\Rightarrow x^2 + y^2 - \frac{71}{2}x + \frac{47}{4}y + \frac{69}{2} = 0$$

$\Rightarrow 4x^2 + 4y^2 - 142x + 47y + 138 = 0$ **is the required circle equation .**

Example. 4: Find the equation of a circle which passes through the point $A(3,1)$, $B(2,2)$ and has its centre on the line $x+y-3=0$

Solution: Let $A(3,1)$, $B(2,2)$ and

$x+y-3=0$ ......(i)

Let $C(h,k)$ be the centre of the required circle.

Then $|AC|$, $|BC|$ is radius of the circle.

So $|AC|=|BC|$

$\Rightarrow \sqrt{(h-3)^2+(k-1)^2}=\sqrt{(h-2)^2+(k-2)^2}$ squaring

$\Rightarrow h^2-6h+9+k^2-2k+1=h^2-4h+4+k^2-4k+4$

$\Rightarrow -6h+4h-2k+4k=8-10$

$\Rightarrow -2h+2k=-2$ *divided by* -2

$\Rightarrow h-k=1$ ....(ii)

Since the centre $C(h,k)$ is lie on the line (i),

then put $x=h$, and $y=k$ in (i)

$h+k-3=0 \Rightarrow h+k=3$.........(iii)

Now adding (ii) and (iii)

$$\begin{array}{r} h-k=1 \\ \underline{h\pm k=3} \\ 2h=4 \end{array}$$

$\Rightarrow h=2$, Put $h=2$ in (iii)

$2+k=3 \Rightarrow k=3-2=1$

Thus centre is $C(2,1)$. Now radius is

$r=|AC|=\sqrt{(2-3)^2+(1-1)^2}=\sqrt{(1)^2+(0)^2}=\sqrt{1}$

$\Rightarrow r=1$

Now equation of the circle is

$(x-h)^2+(y-k)^2=r^2$ put values $h=2$, $k=1$

$$\Rightarrow (x-2)^2+(y-1)^2=(1)^2$$
$$\Rightarrow x^2-4x+4+y^2-2y+1=1$$
$$\Rightarrow x^2+y^2-4x-2y+4=0 \qquad Ans.$$

Example.5: Find the equation of the circle which passes through the two points $A(0,-1)$ and $B(3,-3)$ and $3x-2y-2=0$ is tangent line to the circle at a point $(0,-1)$.

Solution: Let $A(0,-1)$ and $B(3,-3)$ and
$$3x-2y-2=0 \ldots\ldots(i)$$

Let $C(h,k)$ be the centre of the required circle .

Then $|AC|$, $|BC|$ is radius of the circle .

So $|AC|=|BC|$

$$\Rightarrow \sqrt{(h-0)^2+(k-(-1))^2}=\sqrt{(h-3)^2+(k-(-(-3))^2}$$
$$\Rightarrow \sqrt{(h-0)^2+(k+1)^2}=\sqrt{(h-3)^2+(k+3)^2} \text{ squaring}$$
$$\Rightarrow h^2+k^2+2k+1=h^2-6h+9+k^2+6k+9$$
$$\Rightarrow 6h+2k-6k=18-1 \quad \Rightarrow 6h-4k=17 \ldots\ldots(ii)$$

Now slope of $\overline{AC}=m_1=\dfrac{y_2-y_1}{x_2-x_1}$

$$\Rightarrow m_1=\frac{k-(-1)}{h-0}=\frac{k+1}{h} \ldots.(iii)$$

Now from (i) $\Rightarrow 3x-2=2y$

$$\Rightarrow y=\frac{3}{2}x-\frac{2}{3}\ldots\ldots.(iv)$$

Then the slope of line (i) is $m_2=\dfrac{3}{2}$

Since the line (i) is tangent to the circle at $B(0,-1)$ then the radius AC is perpendicular to line (i) .

As $m_1.m_2=-1$ put values

$$\left(\frac{k+1}{h}\right)\left(\frac{3}{2}\right)=-1 \quad => \quad 3(k+1)=-2h$$

$$=> \quad 3k+3=-2h \quad => \quad 2h+3k=-3 \;.....(v)$$

Eq (v) is multiplied by 3

$$=> \; 6h+9k=-9 \;...(vi)$$

Subtract (ii) from (vi)

$$\begin{array}{r} 6h+9k=-9 \\ \underline{\pm 6h \; \mp 4k=\pm 17} \\ 13k=-26 \end{array}$$

$$=> \; k=-\frac{26}{13}=-2.$$

Put $k=-2$ **in (v)**

$$2h+3(-2)=-3 \quad => \quad 2h-6=-3$$

$$=> \; 2h=-3+6 \quad => h=\frac{3}{2}$$

Thus the centre is $C\left(\frac{3}{2},\,-2\right)$.

Now radius is $r=|AC|=\sqrt{\left(\frac{3}{2}-0\right)^2+(-2-(-1))^2}$

$$=\sqrt{\left(\frac{3}{2}\right)^2+(-2+1)^2}=\sqrt{\frac{9}{4}+(-1)^2}$$

$$=> \; r=\sqrt{\frac{9}{4}+1}=\sqrt{\frac{13}{4}}$$

Now equation of the circle is

$(x-h)^2+(y-k)^2=r^2$ **put values** $h=\frac{3}{2}$, $k=-2$

$$=> \left(x-\frac{3}{2}\right)^2+(y-(-2))^2=\left(\sqrt{\frac{13}{4}}\right)^2$$

$$\Rightarrow \left(x-\frac{3}{2}\right)^2+(y+2)^2=\frac{13}{4}$$

$$\Rightarrow x^2-3x+\frac{9}{4}+y^2+4y+4=\frac{13}{4}$$

$$\Rightarrow x^2+y^2-3x+4y+\frac{9}{4}+4-\frac{13}{4}=0$$

$$\Rightarrow x^2+y^2-3x+4y+\frac{9+16-13}{4}=0$$

$$\Rightarrow x^2+y^2-3x+4y+\frac{12}{4}=0$$

$$\Rightarrow x^2+y^2-3x+4y+3=0 \quad \text{Ans.}$$

Exercise 8.1:

Q.1: In each case, find an equation of a circle, when the centre and radius are the following:

(a) $(0, 0)$, $r=4$

Solution: The equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \quad \ldots\text{(i)}$$

Put $h=0$, $k=0$ and $r=4$ in (i)

$$(x-0)^2+(y-0)^2=(4)^2$$

$$\Rightarrow x^2+y^2=16 \quad \textit{Ans.}$$

(b) $(3, 2)$, $r=1$

Solution: The equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \quad \ldots\text{(i)}$$

Put $h=3$, $k=2$ and $r=1$ in (i)

$$(x-3)^2+(y-2)^2=(1)^2$$

$$\Rightarrow (x-3)^2+(y-2)^2=1 \quad \textit{Ans.}$$

(c) $(-4, -3)$, $r=4$

Solution: The equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \quad \ldots\text{(i)}$$

Put $h=-4$, $k=-3$ and $r=4$ in (i)

$$(x-(-4))^2+(y-(-3))^2=(4)^2$$

$$\Rightarrow (x+4)^2+(y+3)^2=16 \quad Ans.$$

(d) $(-a\ ,-b)$, $r=a+b$

Solution: The equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \quad ....(i)$$

Put $h=-a$, $k=-b$ and $r=a+b$ in (i)

$$(x-(-a))^2+(y-)-b))^2=(a+b)^2$$

$$\Rightarrow (x+a)^2+(y+b)^2=(a+b)^2 \quad Ans.$$

Q.2: In each case, determine the equation of a circle using the given information.

(a) $C(0\ ,0)$, tangent to the line $x=-5$

Not: Tangent to the circle: The straight line which touch a circle at only one point is called tangent to the circle at that point. The distance of the centre from tangent to the circle is the radius of the circle.

Solution: The equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \quad ....(i)$$

Since the radius of the circle is equal the distance of centre $C(0\ ,0)$ from the tangent $x=-5$.

As the distance from origin to $x=-5$ on the x- axis is equal to 5. Then $r=5$ and $h=0$, $k=0$ put in (i)

$$(x-0)^2+(y-0)^2=(5)^2$$

$$\Rightarrow x^2+y^2=25 \quad Ans.$$

(b) $C(0\ ,0)$, tangent to the line $y=6$

Solution: The equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \quad ....(i)$$

Since the radius of the circle is equal the distance of centre $C(0\ ,0)$ from the tangent $y=6$.

As the distance from origin to $y=6$ on the y- axis is equal to 6. Then $r=6$ and $h=0$, $k=0$ put in (i)

$$(x-0)^2+(y-0)^2=(6)^2$$

$\Rightarrow \quad x^2+y^2=36 \quad Ans.$

(c) $C(6\ ,\ -6)$, **circumference passes through the origin.**

Solution: **The equation of the circle is given by**

$$(x-h)^2+(y-k)^2=r^2 \quad \text{....(i)}$$

Since the circle passes through the origin $O(0\ ,\ 0)$

Then $r=|OC|=\sqrt{(6-0)^2+(-6+0)^2}$

$\Rightarrow r=\sqrt{36+36} \quad \Rightarrow r^2=72$

Put $h=6\ ,\ k=-6 \ \ and \ \ r^2=72$ **in (i)**

$$(x-6)^2+(y-(-6))^2=72$$

$\Rightarrow \quad (x-6)^2+(y+6)^2=72 \quad Ans.$

(d) $C(-9\ ,\ -6)$, **circumference passes through the point** $P(-20\ ,\ 8)$.

Solution: **The equation of the circle is given by**

$$(x-h)^2+(y-k)^2=r^2 \quad \text{....(i)}$$

Since the circle passes through the point $P(-20\ ,\ 8)$

Then $r=|CP|=\sqrt{(-20-(-9))^2+(8-(-6))^2}$

$\Rightarrow r=\sqrt{(-20+9)^2+(8+6)^2}=\sqrt{(-11)^2+(14)^2}$

$\Rightarrow r=\sqrt{121+196}=\sqrt{217} \quad \Rightarrow r^2=317$

Put $h=-9\ ,\ k=-6 \ \ and \ \ r^2=317$ **in (i)**

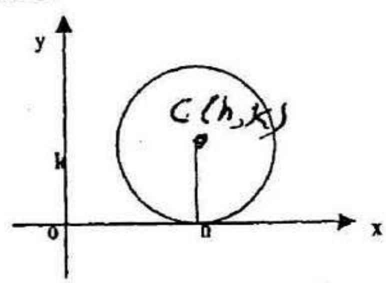
$$(x-(-9))^2+(y-(-6))^2=317$$

$\Rightarrow (x+9)^2+(y+6)^2=317 \quad Ans.$

Note: **If a circle has tangent to the x – axis.**

Since the equation of x –axis is $y=0$.

Then the radius of the circle is the distance from $C(h\ ,\ k)$



and tangent to the circle is $r = |k|$.

(e) $C(-5\ ,\ 4)$, tangent to the x – axis .

Solution: The equation of the circle is given by

$$(x-h)^2 + (y-k)^2 = r^2 \quad ....(i)$$

Then the radius of the circle is the distance from centre $C(-5\ ,\ 4)$ and tangent to the circle is $r = |k| = 4$.

Put $h = -5$, $k = 4$, $r = 4$ in (i)

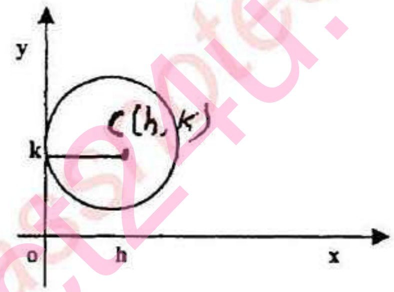
$$(x-(-5))^2 + (y-4)^2 = (4)^2$$

$$\Rightarrow (x+5)^2 + (y-4)^2 = 16 \quad Ans.$$

Note: If a circle has tangent to the y – axis.

Since the equation of y –axis is $x = 0$.

Then the radius of the circle is the distance from $C(h\ ,\ k)$



and tangent to the circle is $r = |h|$.

(f) $C(5\ ,\ 3)$, tangent to the y – axis .

Solution: The equation of the circle is given by

$$(x-h)^2 + (y-k)^2 = r^2 \quad ....(i)$$

Then the radius of the circle is the distance from $C(5\ ,\ 3)$ and tangent to the circle is

$r = |h| = 5$.

Put $h = 5$, $k = 3$, $r = 5$ in (i)

$$(x-5)^2 + (y-3)^2 = (5)^2$$

$$\Rightarrow (x-5)^2 + (y-3)^2 = 25 \quad Ans.$$

Q.3: In each case, find the centre $C(-g\ ,\ -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$ of the following :

(a) $x^2 + y^2 - 8x - 6y + 9 = 0 \quad ......(i)$

Solution: General equation of the circle is

$x^2+y^2+2gx+2fy+c=0$ ....(ii)

Comparing (i) and (ii)

$2g=-8 \Rightarrow g=-4$

$2f=-6 \Rightarrow f=-3$ and $c=9$

Now centre is $C(-g,-f)=C(-(-4),-(-3))=C(4,3)$

Radius: $r=\sqrt{g^2+f^2-c}=\sqrt{(-4)^2+(-3)^2-9}$

$\Rightarrow r=\sqrt{16+9-9}=\sqrt{16}=4$ **Ans.**

(b) $4x^2+4y^2+16x-12y-7=0$

Solution: $4x^2+4y^2+16x-12y-7=0$ **, divided by 4**

$\Rightarrow x^2+y^2+4x-3y-\frac{7}{4}=0$ ......(i)

General equation of the circle is

$x^2+y^2+2gx+2fy+c=0$ ....(ii)

Comparing (i) and (ii)

$2g=4 \Rightarrow g=2$

$2f=-3 \Rightarrow f=-\frac{3}{2}$ and $c=-\frac{7}{4}$

Now centre is $C(-g,-f)=C\left(-2,-\left(-\frac{3}{2}\right)\right)=C\left(-2,\frac{3}{2}\right)$

Radius: $r=\sqrt{g^2+f^2-c}=\sqrt{(2)^2+\left(-\frac{3}{2}\right)^2-\left(-\frac{7}{4}\right)}$

$\Rightarrow r=\sqrt{4+\frac{9}{4}+\frac{7}{4}}=\sqrt{\frac{16+9+7}{4}}=\frac{\sqrt{32}}{2}$

$\Rightarrow r=\frac{\sqrt{16\times 2}}{2}=\frac{4\sqrt{2}}{2}=2\sqrt{2}$ **Ans.**

(c) $x^2+y^2+4x-6y+13=0$ ......(i)

Solution: General equation of the circle is

$x^2+y^2+2gx+2fy+c=0$ ....(ii)

Comparing (i) and (ii)

$2g = 4 \Rightarrow g = 2$

$2f = -6 \Rightarrow f = -3$ **and** $c = 13$

Now centre is $C(-g, -f) = C(-2, -(-3)) = C(-2, 3)$

Radius: $r = \sqrt{g^2 + f^2 - c} = \sqrt{(2)^2 + (-3)^2 - 19}$

$\Rightarrow r = \sqrt{4 + 9 - 13} = \sqrt{0} = 0$ **Ans.**

(d) $x^2 + y^2 - x - 8y + 18 = 0$ **....(i)**

Solution: **General equation of the circle is**

$x^2 + y^2 + 2gx + 2fy + c = 0$ **..........(ii)**

Comparing (i) and (ii)

$2g = -1 \Rightarrow g = -\frac{1}{2}$

$2f = -8 \Rightarrow f = -4$ **and** $c = 18$

Now centre is $C(-g, -f) = C\left(-\left(-\frac{1}{2}\right), -(-4)\right) = C\left(\frac{1}{2}, 4\right)$

Radius: $r = \sqrt{g^2 + f^2 - c} = \sqrt{\left(-\frac{1}{2}\right)^2 + (-4)^2 - 18} = \sqrt{\frac{1}{4} + 16 - 18}$

$\Rightarrow r = \sqrt{\frac{1}{4} - 2} = \sqrt{\frac{1-7}{4}} = \sqrt{-\frac{7}{4}}$

The circle is imaginary. Ans.

Note: The book answer is wrong.

Q. 4 In each case, find an equation of a circle which passes through the three points:

(a) $A(-3, 0)$, $B(5, 4)$, $C(6, -3)$

Solution: Let the equation of the required circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$ **.....(a)**

Since (a) passes through the points $A(-3, 0)$, $B(5, 4)$ **and** $C(6, -3)$.

Take $A(-3, 0)$: **Put** $x = -3$, $y = 0$ **in (a)**

$(-3)^2 + 0 + 2g(-3) + 0 + c = 0$

$\Rightarrow -6g + c + 9 = 0 \Rightarrow -6g + c = -9$ **......(i)**

Take $B(5\ ,4)$**: Put** $x=5$, $y=4$ **in (a)**

$$(5)^2+(4)^2+(2g)(5)+(2f)(4)+c=0$$

$$\Rightarrow\ 10g+8f+c+25+16=0$$

$$\Rightarrow\ 10g+8f+c=-41\ ....(ii)$$

Take $C(6\ ,-3)$ **: Put** $x=6$, $y=-3$ **in (a)**

$$(6)^2+(-3)^2+(2g)(6)+(2f)(-3)+c=0$$

$$\Rightarrow\ 12g-6f+c+36+9=0$$

$$\Rightarrow\ 12g-6f+c=-45\ ....(iii)$$

The system of linear equations (i) , (ii), (iii) in the matrix form is $AX=b$

$$\begin{bmatrix}-6&0&1\\10&8&1\\12&-6&1\end{bmatrix}\begin{bmatrix}g\\f\\c\end{bmatrix}=\begin{bmatrix}-9\\-41\\-45\end{bmatrix}$$

Now the augmented matrix $A/b=\begin{bmatrix}-6&0&1&-9\\10&8&1&-41\\12&-6&1&-45\end{bmatrix}$

we change it into echelon form

$$\underset{\to}{R}\begin{bmatrix}1&0&-\frac{1}{6}&\frac{3}{2}\\10&8&1&-41\\12&-6&1&-45\end{bmatrix}-\frac{1}{6}R_3$$

$$\underset{\to}{R}\begin{bmatrix}1&0&-\frac{1}{6}&\frac{3}{2}\\0&8&\frac{8}{3}&-56\\0&-6&3&-63\end{bmatrix}\begin{matrix}R_2-10R_1\\R_3-12R_1\end{matrix}$$

$$\underset{\neg}{R}\begin{bmatrix} 1 & 0 & -\frac{1}{6} & \frac{3}{2} \\ 0 & 1 & \frac{1}{3} & -7 \\ 0 & -6 & 3 & -63 \end{bmatrix}\frac{1}{8}R_2$$

$$\underset{\neg}{R}\begin{bmatrix} 1 & 0 & -\frac{1}{6} & \frac{3}{2} \\ 0 & 1 & \frac{1}{3} & -7 \\ 0 & 0 & 5 & -105 \end{bmatrix}R_3 + 6R_1$$

From R_3 => $5c = -105$ => $c = -\frac{105}{5} = -21$ **...(iv)**

From R_2 => $f + \frac{1}{3}c = -7$ **.....(v)**

From R_1 => $g - \frac{1}{6}c = \frac{3}{2}$ **........(vi)**

Put $c = -21$ **in (v)**

$f + \left(\frac{1}{3}\right)(-21) = -7$ => $f - 7 = -7$

=> $f = -7 + 7 = 0$

Put $c = -21$ **in (vi)**

$g - \left(\frac{1}{6}\right)(-21) = \frac{3}{2}$ => $g + \frac{7}{2} = \frac{3}{2}$ => $g = \frac{3}{2} - \frac{7}{2}$

=> $g = \frac{-4}{2} = -2$

Put values of g , f and c in (a)

$x^2+y^2-2(2)x+0.y-21=0$

$\Rightarrow x^2+y^2-4x-21=0$ *Ans.*

(b) $A(7\,,-1)\,,\; B(5,\,3)\,,\; D(-4\,,6)$

Note: This question we can solve by another method. The above method is very long.

Solution: Let $C(h\,,k)$ be the centre of the required circle.

Then $|AC|$, $|BC|$ *and* $|DC|$ is radius of the circle.

So $|AC|=|BC|$

$\Rightarrow \sqrt{(h-7)^2+(k+1)^2}=\sqrt{(h-5)^2+(k-3)^2}$ squaring

$\Rightarrow h^2-14h+49+k^2+2k+1=h^2-10h+25+k^2-6k+9$

$\Rightarrow -14h+10h+2k+6k=34-50$

$\Rightarrow -4h+8k=-16$ *Divided by* -4

$\Rightarrow h-2k=4$ ....(i)

Also $|AC|=|DC|$

$\Rightarrow \sqrt{(h-7)^2+(k+1)^2}=\sqrt{(h-(-4))^2+(k-6)^2}$

$\Rightarrow \sqrt{(h-7)^2+(k+1)^2}=\sqrt{(h+4)^2+(k-6)^2}$ squaring

$\Rightarrow h^2-14h+49+k^2+2k+1=h^2+8h+16+k^2-12k+36$

$\Rightarrow -14h-8h+2k+12k=52-50$

$\Rightarrow -22h+14k=2$ *Divided by* -2

$\Rightarrow 11h-7k=-1$ ....(ii)

Eq(i) is multiplied by 11 $\Rightarrow 11h-22k=44$ ....(iii)

Now (iii) – (ii)

$$\begin{array}{r} 11h-22k=44 \\ \pm 11h\mp 7k=\mp 1 \\ \hline -15k=45 \end{array}$$

$\Rightarrow k=-3$ put $k=-3$ in (i)

$h-2(-3)=4 \quad => \quad h+6=4$

$=> \quad h=4-6=-2$

Thus the centre is $C(-2\ ,-3)$.

Now radius is $r=|AC|=\sqrt{(-2-7)^2+(-3-(-1))^2}$

$=> \quad r=\sqrt{(-9)^2+(-3+1)^2}=\sqrt{81+(-2)^2}=\sqrt{81+4}$

$=> \quad r=\sqrt{85}$

Now equation of the circle is

$(x-h)^2+(y-k)^2=r^2$ **put values** $h=-2\ ,\ k=-3$

$=> (x-(-2))^2+(y-(-3))^2=\left(\sqrt{85}\right)^2$

$=> \quad (x+2)^2+(y+3)^2=85$

$=> \quad x^2+4x+4+y^2+6y+9=85$

$=> \quad x^2+y^2+4x+6y-72=0 \qquad$ *Ans.*

(c) $A(1\ ,2)\ ,\ B(3\ ,-4)\ ,\ D(5\ ,-6)$

<u>Solution</u>: Let $C(h\ ,k)$ **be the centre of the required circle.**

Then $|AC|\ ,\ |BC|$ *and* $|DC|$ **is radius of the circle.**

So $|AC|=|BC|$

$=> \quad \sqrt{(h-1)^2+(k-2)^2}=\sqrt{(h-3)^2+(k-(-4))^2}$

$=> \quad \sqrt{(h-1)^2+(k-2)^2}=\sqrt{(h-3)^2+(k+4)^2}$ **squaring**

$=> \quad h^2-2h+1+k^2-4k+4=h^2-6h+9+k^2+8k+16$

$=> -2h+6h-4k-8k=25-5$

$=> \quad 4h-12k=20 \qquad$ *Divided by 4*

$=> \quad h-3k=5 \quad ....(i)$

Also $|AC|=|DC|$

$=> \quad \sqrt{(h-1)^2+(k-2)^2}=\sqrt{(h-5)^2+(k-(-6))^2}$

$=> \quad \sqrt{(h-1)^2+(k-2)^2}=\sqrt{(h-5)^2+(k+6)^2}$ **squaring**

$$\Rightarrow \quad h^2-2h+1+k^2-4k+4=h^2-10h+25+k^2+12k+36$$

$$\Rightarrow -2h+10h-4k-12k=61-5$$

$$\Rightarrow \quad 8h-16k=56 \quad \text{Divided by } 8$$

$$\Rightarrow \quad h-2k=7 \quad \ldots\ldots(ii)$$

Subtract (i) from (ii)

$$\begin{array}{r} h-2k=7 \\ \underline{\pm h \mp 3k=\pm 5} \\ k=2 \end{array}$$

Put $k=2$ **in (ii)**

$$h-2(2)=7 \quad \Rightarrow \quad h=7+4=11$$

Thus the centre is $C(11\,,\,2)$

Now radius is $r=|AC|=\sqrt{(11-1)^2+(2-2)^2}$

$$\Rightarrow \quad r=\sqrt{(10)^2+0}=\sqrt{(10)^2}=10$$

Now equation of the circle is

$$(x-h)^2+(y-k)^2=r^2 \quad \text{put values} \quad h=11 \quad , \quad k=2$$

$$\Rightarrow \quad (x-11)^2+(y-2)^2=(10)^2$$

$$\Rightarrow \quad x^2-22x+121+y^2-4y+4=100$$

$$\Rightarrow \quad x^2+y^2-22x-4y+125-100=0$$

$$\Rightarrow \quad x^2+y^2-22x-4y+25=0 \quad \text{Ans.}$$

<u>Q.5</u>: In each case, find an equation of a circle which

(a) contains the point $(2\,,\,6)$, $(6\,,\,4)$ and has its centre on the line $3x+2y-1=0$

<u>Solution:</u> Let $A(2\,,\,6)$, $B(6\,,\,4)$ **and**

$$3x+2y-1=0 \quad \ldots\ldots(i)$$

Let $C(h\,,k)$ **be the centre of the required circle.**

Then $|AC|$, $|BC|$ **is radius of the circle.**

So $|AC|=|BC|$

$$\Rightarrow \quad \sqrt{(h-2)^2+(k-6)^2}=\sqrt{(h-6)^2+(k-4)^2} \quad \text{squaring}$$

$\Rightarrow \quad h^2 - 4h + 4 + k^2 - 12k + 36 = h^2 - 12h + 36 + k^2 - 8k + 16$

$\Rightarrow -4h + 12h - 12k + 8k = 52 - 40$

$\Rightarrow \quad 8h - 4k = 12 \qquad \textit{divided by } 4$

$\Rightarrow \quad 2h - k = 3 \quad ....(ii)$

Since the centre $C(h, k)$ is lie on the line (i),

then put $x = h$, and $y = k$ in (i)

$3h + 2k - 1 = 0 \quad \Rightarrow \quad 3h + 2k = 1 \quad ....(iii)$

Eq (ii) is multiplied by 2 $\quad \Rightarrow \quad 4h - 2k = 6$ **.......(iv)**

Adding (ii) and (iii)

$$\begin{array}{r} 3h + 2k = 1 \\ \underline{4h - 2k = 6} \\ 7h = 7 \end{array}$$

$\Rightarrow \quad h = 1$, **Put** $h = 1$ **in (ii)**

$2(1) - k = 3 \quad \Rightarrow \quad 2 - 3 = k$

$\Rightarrow k = -1$

Thus centre is $C(1, -1)$. **Now radius is**

$$r = |AC| = \sqrt{(2-1)^2 + (6+1)^2} = \sqrt{(1)^2 + (7)^2} = \sqrt{1+49}$$

$\Rightarrow r = \sqrt{50}$

Now equation of the circle is

$(x-h)^2 + (y-k)^2 = r^2$ **put values** $h = 1$, $k = -1$

$\Rightarrow \quad (x-1)^2 + (y-(-1))^2 = (\sqrt{50})^2$

$\Rightarrow \quad (x-1)^2 + (y+1)^2 = 50$

$\Rightarrow \quad x^2 - 2x + 1 + y^2 + 2y + 1 = 50$

$\Rightarrow \quad x^2 + y^2 - 2x + 2y + 2 - 50 = 0$

$\Rightarrow \quad x^2 + y^2 - 2x + 2y - 48 = 0$ **Ans.**

(b) contains the point $(4, 1)$, $(6, 5)$ and has its centre on the line $4x + y - 16 = 0$

Solution: Let $A(4,1)$, $B(6,5)$ and

$4x+y-16=0$ ......(i)

Let $C(h,k)$ be the centre of the required circle.

Then $|AC|$, $|BC|$ is radius of the circle.

So $|AC|=|BC|$

$\Rightarrow \sqrt{(h-4)^2+(k-1)^2}=\sqrt{(h-6)^2+(k-5)^2}$ squaring

$\Rightarrow h^2-8h+16+k^2-2k+1=h^2-12h+36+k^2-10k+25$

$\Rightarrow -8h+12h-2k+10k=61-17$

$\Rightarrow 4h+8k=44$ *divided by 4*

$\Rightarrow h+2k=11$ ....(ii)

Since the centre $C(h,k)$ is lie on the line (i),

then put $x=h$, and $y=k$ in (i)

$4h+k-16=0 \Rightarrow 4h+k=16$ ....(iii)

Eq (iii) is multiplied by 2 $\Rightarrow 8h+2k=32$ ....(iv)

Now (iv) – (ii)

$$\begin{array}{r} 8h+2k=32 \\ \pm h \pm 2k = \pm 11 \\ \hline 7h=21 \end{array}$$

$\Rightarrow h=3$, Put $h=3$ in (ii)

$3+2k=11 \Rightarrow 2k=11-3$

$\Rightarrow 2k=8 \Rightarrow k=4$

Thus centre is $C(3,4)$. Now radius is

$r=|AC|=\sqrt{(3-4)^2+(4-1)^2}=\sqrt{(-1)^2+(3)^2}=\sqrt{1+9}$

$\Rightarrow r=\sqrt{10}$

Now equation of the circle is

$(x-h)^2+(y-k)^2=r^2$ put values $h=3$, $k=4$

$$\Rightarrow \quad (x-3)^2+(y-4)^2=\left(\sqrt{10}\right)^2$$

$$\Rightarrow \quad x^2-6x+9+y^2-16y+16=10$$

$$\Rightarrow \quad x^2+y^2-6x-16y+25-10=0$$

$$\Rightarrow \quad x^2+y^2-6x-16y+15=0 \quad \textbf{Ans.}$$

Q.6: **Find an equation of a circle which passes through the points :**

(a) $(0\,,\,0)$, $(0\,,\,3)$ **and the line** $4x-5y=0$ **is tangent to at** $(0\,,\,0)$ **.**

Solution: Let $A(0\,,\,0)$, $B(0\,,\,3)$ **and**

$4x-5y=0$ **......(i)**

Let $C(h\,,k)$ **be the centre of the required circle .**

Then $|AC|$, $|BC|$ **is radius of the circle .**

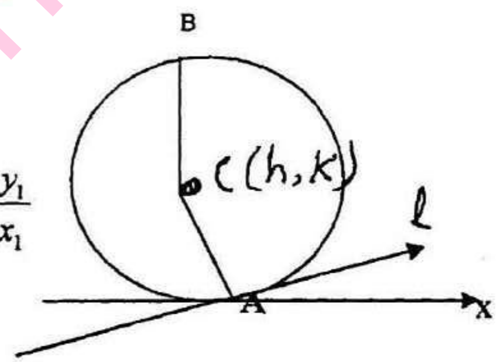
So $|AC|=|BC|$

$$\Rightarrow \quad \sqrt{(h-0)^2+(k-0)^2}=\sqrt{(h-0)^2+(k-3)^2} \quad \textbf{squaring}$$

$$\Rightarrow \quad h^2+k^2=h^2+k^2-6k+9$$

$$\Rightarrow 6k=9 \qquad \Rightarrow \quad k=\frac{9}{6}$$

$$\Rightarrow \quad k=\frac{3}{2}$$



Now slope of $\overline{AC}=m_1=\dfrac{y_2-y_1}{x_2-x_1}$

$$\Rightarrow \quad m_1=\frac{k-0}{h-0}=\frac{k}{h} \quad \textbf{....(ii)}$$

put $k=\dfrac{3}{2}$ **in (ii)**

$$\Rightarrow \quad m_1=\frac{3}{2h} \quad \textbf{....(iii)}$$

Now from (i) => $5y = 4x$ Or $y = \frac{4}{5}x$ .....(iv)

Then the slope of line is $m_2 = \frac{4}{5}$

Since the line (i) is tangent to the circle at $A(0\,,\,0)$ then the radius AC is perpendicular to line (i) .

As $m_1.m_2 = -1$ put values

$$\left(\frac{3}{2h}\right)\left(\frac{4}{5}\right) = -1 \quad => \quad 12 = -10h$$

$$=> \quad h = -\frac{12}{10} = -\frac{6}{5}$$

Thus the centre is $C\left(-\frac{6}{5}\,,\,\frac{3}{2}\right)$.

Now radius is $r = |AC| = \sqrt{\left(-\frac{6}{5}-0\right)^2 + \left(\frac{3}{2}-0\right)^2}$

$$=> \quad r = \sqrt{\frac{36}{25}+\frac{9}{4}} = \sqrt{\frac{144+225}{100}} = \sqrt{\frac{369}{100}}$$

Now equation of the circle is

$(x-h)^2 + (y-k)^2 = r^2$ put values $h = -\frac{6}{5}$, $k = \frac{3}{2}$

$$=> \left(x - \left(-\frac{6}{5}\right)\right)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\sqrt{\frac{369}{100}}\right)^2$$

$$=> \left(x + \frac{6}{5}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{369}{100}$$

$$=> x^2 + \frac{12}{5}x + \frac{36}{25} + y^2 - 3y + \frac{9}{4} = \frac{369}{100}$$

(b) $(0\,,\,-1)$, $(3\,,\,0)$ and the line $3x + y = 9$ is tangent to at $(3\,,\,0)$.

<u>Solution:</u> Let $A(0\,,\,-1)$, $B(3\,,\,0)$ and

$$3x + y = 9 \quad \text{......(i)}$$

Let $C(h, k)$ **be the centre of the required circle.**

Then $|AC|$, $|BC|$ **is radius of the circle.**

So $|AC| = |BC|$

$$\Rightarrow \sqrt{(h-0)^2 + (k-(-1))^2} = \sqrt{(h-3)^2 + (k-0)^2}$$

$$\Rightarrow \sqrt{(h-0)^2 + (k+1)^2} = \sqrt{(h-3)^2 + (k-0)^2} \text{ squaring}$$

$$\Rightarrow h^2 + k^2 + 2k + 1 = h^2 - 6h + 9 + k^2$$

$$\Rightarrow 6h + 2k = 8 \quad \Rightarrow 3h + k = 4 \quad \text{.......}(ii)$$

Now slope of $\overline{BC} = m_1 = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\Rightarrow m_1 = \frac{k-0}{h-3} = \frac{k}{h-3} \quad \text{....(iii)}$$

Now from (i) $\Rightarrow y = -3x + 9$ **.......(iv)**

Then the slope of line is $m_2 = -3$

Since the line (i) is tangent to the circle at $B(0, 3)$ **then the radius** BC **is perpendicular to line (i).**

As $m_1.m_2 = -1$ **put values**

$$\left(\frac{k}{h-3}\right)(-3) = -1 \quad \Rightarrow 3k = h - 3$$

$$\Rightarrow -h + 3k = -3 \quad \text{.....}(v)$$

Eq (v) is multiplied by 3 $\Rightarrow -3h + 9k = -9$ **...(vi)**

Adding (ii) and (vi)

$$\begin{array}{r} 3h + k = 4 \\ -3h + 9k = -9 \\ \hline 10k = -5 \end{array}$$

$$\Rightarrow k = -\frac{5}{10} = -\frac{1}{2}.$$

Put $k = -\dfrac{1}{2}$ **in (v)**

$$-h+3\left(-\frac{1}{2}\right)=-3 \quad => \quad -h-\frac{3}{2}=-3$$

$$=> \quad -h=-3+\frac{3}{2} \quad => -h=\frac{-6+3}{2}$$

$$=> \quad -h=-\frac{3}{2} \quad => \quad h=\frac{3}{2}$$

Thus the centre is $C\left(\frac{3}{2}, -\frac{1}{2}\right)$.

Now radius is $r=|AC|=\sqrt{\left(\frac{3}{2}-0\right)^2+\left(-\frac{1}{2}-(-1)\right)^2}$

$$=\sqrt{\left(\frac{3}{2}\right)^2+\left(-\frac{1}{2}+1\right)^2}=\sqrt{\frac{9}{4}+\left(\frac{1}{2}\right)^2}$$

$$=> \quad r=\sqrt{\frac{9}{4}+\frac{1}{4}}=\sqrt{\frac{10}{4}}$$

Now equation of the circle is

$(x-h)^2+(y-k)^2=r^2$ **put values** $h=\frac{3}{2}$, $k=-\frac{1}{2}$

$$=> \quad \left(x-\frac{3}{2}\right)^2+\left(y-\left(-\frac{1}{2}\right)\right)^2=\left(\sqrt{\frac{10}{4}}\right)^2$$

$$=> \quad \left(x-\frac{3}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=\frac{10}{4}$$

$$=> x^2-3x+\frac{9}{4}+y^2+y+\frac{1}{4}=\frac{10}{4}$$

$$=> \quad x^2+y^2-3x+y+\frac{9}{4}+\frac{1}{4}-\frac{10}{4}=0$$

$$=> \quad x^2+y^2-3x+y+\frac{10-10}{4}=0$$

$$=> \quad x^2+y^2-3x+y=0 \quad \textbf{Ans.}$$

Note: Concentric circle: If one circle having a common centre with another circle is called concentric circle .

Q.7: Find an equation of a circle that is concentric to circle

(a) $2x^2+2y^2+16x-7y=0$ and is tangent to y –axis .

Solution: $2x^2+2y^2+16x-7y=0$ divided by 2

$$\Rightarrow x^2+y^2+8x-\frac{7}{2}y=0 \quad ....(i)$$

As we know that the general equation of the circle is

$$x^2+y^2+2gx+2fy+c=0 \quad ....(ii)$$

Comparing (i) and (ii) ,

$$2g=8 \Rightarrow g=4$$

$$2f=-\frac{7}{2} \Rightarrow f=-\frac{7}{4}$$

Now the centre is $C(-g\ ,-f)=C\left(-4,\ \frac{7}{4}\right)$

Then the centre of the given circle (i) as well as required circle is $C\left(-4\ ,\frac{7}{4}\right)$.

Since y –axis is the tangent to the required circle
Then the radius of the circle is the distance from centre $C\left(-4\ ,\frac{7}{4}\right)$ and tangent to the circle is

$$r=|h|=|-4|=4\ .$$

The equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \quad ....(i)$$

Put $h=-4$, $k=\frac{7}{4}$, $r=4$ in (i)

$$(x-(-4))^2+\left(y-\frac{7}{4}\right)^2=(4)^2$$

$$\Rightarrow (x+4)^2+\left(y-\frac{7}{4}\right)^2=16 \quad Ans.$$

(b) $x^2+y^2-8x+4=0$ **and is tangent to the line** $x+2y+6=0$

<u>Solution</u>: $x^2+y^2-8x+4=0$ ...(i)

$x+2y+6=0$ ......(ii)

As we know that the general equation of the circle is

$x^2+y^2+2gx+2fy+c=0$ ....(iii)

Comparing (i) and (iii),

$2g=-8 \Rightarrow g=-4$

$2f=0 \Rightarrow f=0$

Now the centre is $C(-g,-f)=C(4,0)$

Then the centre of given circle (i) as well as required circle is $C(4,0)$.

Since line (ii) is the tangent to the required circle
Then the radius of the circle is the perpendicular distance from centre $C(4,0)$ **and tangent line** $x+2y+6=0$.

$$r=d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}} \text{........(iv)}$$

Put $x_1=4$, $y_1=0$, $a=1$, $b=2$, $c=6$ **in (iv)**

$$r=\frac{|(1)(4)+(2)(0)+6|}{\sqrt{(1)^2+(2)^2}}=\frac{|4+0+6|}{\sqrt{1+4}}$$

$$\Rightarrow r=\frac{10}{\sqrt{5}}$$

The equation of the circle is given by

$(x-h)^2+(y-k)^2=r^2$ ....(i)

Put $h=4$, $k=0$, $r=\frac{10}{\sqrt{5}}$ in (i)

$$(x-4)^2+(y-0)^2=\left(\frac{10}{\sqrt{5}}\right)^2$$

$$\Rightarrow (x-4)^2+(y-0)^2=\frac{100}{5}$$

$$\Rightarrow (x-4)^2+(y-0)^2=20 \qquad Ans.$$

(c) $x^2+y^2+6x-10y+33=0$ **and touching the x –axis .**

Solution: $x^2+y^2+6x-10y+33=0$ ….(i)

As we know that the general equation of the circle is

$x^2+y^2+2gx+2fy+c=0$ ……..(ii)

Comparing (i) and (ii) ,

$2g=6 \Rightarrow g=3$

$2f=-10 \Rightarrow f=-5$

Now the centre is $C(-g\ ,-f)=C(-3,\ 5)$

Then the centre of given circle (i) as well as required circle is $C(-3\ ,5)$.

Since the required circle touching the x – axis

then x –axis is the tangent to the required circle

Then the radius of the circle is the distance from centre $C(-3\ ,5)$ and tangent to the circle is

$r=|k|=|5|=5$.

The equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \quad ….(i)$$

Put $h=-3$, $k=5$, $r=5$ in (i)

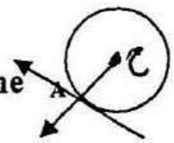
$$(x-(-3))^2+(y-5)^2=(5)^2$$

$$\Rightarrow (x+3)^2+(y-5)^2=25$$

Tangent and normal : If a line touch a circle at only one point is called tangent line . The point where tangent meets the circle is called point of contact .

The line which is perpendicular to the tangent line is called normal .

tangent line



Normal line

<u>Example 7</u>: Find the points of intersection of the line $3x-4y+20=0$ **and the circle** $x^2+y^2=25$

<u>Solution</u>: $3x-4y+20=0$ **.....(i) and**

$x^2+y^2=25$ **.....(ii)**

From (i) $\Rightarrow 3x=4y-20 \Rightarrow x=\dfrac{4y-20}{3}$ .....(*iii*)

Put (iii) in (i) $\Rightarrow \left(\dfrac{4y-20}{3}\right)^2+y^2=25$

$\Rightarrow \dfrac{16y^2+400-160y}{9}+y^2=25$

$\Rightarrow 16y^2-160y+400+9y^2=225$

$\Rightarrow 25y^2-160y+400-225=0$

$\Rightarrow 25y^2-160y+175=0$, *divided by* 5

$\Rightarrow 5y^2-32y+35=0$

$5y^2-25y-7y+35=0$

$\Rightarrow 5y(y-5)-7(y-5)=0$

$\Rightarrow (y-5)(5y-7)=0$

$\Rightarrow y-5=0 \quad and \quad 5y-7=0$

$\Rightarrow y=5 \quad and \quad y=\dfrac{7}{5}$

Put $y=5$ **in (iii)**

$x=\dfrac{4(5)-20}{3}=\dfrac{20-20}{3}=0$ **Put** $y=\dfrac{7}{5}$ **in (iii)**

$$x = \frac{4\left(\frac{7}{5}\right) - 20}{3} = \frac{28 - 100}{5 \times 3} = -\frac{72}{15} = -\frac{24}{5}$$

Thus the point of intersections are $(0\ ,\ 5)$ and $\left(-\frac{24}{5}\ ,\ \frac{7}{5}\right)$.

<u>Condition of tangency</u>: The line $y = mx + c$ ...(i) touch the circle $x^2 + y^2 = a^2$ ....(ii) if $c = \pm a\sqrt{1+m^2}$ ....(iii)

Eq(iii) is the condition for a line which touch the given circle.

Put (iii) in (i) => $y = mx \pm a\sqrt{1+m^2}$ .....(iv)

Eq (iv) shows the equation of tangent line in slope form.

<u>Example 9</u>: Find the coordinates of the middle point of the chord which the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ cuts off on the line $x - y + 2 = 0$

<u>Solution</u>: $x^2 + y^2 + 4x - 2y - 3 = 0$ **.....(i)**

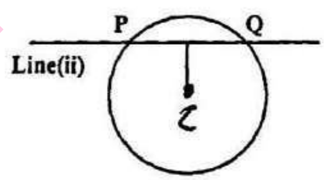
$x - y + 2 = 0$ **......(ii)**

As from (i) then $2g = 4$ => $g = 2$

and $2f = -2$ => $f = -1$

Then the centre is $C(-g\ ,-f) = C(-2\ ,\ 1)$.

Let the line (ii) cuts the circle (i) at points P and Q.



Let $M(x_1\ ,\ y_1)$ is the middle point of the chord PQ.

Join C and M.

Then line CM is perpendicular to the chord PQ.

Since $M(x_1\ ,\ y_1)$ lie on line (ii). Then eq(ii) becomes

$x_1 - y_1 + 2 = 0$ **.....(iii)**

Now from (ii) => $y = x + 2$

Then the slope line (ii) is $m_1 = 1$.

Now slope of CM is $m_2 = \frac{y_1 - 1}{x_1 + 2}$

Since CM is perpendicular to the chord PQ , then

$m_1.m_2 = -1 \quad => \left(\frac{y_1 - 1}{x_1 + 2}\right)(1) = -1$

$=> \; y_1 - 1 = -x_1 - 2$

$=> \; x_1 + y_1 + 1 = 0 \; \text{........}(iv)$

Now adding (iii) and (iv)

$$\begin{array}{c} x_1 - y_1 + 2 = 0 \\ \underline{x_1 + y_1 + 1 = 0} \\ 2x_1 - 3 = 0 \end{array}$$

$=> \; x_1 = -\frac{3}{2}$, put $x_1 = -\frac{3}{2}$ in(iv)

$=> -\frac{3}{2} + y_1 + 1 = 0 \quad => \; y_1 = \frac{3}{2} - 1 = \frac{3-2}{2} = \frac{1}{2}$

Thus the middle point is $M\left(-\frac{3}{2}, \frac{1}{2}\right)$ Ans.

Point of contact : If $y = mx \pm a\sqrt{1+m^2}$ be a tangent line to the circle $x^2 + y^2 = a^2$, then the point of contact is given by $A(x_1, y_1) = A\left(\frac{-am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$.

Equation of tangent and normal to a circle at a point :

The equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

Then the equation of tangent to the circle (i) at the point $A(x_1, y_1)$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0 \; \text{......(a)}$$

The equation of normal to the circle (i) at the point $A(x_1, y_1)$ is $y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$ ......(b)

The equation of tangent to the circle $x^2 + y^2 = a^2$ at the point $A(x_1 , y_1)$ is $x x_1 + yy_1 = a^2$ .....(c)

The equation of normal to the circle $x^2 + y^2 = a^2$ at the point $A(x_1 , y_1)$ is $xy_1 - yx_1 = 0$ ....(d)

Example .11: Find the equations of the tangent and normal to the circle $x^2 + y^2 = 25$ at a point $(3 , 4)$.

Solution: As we know that the equation of tangent to the circle $x^2 + y^2 = a^2$ at the point $A(x_1 , y_1)$ is

$x x_1 + yy_1 = a^2$ .....(i)

Here put $a^2 = 25$, $x_1 = 3$, $y_1 = 4$ in (i)

$3x + 4y = 25$ is the required tangent equation .

The equation of normal to the circle $x^2 + y^2 = a^2$ at the point $A(x_1 , y_1)$ is $xy_1 - yx_1 = 0$ ....(ii)

Put $x_1 = 3$, $y_1 = 4$ in (ii)

=> $4x - 3y = 0$ is the required normal equation .

Two tangents to the given circle : As $y = mx \pm a\sqrt{1+m^2}$ ...(A) is any tangent to the circle $x^2 + y^2 = a^2$ .Then the tangent line passes through the point $A(x_1 , y_1)$ is

$y_1 = mx_1 + a\sqrt{1+m^2}$

=> $y_1 - m_1x = a\sqrt{1+m^2}$ squaring both sides

=> $(y_1 - mx_1)^2 = \left(a\sqrt{1+m^2}\right)^2$

=> $y_1^2 + m^2x_1^2 - 2x_1y_1m = a^2(1+m^2)$

=> $\left(x_1^2 - a^2\right)m^2 - 2x_1y_1m + y_1^2 - a^2 = 0$ ....(B)

Eq(B) is the quadratic equation of m , from eq(B) we find two values of m , then put these values in (A) ,we get two tangent lines .

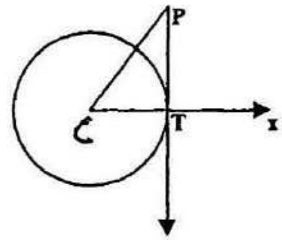
Length of tangent to a circle from a given external point:

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(i) is the equation of circle

Let $P(x_1 , y_1)$ is external point.

Let PT be the one of two tangents drawn from point P to the circle (i) .

Then the length of tangent PT is



$|PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ .....(a)

The length of tangent drawn from the point $P(x_1 , y_1)$ to the circle $x^2 + y^2 = a^2$ is $|PT| = \sqrt{x_1^2 + y_1^2 - a^2}$ ....(b)

<u>Example :13</u>: Find the length of the tangent drawn from the point $P(3 , 4)$ on the circles

(a) $x^2 + y^2 = 9$

(b) $x^2 + y^2 - 2x - y - 3 = 0$

<u>Solution</u>: (a) $x^2 + y^2 = 9$ ....(i)

The length of tangent drawn from the point $P(x_1 , y_1)$ to the circle $x^2 + y^2 = a^2$ is

$|PT| = \sqrt{x_1^2 + y_1^2 - a^2}$ ....(ii) , Put $x_1 = 3$, $y_1 = 4$, $a^2 = 9$ in (ii)

=> $|PT| = \sqrt{(3)^2 + (4)^2 - 9} = \sqrt{9 + 16 - 9} = \sqrt{16} = 4$

(b) $x^2 + y^2 - 2x - y - 3 = 0$ .....(i)

The length of tangent drawn from the point $P(x_1 , y_1)$ to the Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$|PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ ....(ii)

Put $2g = -2$, $2f = -1$, $c = -3$, $x_1 = 3$, $y_1 = 4$ in (ii)

$|PT| = \sqrt{(3)^2 + (4)^2 + (-2)(3) + (-1)(4) - 3}$

$= \sqrt{9 + 16 - 6 - 4 - 3} = \sqrt{12}$ *Ans.*

Exercise 8.2:

Q.1: In each case, find the tangent and normal equations

(a) At a point $(1, 2)$ to the circle $x^2 + y^2 = 5$

(b) At a point $(-3, -2)$ to the circle $x^2 + y^2 = 13$

(c) At a point $(4, 1)$ to the circle $x^2 + y^2 - 4x + 2y - 3 = 0$

Solution: (a) $x^2 + y^2 = 5$ ....(i) and $(1, 2)$.

As we know that the equation of tangent to the circle

$x^2 + y^2 = a^2$ at the point $A(x_1, y_1)$ is

$x x_1 + y y_1 = a^2$ .....(a)

Here put $a^2 = 5$, $x_1 = 1$, $y_1 = 2$ in (a)

$x + 2y = 5$ is the required tangent equation.

The equation of normal to the circle $x^2 + y^2 = a^2$ at the point $A(x_1, y_1)$ is $x y_1 - y x_1 = 0$ ....(b)

Put $x_1 = 1$, $y_1 = 2$ in (b)

=> $x - 2y = 0$ is the required normal equation.

(b) $x^2 + y^2 = 13$ ....(i) and $(-3, -2)$.

As we know that the equation of tangent to the circle

$x^2 + y^2 = a^2$ at the point $A(x_1, y_1)$ is

$x x_1 + y y_1 = a^2$ .....(a)

Here put $a^2 = 13$, $x_1 = -3$, $y_1 = -2$ in (a)

$-3x - 2y = 13$

=> $3x + 2y = -13$ is the required tangent equation.

The equation of normal to the circle $x^2 + y^2 = a^2$ at the point $A(x_1, y_1)$ is $x y_1 - y x_1 = 0$ ....(b)

Put $x_1 = -3$, $y_1 = -2$ in (b)

=> $-3x + 2y = 0$ is the required normal equation.

(c) $x^2 + y^2 - 4x + 2y - 3 = 0$ ....(i) and $(4, 1)$

Here $2g = -4$ => $g = -2$, $2f = 2$, $f = 1$, $c = -3$

Then the equation of tangent to the circle (i) at the point

$A(x_1 , y_1)$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0 \text{ ......(a)}$$

Put $g = -2$, $f = 1$, $c = -3$, $x_1 = 4$, $y_1 = 1$ **in (a)**

=> $4x + 1.y - 2(x + 4) + 1(y + 1) - 3 = 0$

=> $4x + y - 2x - 8 + y + 1 - 3 = 0$

=> $2x + 2y - 10 = 0$ => $2(x + y - 5) = 0$

=> $x + y - 5 = 0$ *both sides divided by* 2

is the required equation of tangent.

The equation of normal to the circle (i) at the point

$A(x_1 , y_1)$ **is** $y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$ **......(b)**

Put $g = -2$, $f = 1$, $c = -3$, $x_1 = 4$, $y_1 = 1$ **in in (b)**

$$y - 1 = \frac{1+1}{4-2}(x - 4)$$

=> $y - 1 = \frac{2}{2}(x - 4)$ => $y - 1 = 1(x - 4)$

=> $y - 1 = x - 4$

=> $0 = x - y - 4 + 1$

=> $x - y - 3 = 0$ *is the required eqution of normal.*

Q.2: In each case, find the tangent and normal equations

(a) At a point $(\cos 60^o , \sin 60^o)$ **to the circle** $36(x^2 + y^2) = 13$

(b) At a point $(2\cos 45^o , 2\sin 45^o)$ **to the circle** $x^2 + y^2 = 4$

(c) At a point $(\cos 30^o , \sin 30^o)$ **to the circle** $x^2 + y^2 = 1$

Solution: (a) $36(x^2 + y^2) = 13$

$$\Rightarrow x^2 + y^2 = \frac{13}{36} \text{ ......(i)}$$

and point $(\cos 60^o , \sin 60^o) = \left(\frac{1}{2} , \frac{\sqrt{3}}{2}\right)$

As we know that the equation of tangent to the circle
$x^2+y^2=a^2$ at the point $A(x_1, y_1)$ is
$xx_1+yy_1=a^2$ .....(a)

Here put $a^2=\frac{13}{36}$, $x_1=\frac{1}{2}$, $y_1=\frac{\sqrt{3}}{2}$ in (a)

$\frac{1}{2}x+\frac{\sqrt{3}}{2}y=\frac{13}{36}$ => $x+\sqrt{3}y=2\times\frac{13}{36}$

=> $x+\sqrt{3}y=\frac{13}{18}$

=> $18x+18\sqrt{3}y=2$

is the required tangent equation .

The equation of normal to the circle $x^2+y^2=a^2$ at the point $A(x_1, y_1)$ is $xy_1-yx_1=0$ ....(b)

put $a^2=\frac{13}{36}$, $x_1=\frac{1}{2}$, $y_1=\frac{\sqrt{3}}{2}$ in (b)

=> $\frac{1}{2}x-\frac{\sqrt{3}}{2}y=0$

=> $x-\sqrt{3}y=0$ is the required normal equation .

(b) $x^2+y^2=4$ ....(i) and

Point is $(2\cos 45^o, 2\sin 45^o)=\left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)=(\sqrt{2}, \sqrt{2})$

As we know that the equation of tangent to the circle
$x^2+y^2=a^2$ at the point $A(x_1, y_1)$ is
$xx_1+yy_1=a^2$ .....(a)

Here put $a^2=4$, $x_1=\sqrt{2}$, $y_1=\sqrt{2}$ in (a)

$\sqrt{2}x+\sqrt{2}y=4$ => $x+y=\frac{4}{\sqrt{2}}$

=> $x+y=\sqrt{2}$

is the required tangent equation .

The equation of normal to the circle $x^2 + y^2 = a^2$ at the point

$A(x_1 , y_1)$ is $xy_1 - yx_1 = 0$ ....(b)

Put $x_1 = \sqrt{2}$, $y_1 = \sqrt{2}$ in (b)

=> $\sqrt{2}x - \sqrt{2}y = 0$

=> $x - y = 0$ is the required normal equation .

(c) $x^2 + y^2 = 1$ .....(i) and the point is

$$(\cos 30^o , \sin 30^o) = \left(\frac{\sqrt{3}}{2} , \frac{1}{2}\right)$$

As we know that the equation of tangent to the circle $x^2 + y^2 = a^2$ at the point $A(x_1 , y_1)$ is

$x x_1 + yy_1 = a^2$ .....(a)

Here put $a^2 = 1$, $x_1 = \frac{\sqrt{3}}{2}$, $y_1 = \frac{1}{2}$ in (a)

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 1 \quad => \sqrt{3}\,x + y = 2$$

is the required tangent equation .

The equation of normal to the circle $x^2 + y^2 = a^2$ at the point

$A(x_1 , y_1)$ is $xy_1 - yx_1 = 0$ ....(b)

Put $x_1 = \frac{\sqrt{3}}{2}$, $y_1 = \frac{1}{2}$ in (b)

=> $\frac{\sqrt{3}}{2}x - \frac{1}{2}y = 0$

=> $\sqrt{3}\,x - y = 0$ is the required equation of normal .

Q.3: For what value of n ,

(a) The line $lx + my + n = 0$ touches the circle $x^2 + y^2 = a^2$

(b) The line $x + y + n = 0$ touches the circle $x^2 + y^2 = 9$

(c) The line $2x + 2y + n = 0$ touches the circle $x^2 + y^2 = 81$

Solution: (a) $lx+my+n=0$ ....(i) and $x^2+y^2=a^2$....(ii)

As we know that the condition which the line $y=mx+c$ ..(a)

touch the circle $x^2+y^2=a^2$ ....(b)

if $c=\pm a\sqrt{1+m^2}$ ....(iii)

From (i) => $my=-lx-n$

=> $y=-\frac{l}{m}x-\frac{n}{m}$ .....(iv)

Then the slope of line (i) is $m=-\frac{l}{m}$.

Also from (iv) and (a) => $c=-\frac{n}{m}$

And from (ii) => $a^2=a^2$

Put $m=-\frac{l}{m}$, $c=-\frac{n}{m}$ and $a=a$ in (iii)

$-\frac{n}{m}=\pm a\sqrt{1+\left(-\frac{l}{m}\right)^2}$ => $-\frac{n}{m}=\pm a\sqrt{1+\frac{l^2}{m^2}}$,

=> $\left(-\frac{n}{m}\right)^2=\left(\pm a\sqrt{\frac{m^2+l^2}{m^2}}\right)^2$

=> $\frac{n^2}{m^2}=a^2\left(\frac{m^2+l^2}{m^2}\right)$ both sides multiplied by m^2

=> $n^2=m^2+l^2$

(b) $x+y+n=0$ .....(i) and $x^2+y^2=9$......(ii)

As we know that the condition which the line $y=mx+c$ ..(a)

touch the circle $x^2+y^2=a^2$ ....(b)

if $c=\pm a\sqrt{1+m^2}$ ....(iii)

From (i) => $y=-x-n$ ...(iv)

, then the slope of line (i) is $m=-1$.

Also from (iv) and (a) $\Rightarrow c = -n$

And from (ii) $\Rightarrow a^2 = 9 \Rightarrow a = 3$

Put $m = -1$, $c = -n$ and $a = 3$ in (iii)

$-n = \pm(3)\sqrt{1+(-1)^2} \Rightarrow -n = \pm 3\sqrt{2}$,

$\Rightarrow (-n)^2 = (\pm 3\sqrt{2})^2$

$\Rightarrow n^2 = 18$ *Ans.*

(c) $2x + 2y + n = 0$ ....(i) and $x^2 + y^2 = 81$ .....(ii)

As we know that the condition which the line $y = mx + c$ ...(a) touch the circle $x^2 + y^2 = a^2$ ....(b)

if $c = \pm a\sqrt{1+m^2}$ ....(iii)

From (i) $\Rightarrow 2y = -2x - n$ divided by 2

$\Rightarrow y = -x - \frac{n}{2}$ ....(iv),

then the slope of line (i) is $m = -1$.

Also from (i) and (a) $\Rightarrow c = -\frac{n}{2}$

And from (ii) $\Rightarrow a^2 = 81 \Rightarrow a = 9$

Put $m = -1$, $c = -\frac{n}{2}$ and $a = 9$ in (iii)

$-\frac{n}{2} = \pm(9)\sqrt{1+(-1)^2} \Rightarrow -n = \pm 18\sqrt{2}$

$\Rightarrow (-n)^2 = (\pm 18\sqrt{2})^2 = (324)(2) = 648$

$\Rightarrow n^2 = 648$ *Ans.*

Q.4: For what value of c

(a) The line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$.

Solution: $y = mx + c$ ....(i) and $x^2 + y^2 = a^2$ ......(ii)

Put (i) in (ii)

$x^2 + (mx + c)^2 = a^2 \Rightarrow x^2 + m^2x^2 + c^2 + 2mcx = a^2$

$\Rightarrow (1+m^2)x^2 + 2mcx + c^2 - a^2 = 0$ ......(iii)

Since eq(iii) is the quadratic equation of x . Then x has two roots . Now line (i) touch the circle (ii) only if x has equal roots (values) .
we know that the roots of quadratic equation are equal if

$b^2 - 4ac = 0$ .....(iv)

From (ii) $a = 1 + m^2$, $b = 2mc$, $c = c^2 - a^2$ put in (iv)

$(2mc)^2 - 4(1+m^2)(c^2 - a^2) = 0$

=> $4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4a^2m^2 = 0$

=> $-4c^2 + 4a^2 + 4a^2m^2 = 0$

=> $-c^2 = -a^2 - a^2m^2$ => $c^2 = a^2(1+m^2)$

=> $c = \pm a\sqrt{1+m^2}$ is the required the required value of c .

(b) The line $y = -x + c$ touches the circle $x^2 + y^2 = 9$.

<u>Solution</u>: $y = -x + c$ ....(i) and $x^2 + y^2 = 9$......(ii)

Put (i) in (ii)

$x^2 + (-x + c)^2 = 9$ => $x^2 + x^2 + c^2 - 2cx = 9$

=> $2x^2 - 2cx + c^2 - 9 = 0$ ......(iii)

Since eq(iii) is the quadratic equation of x . Then x has two roots . Now line (i) touch the circle (ii) only if x has equal roots (values) .
we know that the roots of quadratic equation are equal if

$b^2 - 4ac = 0$ .....(iv)

From (ii) $a = 2$, $b = -2c$, $c = c^2 - 9$ put in (iv)

$(-2c)^2 - 4(2)(c^2 - 9) = 0$

=> $4c^2 - 8c^2 + 72 = 0$

=> $-4c^2 + 72 = 0$

=> $-4c^2 = -72$ => $c^2 = 18$

=> $c = \pm\sqrt{18} = \pm 3\sqrt{2}$ is the required the required value of c .

(c) The line $y = -x - \frac{c}{2}$ touches the circle $x^2 + y^2 = 81$.

Solution: $y=-x-\frac{c}{2}$ ....(i) and $x^2+y^2=81$ ....(ii)

Put (i) in (ii)

$$x^2+\left(-x-\frac{c}{2}\right)^2=81 \Rightarrow x^2+x^2+\frac{c^2}{4}+cx=81$$

$$\Rightarrow 2x^2+cx+\frac{c^2}{4}-81=0 \quad \textit{multiplied by } 4$$

$$\Rightarrow 8x^2+4cx+c^2-324=0 \quad .....(iii)$$

Since eq(iii) is the quadratic equation of x. Then x has two roots. Now line (i) touch the circle (ii) only if x has equal roots (values).

we know that the roots of quadratic equation are equal if

$b^2-4ac=0$ .....(iv)

From (ii) $a=8$, $b=4c$, $c=c^2-324$ put in (iv)

$$(4c)^2-4(8)(c^2-324)=0$$

$$\Rightarrow 16c^2-32c^2+10368=0$$

$$\Rightarrow -16c^2=-10368 \Rightarrow c^2=\frac{10368}{16}$$

$$\Rightarrow c^2=648 \Rightarrow c=\pm\sqrt{648}=\pm\sqrt{324\times 2}$$

$\Rightarrow c=\pm 18\sqrt{2}$ is the required the required value of c.

Q.5: Find the condition at which the line $lx+my+n=0$ touches the circle $x^2+y^2+2gx+2fy+c=0$

Solution: $lx+my+n=0$ .....(i)

$$x^2+y^2+2gx+2fy+c=0 \quad ....(ii)$$

From (i) $\Rightarrow my=-lx-n$

$$\Rightarrow y=-\frac{lx+n}{m} \quad ......(iii)$$

Put (iii) in (ii), we get

$$x^2+\left(-\frac{lx+n}{m}\right)^2+2gx+(2f)\left(-\frac{lx+n}{m}\right)+c=0$$

$$=> x^2+\frac{l^2x^2+n^2+2nlx}{m^2}+2gx-\frac{2fl}{m}x-\frac{2fn}{m}+c=0$$

$$=> x^2+\frac{l^2}{m^2}x^2+\frac{n^2}{m^2}+\frac{2nl}{m^2}x+2gx-\frac{2fl}{m}x-\frac{2fn}{m}+c=0$$

Multiplied by m^2

$$m^2x^2+l^2x^2+n^2+2nl\,x+2gm^2x-2f\,lm\,x-2fnm+m^2c=0$$

$$(m^2+l^2)x^2+2(nl-flm+gm^2)x+(n^2-2fnm+m^2c)=0 \ldots\text{(iv)}$$

Since eq(iv) is the quadratic equation of x. Then x has two roots. Now line (i) touch the circle (ii) only if x has equal roots (values).

we know that the roots of quadratic equation are equal if

$$b^2-4ac=0 \ \ldots\ldots\text{(v)}$$

Here $a=m^2+l^2$, $b=2(nl-flm+gm^2)$ **and**

$c=n^2-2fmn+m^2c$ **in (v)**

$$=> [2(nl-flm+gm^2)]^2-4(m^2+l^2)(n^2-2fmn+m^2c)=0$$

$$=> 4(n^2l^2+f^2l^2m^2+g^2m^4-2fl^2nm-2fglm^3+2g\ln m^2)$$
$$-4(m^2n^2-2fm^3n+m^4c+l^2n^2-2fmnl^2+l^2m^2c)=0$$

$$=> 4\begin{pmatrix} n^2l^2+f^2l^2m^2+g^2m^4-2fl^2nm-2fglm^3+2g\ln m^2 \\ -m^2n^2+2fm^3n-m^4c-l^2n^2+2fmnl^2-l^2m^2c \end{pmatrix}=0$$

$$=> 4\begin{pmatrix} f^2l^2m^2+g^2m^4-2fglm^3+2g\ln m^2 \\ -m^2n^2+2fm^3n-m^4c-l^2m^2c \end{pmatrix}=0$$

$$=> 4m^2\begin{pmatrix} f^2l^2+g^2m^2-2fglm+2g\ln \\ -n^2+2fmn-m^2c-l^2c \end{pmatrix}=0$$

$$=> f^2l^2+g^2m^2-2fglm+2g\ln-n^2+2fmn-m^2c-l^2c=0$$

$$=> -cl^2+f^2l^2-2fglm-m^2c+g^2m^2+2g\ln+2fmn-n^2=0$$

Multiplied by - 1

$$=> cl^2-f^2l^2+2fglm+m^2c-g^2m-2g\ln-2fmn+n^2=0$$

=> $(c-f^2)l^2+2gflm+(c-g^2)m^2-2n(gl+fm)+n^2=0$ ...(A)

Eq (A) is the required condition.

Q.6: For what value of n,

(a) The line $3x+4y+n=0$ touches the circle
$$x^2+y^2-4x-6y-12=0$$

(b) The line $x-2y+n=0$ touches the circle
$$x^2+y^2+3x+6y-5=0$$

(c) The line $2x+y+n=0$ touches the circle
$$x^2+y^2-2x-10y+21=0$$

Solution: (a) $3x+4y+n=0$ .....(i) and
$$x^2+y^2-4x-6y-12=0 \quad ....(ii)$$

From (i) => $4y=-3x-n$

=> $y=-\dfrac{3x+n}{4}$ ......(iii)

Put (iii) in (ii)
$$x^2+\left(-\frac{3x+n}{4}\right)^2-4x-6\left(-\frac{3x+n}{4}\right)-12=0$$

=> $x^2+\dfrac{9x^2+6nx+n^2}{16}-4x+\dfrac{18x}{4}+\dfrac{6n}{4}-12=0$

Multiplied by 16
$$16x^2+9x^2+6nx+n^2-64x+72x+24n-192=0$$

=> $25x^2+(6n+8)x+(n^2+24n-192)=0$ .....(iv)

It is quadratic equation of x,

Now line (i) touch the circle (ii) only if x has equal roots (values).

we know that the roots of quadratic equation are equal if
$$b^2-4ac=0 \quad .....(v)$$

Put $a=25$, $b=6n+8$, $c=n^2+24n-192$ in (v)

$[2(3n+4)]^2-4(25)(n^2+24n-192)=0$

=> $4(9n^2+16+24n)-4(25n^2+600n-4800)=0$

=> $4(9n^2+16+24n-25n^2-600n+4800)=0$

=> $-16n^2-576n+4816=0$ **divided by - 16**

=> $n^2+36n-301=0$

=> $n^2-7n+43n-301=0$

=> $n(n-7)+43(n-7)=0$

=> $(n-7)(n+43)=0$

=> $n-7=0$ *and* $n+43=0$

=> $n=7$ *and* $n=-43$ **Ans.**

(b) $x-2y+n=0$ **.....(i) and** $x^2+y^2+3x+6y-5=0$ **....(ii)**

From (i) => $-2y=-x-n$

=> $y=\dfrac{x+n}{2}$ **.....(iii)**

Put (iii) in (ii) .

$$x^2+\left(\frac{x+n}{2}\right)^2+3x+6\left(\frac{x+n}{2}\right)-5=0$$

=> $x^2+\dfrac{x^2+2xn+n^2}{4}+3x+3x+3n-5=0$

Multiplied by 4 .

=> $4x^2+x^2+2xn+n^2+12x+12x+12n-20=0$

=> $5x^2+2(n+12)x+n^2+12n-20=0$ ......*(iv)*

It is quadratic equation of x , then

$b^2-4ac=0$ **......(v)**

Put $a=5$, $b=2(n+12)$, $c=n^2+12n-20$ **in (v)**

$(2(n+12))^2-4(5)(n^2+12n-20)=0$

=> $4(n^2+24n+144)-4(5)(n^2+12n-20)=0$

=> $4(n^2+24n+144-5n^2-60n+100)=0$

$=> \; -4n^2 - 36n + 244 = 0$ divided by - 4

$=> \; n^2 + 9n - 61 = 0$

Then by quadratic formula $n = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$=> \; n = \dfrac{-9 \pm \sqrt{(9)^2 - 4(1)(-61)}}{2(1)} = \dfrac{-9 \pm \sqrt{81 + 244}}{2}$

$=> \; n = \dfrac{-9 \pm \sqrt{325}}{2} = \dfrac{-9 \pm 18.02}{2}$

$=> n = \dfrac{-9 + 18.02}{2}, \quad n = \dfrac{-9 - 18.02}{2}$

$=> n = \dfrac{9.02}{2} = 4.52 \quad and \quad n = -\dfrac{27.02}{2} = -13.52$

Thus $n = 4.52 \; , \; -13.52$ **Ans.**

(c) $2x + y + n = 0$ **.....(i) and**

$x^2 + y^2 - 2x - 10y + 21 = 0$ **.......(ii)**

From (i) $=> y = -2x - n$ **......(iii)**

Put (iii) in (ii)

$x^2 + (-2x - n)^2 - 2x - 10(-2x - n) + 21 = 0$

$=> \; x^2 + 4x^2 + n^2 + 4nx - 2x + 20x + 10n + 21 = 0$

$=> 5x^2 + (4n + 18)x + (n^2 + 10n + 21) = 0 \; .....(iv)$

It is quadratic equation of x ,

Now line (i) touch the circle (ii) only if x **has equal roots (values) .**

we know that the roots of quadratic equation are equal if

$b^2 - 4ac = 0$ **.....(v)**

Put $a = 5 \; , \; b = 4n + 18 = 2(2n + 9) \; , \; c = n^2 + 10n + 21$ **in (v)**

$[2(2n + 9)]^2 - 4(5)(n^2 + 10n + 21) = 0$

$=> 4(4n^2 + 81 + 36n) - 4(5n^2 + 50n + 105) = 0$

$=> 4(4n^2 + 81 + 36n - 5n^2 - 50n - 105) = 0 \;$ *divided by* 4

$$=> -n^2 - 14n - 24 = 0$$
$$=> -(n^2 + 14n + 24) = 0$$
$$=> n^2 + 14n + 24 = 0$$
$$=> n^2 + 2n + 12n + 24 = 0$$
$$=> n(n+2) + 12(n+2) = 0$$
$$=> (n+2)(n+12) = 0$$
$$=> n+2 = 0 \quad or \quad n+12 = 0$$
$$=> n = -2 \quad or \quad n = -12$$

Q.7: If the tangent length from the point P to the circle $x^2 + y^2 = a^2$ is equal to the perpendicular distance from p to the line $lx + my + n = 0$, then find the locus of P.

Solution: Let $P(x, y)$ be a point on the locus.

Circle equation is $x^2 + y^2 = a^2$ ......(i)

And line is $lx + my + n = 0$ ...(ii)

The length of tangent drawn from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is

$|PT| = \sqrt{x_1^2 + y_1^2 - a^2}$ ....(iii), Since the point is $P(x, y)$.

Put $x_1 = x$, $y_1 = y$, $a^2 = a^2$ in (iii)

Length of tangent $= \sqrt{x^2 + y^2 - a^2}$ .....(a)

Now perpendicular distance of point $P(x, y)$ from line (ii) is

$$d = \frac{|lx + my + n|}{\sqrt{l^2 + m^2}} \text{.....}(b)$$

Given that the length of tangent = perpendicular distance d

$$=> \sqrt{x^2 + y^2 - a^2} = \frac{|lx + my + n|}{\sqrt{l^2 + m^2}} \quad \textit{sqauring both sides}$$

$$=> x^2 + y^2 - a^2 = \frac{(lx + my + n)^2}{l^2 + m^2}$$

$$=> (l^2 + m^2)(x^2 + y^2 - a^2) = (lx + my + n)^2 \quad \textbf{Ans.}$$

Q.8: (a) Find the locus of the point P,

(a) if the length of the tangent line from the point P to the circle $x^2 + y^2 = 9$ is equal to the perpendicular distance from P to the line $3x + 4y + 3 = 0$.

(b) If the length of the tangent line from the point P to the circle $x^2 + y^2 = 25$ is equal to the perpendicular distance from P to the line $4x + 3y + 3 = 0$.

Solution: (a) Let $P(x\ ,\ y)$ be a point on the locus .

Circle equation is $x^2 + y^2 = 9$ ......(i)

And line is $3x + 4y + 3 = 0$ ...(ii)

The length of tangent drawn from the point $P(x_1\ ,\ y_1)$ to the circle $x^2 + y^2 = a^2$ is

$|PT| = \sqrt{x_1^2 + y_1^2 - a^2}$ ....(iii) , Since the point is $P(x\ ,\ y)$.

Put $x_1 = x\ ,\ y_1 = y\ ,\ a^2 = 9$ in (iii)

Length of tangent $= \sqrt{x^2 + y^2 - 9}$ .....(a)

Now perpendicular distance of point $P(x\ ,\ y)$ from line (ii) is

$$d = \frac{|3x + 4y + 3|}{\sqrt{(3)^2 + (4)^2}} = \frac{|3x + 4y + 3|}{\sqrt{9 + 16}} = \frac{|3x + 4y + 3|}{5} \quad .....(b)$$

Given that the length of tangent = perpendicular distance d

$$\Rightarrow \sqrt{x^2 + y^2 - 9} = \frac{|3x + 4y + 3|}{5} \quad \textit{sqauring both sides}$$

$$\Rightarrow x^2 + y^2 - 9 = \frac{(3x + 4y + 3)^2}{25}$$

$$\Rightarrow 25(x^2 + y^2 - 9) = (3x + 4y + 3)^2 \quad \text{Ans.}$$

(b) Let $P(x\ ,\ y)$ be a point on the locus .

Circle equation is $x^2 + y^2 = 25$ ......(i)

And line is $4x + 3y + 3 = 0$ ...(ii)

The length of tangent drawn from the point $P(x_1\ ,\ y_1)$ to the circle $x^2+y^2=a^2$ is

$|PT|=\sqrt{x_1^2+y_1^2-a^2}$ ....(iii) , Since the point is $P(x\ ,\ y)$.

Put $x_1=x\ ,\ y_1=y\ ,\ a^2=25$ in (iii)

Length of tangent $=\sqrt{x^2+y^2-25}$ .....(a)

Now perpendicular distance of point $P(x\ ,\ y)$ from line (ii) is

$$d=\frac{|4x+3y+3|}{\sqrt{(3)^2+(4)^2}}=\frac{|4x+3y+3|}{\sqrt{9+16}}=\frac{|4x+3y+3|}{5} \quad .....(b)$$

Given that the length of tangent = perpendicular distance d

$$\Rightarrow \sqrt{x^2+y^2-25}=\frac{|4x+3y+3|}{5} \quad \textit{sqauring both sides}$$

$$\Rightarrow x^2+y^2-25=\frac{(4x+3y+3)^2}{25}$$

$$\Rightarrow 25(x^2+y^2-25)=(4x+3y+3)^2 \quad \textbf{Ans.}$$

Q.9: (a) The length of the tangent from (f , g) to the circle $x^2+y^2=6$ is twice the length of the tangent to the circle $x^2+y^2+3x+3y=0$. Prove that $f^2+g^2+4f+4g+2=0$

(b) The length of the tangent from (f , g) to the circle $x^2+y^2=4$ is 4 times the length of the tangent to the circle $x^2+y^2+2x+2y=0$. Prove that

$$15f^2+15g^2+32f+32g+4=0$$

Solution: (a) $x^2+y^2=6$ ....(i)

$x^2+y^2+3x+3y=0$ .....(ii)

The length of tangent from circle $x^2+y^2=6$:

As the length of tangent drawn from the point $P(x_1\ ,\ y_1)$ to the circle $x^2+y^2=a^2$ is

$|PT|=\sqrt{x_1^2+y_1^2-a^2}$ ....(iii) , Since the point is $P(f\ ,\ g)$.

Put $x_1=f\ ,\ y_1=g\ ,\ a^2=6$ in (iii)

Length of tangent $=\sqrt{f^2+g^2-6}$ .....(a)

The length of tangent from circle $x^2 + y^2 + 3x + 3y = 0$ **:**

Now the length of tangent drawn from the point $P(x_1, y_1)$ **to the circle** $x^2 + y^2 + 2gx + 2fy + c = 0$ **is**

$|PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ **....(iv) ,**

Since the point is $P(f, g)$.

Put $x_1 = f$, $y_1 = g$, $2g = 3$, $2f = 3$, $c = 0$ **in (iv)**

Length of tangent $= \sqrt{f^2 + g^2 + 3f + 3g}$ **.....(b)**

According to the given condition

$\sqrt{f^2 + g^2 - 6} = 2\left(\sqrt{f^2 + g^2 + 3f + 3g}\right)$ **squaring both sides**

$\Rightarrow f^2 + g^2 - 6 = 4(f^2 + g^2 + 3f + 3g)$

$\Rightarrow 0 = 4f^2 + 4g^2 + 12f + 12g - f^2 - g^2 + 6$

$\Rightarrow 3f^2 + 3g^2 + 12f + 12g + 6 = 0$ *divided by* 3

$\Rightarrow f^2 + g^2 + 4f + 4g + 2 = 0$ *proved.*

(b) $x^2 + y^2 = 4$ **....(i)**

$x^2 + y^2 + 2x + 2y = 0$ **.....(ii)**

The length of tangent from circle $x^2 + y^2 = 4$ **:**

As the length of tangent drawn from the point $P(x_1, y_1)$ **to the circle** $x^2 + y^2 = a^2$ **is**

$|PT| = \sqrt{x_1^2 + y_1^2 - a^2}$ **....(iii) , Since the point is** $P(f, g)$.

Put $x_1 = f$, $y_1 = g$, $a^2 = 4$ **in (iii)**

Length of tangent $= \sqrt{f^2 + g^2 - 4}$ **.....(a)**

The length of tangent from circle $x^2 + y^2 + 2x + 2y = 0$ **:**

Now the length of tangent drawn from the point $P(x_1, y_1)$ **to the circle** $x^2 + y^2 + 2gx + 2fy + c = 0$ **is**

$|PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ **....(iv) ,** Since the point is $P(f\ ,\ g)$.

Put $x_1 = f\ ,\ y_1 = g\ ,\ 2g = 2\ ,\ 2f = 2\ ,\ c = 0$ in (iv)

Length of tangent = $\sqrt{f^2 + g^2 + 2f + 2g}$ .....(b)

According to the given condition

$\sqrt{f^2 + g^2 - 4} = 4\left(\sqrt{f^2 + g^2 + 2f + 2g}\right)$ squaring both sides

$\Rightarrow f^2 + g^2 - 4 = 16\left(f^2 + g^2 + 2f + 2g\right)$

$\Rightarrow 0 = 16f^2 + 16g^2 + 32f + 32g - f^2 - g^2 + 4$

$\Rightarrow 15f^2 + 15g^2 + 32f + 32g + 4 = 0$ *proved.*

Q.10: Find the equations of two tangents to the

(b) circle $x^2 + y^2 = 4$, which are parallel to the straight line $x + 2y + 3 = 0$.

(b) circle $x^2 + y^2 = 25$, which are parallel to the straight line $3x + 4y + 3 = 0$.

> **Note :** Two lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ are parallel lines because both lines have same slopes .

Solution: (a) $x^2 + y^2 = 4$ .....(i)

$x + 2y + 3 = 0$ .....(ii)

Now the line which is parallel to (ii) is

$x + 2y + t = 0$ .....(iii)

Centre of circle (i) is $C(0\ ,\ 0)$.

As the line (iii) is tangent to the circle (i) , then the perpendicular distance d from the centre $C(0\ ,\ 0)$ is

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \quad \text{........(iv)} ,$$

Put $a = 1\ ,\ b = 2\ ,\ c = t\ ,\ x_1 = 0\ ,\ y_1 = 0$ in (iv)

$$d = \frac{|(1)(0)+(2)(0)+t|}{\sqrt{(1)^2+(2)^2}} = \frac{|t|}{\sqrt{1+4}} = \frac{|t|}{\sqrt{5}}$$

$$\Rightarrow \quad d = \pm\frac{t}{\sqrt{5}}$$

But the perpendicular distance from the centre to the tangent line is equal to radius of the circle.

From (i) the radius of the circle is $r = 2$

Then $d = r \quad \Rightarrow \quad \pm\frac{t}{\sqrt{5}} = 2 \quad or \quad t = \pm 2\sqrt{5}$

Put $t = 2\sqrt{5}$ *and* $t = -2\sqrt{5}$ **in (iii)**

$$x + 2y + 2\sqrt{5} = 0 \ldots.(a) \qquad and \qquad x + 2y - 2\sqrt{5} = 0 \ldots..(b)$$

Eq (a) and (b) are the required tangent lines.

(b) $x^2 + y^2 = 25$ **.....(i)**

$$3x + 4y + 3 = 0 \quad \text{.....(ii)}$$

Now the line which is parallel to (ii) is

$$3x + 4y + t = 0 \quad \text{.....(iii)}$$

Centre of circle (i) is $C(0\ ,\ 0)$.

As the line (iii) is tangent to the circle (i), then the perpendicular distance d from the centre $C(0\ ,\ 0)$ **is**

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} \quad \text{........(iv)},$$

Put $a = 3$, $b = 4$, $c = t$, $x_1 = 0$, $y_1 = 0$ **in (iv)**

$$d = \frac{|(3)(0)+(4)(0)+t|}{\sqrt{(3)^2+(4)^2}} = \frac{|t|}{\sqrt{9+16}} = \frac{|t|}{\sqrt{25}}$$

$$\Rightarrow \quad d = \pm\frac{t}{5}$$

But the perpendicular distance from the centre to the tangent line is equal to radius of the circle.

From (i) the radius of the circle is $r = 5$

Then $d = r \Rightarrow \pm\frac{t}{5} = 5$ or $t = \pm 25$

Put $t = 25$ and $t = -25$ in (iii)

$3x + 4y + 25 = 0 ....(a)$ and $3x + 4y - 25 = 0 .....(b)$

Eq (a) and (b) are the required tangent lines.

Q.11:(a) Prove that the lines $x = 8$ and $y = 7$ touch the circle $x^2 + y^2 - 6x - 4y - 12 = 0$. Find also the contact point.

Solution: Given $x = 8 \Rightarrow x - 8 = 0$ .....(i)

$y = 7 \Rightarrow y - 7 = 0$ .....(ii)

And circle $x^2 + y^2 - 6x - 4y - 12 = 0$ .....(iii)

We know that a line touch a circle only if the perpendicular distance of the line from the centre is equal to the radius of the circle.

first we find the centre of the given circle (iii)

$2g = -6 \Rightarrow g = -3$

$2f = -4 \Rightarrow f = -2$ and $c = -12$

Now centre of the circle $C(-g, -f) = C(3, 2)$.

Radius of the circle is $r = \sqrt{g^2 + f^2 - c}$

$\Rightarrow r = \sqrt{(-3)^2 + (-2)^2 - (-12)} = \sqrt{9 + 4 + 12} = \sqrt{25}$

$\Rightarrow r = 5$ ......(a)

Now the perpendicular distance of the line $x + 0.y - 8 = 0$

From centre $C(3, 2)$ is

$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$, put $a = 1, b = 0, c = -8, x_1 = 3, y_1 = 2$

$\Rightarrow d = \frac{|(1)(3) + (0)(2) - 8|}{\sqrt{(1)^2 + (0)^2}} = \frac{|3 - 8|}{\sqrt{1}} = \frac{|-5|}{1} = 5$ .....(b)

From (a) and (b) $d = r$.

Hence the line (i) touch the circle.

Now we find the points of contact (i) and (iii):

put $x = 8$ in (iii)

$$(8)^2 + y^2 - 6(8) - 4y - 12 = 0$$

$$\Rightarrow 64 + y^2 - 48 - 4y - 12 = 0$$

$$\Rightarrow y^2 - 4y + 4 = 0$$

$$\Rightarrow (y-2)^2 = 0 \quad \Rightarrow \quad y - 2 = 0$$

$$\Rightarrow \quad y = 2$$

Thus the point of contact of (i) and (iii) is $(8\ ,\ 2)$.

Now the perpendicular distance of the line $0.x + y - 7 = 0$

From centre $C(3\ ,\ 2)$ **is**

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$ **, put** $a = 0,\ b = 1\ ,\ c = -7\ ,\ x_1 = 3\ ,\ y_1 = 2$

$$\Rightarrow d = \frac{|(0)(3) + (1)(2) - 7|}{\sqrt{(1)^2 + (0)^2}} = \frac{|2-7|}{\sqrt{1}} = \frac{|-5|}{1} = 5 \quad .....(b)$$

From (a) and (b) $d = r$ **.**

Hence the line (ii) touch the circle .

Now we find the points of contact (ii) and (iii):

Put $y = 7$ **in (iii)**

$$x^2 + (7)^2 - 6x - 4(7) - 12 = 0$$

$$\Rightarrow x^2 + 49 - 6x - 28 - 12 = 0$$

$$\Rightarrow x^2 - 6x + 9 = 0 \quad \Rightarrow \quad (x-3)^2 = 0$$

$$\Rightarrow x - 3 = 0 \quad \Rightarrow \quad x = 3$$

Thus the point of contact of (ii) and (iii) is $(3\ ,\ 7)$.

(b) $x + y - 1 = 0$ *and* $x - y + 1 = 0$ **touch the circle** $x^2 + y^2 - 4x - 2y + 3 = 0$ **. Find also the contact point .**

<u>Solution</u> : Given $x + y - 1 = 0 \quad .....(i)$

$$x - y + 1 = 0 \quad .....(ii)$$

And circle $x^2 + y^2 - 4x - 2y + 3 = 0 \quad .....(iii)$

We know that a line touch a circle only if the perpendicular distance of the line from the centre is equal to the radius of the circle .

first we find the centre of the given circle (iii)

$2g=-4 \Rightarrow g=-2$

$2f=-2 \Rightarrow f=-1$ and $c=3$

Now centre of the circle $C(-g\ ,\ -f)=C(2,\ 1)$.

Radius of the circle is $r=\sqrt{g^2+f^2-c}$

$\Rightarrow r=\sqrt{(-2)^2+(-1)^2-3}=\sqrt{4+1-3}$

$\Rightarrow r=\sqrt{2}$ ......(a)

Now the perpendicular distance of the line $x+y-1=0$

From centre $C(2\ ,\ 1)$ is

$d=\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$, put $a=1,\ b=1\ ,\ c=-1\ ,\ x_1=2\ ,\ y_1=1$

$\Rightarrow d=\dfrac{|(1)(2)+(1)(1)-1|}{\sqrt{(1)^2+(1)^2}}=\dfrac{|2+1-1|}{\sqrt{2}}=\dfrac{|2|}{\sqrt{2}}=\dfrac{2}{\sqrt{2}}=\sqrt{2}$ .....(b)

From (a) and (b) $d=r$.

Hence the line (i) touch the circle .

Now we find the points of contact of (i) and (iii) :

From (i) $\Rightarrow y=-x+1$ ......(iv) put (iv) in (iii)

$x^2+(1-x)^2-4x-2(1-x)+3=0$

$\Rightarrow x^2+1+x^2-2x-4x-2+2x+3=0$

$\Rightarrow 2x^2-4x+2=0 \Rightarrow x^2-2x+1=0$

$\Rightarrow (x-1)^2=0 \Rightarrow x-1=0$

$\Rightarrow x=1$

Put $x=1$ in (iv) $\Rightarrow y=-1+1=0$

Thus the point of contact of (i) and (iii) is $(1\ ,\ 0)$.

Now the perpendicular distance of the line $x-y+1=0$

from centre $C(2\ ,\ 1)$ is

$d=\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$, put $a=1,\ b=-1\ ,\ c=1\ ,\ x_1=2\ ,\ y_1=1$

$$=> d = \frac{|(1)(2)+(-1)(1)+1|}{\sqrt{(1)^2+(-1)^2}} = \frac{|2-1+1|}{\sqrt{2}} = \frac{|2|}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \quad .....(b)$$

From (a) and (b) $d = r$.

Hence the line (ii) touch the circle .

Now we find the points of contact of (ii) and (iii) :

From (ii) => $x = y - 1$ .....(v)

Put (v) in (iii)

$$(y-1)^2 + y^2 - 4(y-1) - 2y + 3 = 0$$

=> $y^2 - 2y + 1 + y^2 - 4y + 4 - 2y + 3 = 0$

=> $2y^2 - 8y + 8 = 0$

=> $y^2 - 4y + 4 = 0$ => $(y-2)^2 = 0$

=> $y - 2 = 0$ => $y = 2$

Put $y = 2$ in (v) => $x = 2 - 1 = 1$

Thus the point of contact of (ii) and (iii) is $(1 , 2)$.

Q.12: Find the equations of the tangents

(a) to the circle $x^2 + y^2 = 2$, which make an angle 45^o with x –axis .

(b) to the circle $3x^2 + 3y^2 = 1$, which make an angle 30^o with x –axis .

(c) to the circle $x^2 + y^2 = 4$, which make an angle 60^o with x –axis .

Solution: (a) $x^2 + y^2 = 2$ .....(i)

As we know that the equation of tangent line is

$$y = mx \pm a\sqrt{1+m^2} \quad ......(ii)$$

Since the tangent line makes an angle 45^o with x –axis , then the slope of tangent line is

$$m = \tan 45^o = 1$$

and from (i) => $a^2 = 2$ => $a = \sqrt{2}$

put values in (ii)

$$y = x \pm \sqrt{2}\left(\sqrt{1+(1)^2}\right) = x \pm \left(\sqrt{2}\right)\left(\sqrt{2}\right)$$

$\Rightarrow \quad y = x \pm 2 \quad$ *Ans.*

(b) $3x^2 + 3y^2 = 1 \quad \Rightarrow \quad x^2 + y^2 = \frac{1}{3}$ .....(i)

As we know that the equation of tangent line is

$$y = mx \pm a\sqrt{1+m^2} \quad \text{......(ii)}$$

Since the tangent line makes an angle 30^o **with x –axis , then the slope of tangent line is**

$$m = \tan 30^o = \frac{1}{\sqrt{3}}$$

and from (i) $\Rightarrow \quad a^2 = \frac{1}{3} \quad \Rightarrow \quad a = \frac{1}{\sqrt{3}}$

put values in (ii)

$$y = \frac{1}{\sqrt{3}}x \pm \frac{1}{\sqrt{3}}\left(\sqrt{1+\left(\frac{1}{\sqrt{3}}\right)^2}\right) = \frac{1}{\sqrt{3}}x \pm \frac{1}{\sqrt{3}}\left(\sqrt{1+\frac{1}{3}}\right)$$

$$\Rightarrow \quad y = \frac{1}{\sqrt{3}}x \pm \frac{1}{\sqrt{3}}\left(\sqrt{\frac{4}{3}}\right)$$

$$\Rightarrow \quad y = \frac{1}{\sqrt{3}}x \pm \frac{2}{3} \quad \textbf{Ans.}$$

(c) $x^2 + y^2 = 4$ .....(i)

As we know that the equation of tangent line is

$$y = mx \pm a\sqrt{1+m^2} \quad \text{......(ii)}$$

Since the tangent line makes an angle 60^o **with x –axis , then the slope of tangent line is**

$$m = \tan 60^o = \sqrt{3}$$

and from (i) $\Rightarrow \quad a^2 = 4 \quad \Rightarrow \quad a = 2$

put values in (ii)

$$y=\sqrt{3}\ x\pm 2\left(\sqrt{1+\left(\sqrt{3}\right)^2}\right)=\sqrt{3}x\pm 2\left(\sqrt{4}\right)$$

$\Rightarrow\ y=\sqrt{3}\ x\pm 4 \quad Ans.$

Example .15: If $A(-3,\ 4)$ and $B(1\ ,\ 5)$ are the end points of The chord AB of the circle $x^2+y^2+x-5y-2=0$, then show that

(a) The line from the centre of the circle is perpendicular to AB , also bisects the chord AB .

(b) The line from the centre of the circle to the mid point of the chord AB is perpendicular to the chord AB .

(c) The perpendicular bisector CD of the chord AB passes through the centre of the given circle .

Solution: $x^2+y^2+x-5y-2=0$ .....(i)

The general equation of the circle is

$x^2+y^2+2gx+2fy+c=0$ ....(ii)

Comparing (i) and (ii) , then

$2g=1 \Rightarrow g=\frac{1}{2}$

$2f=-5 \Rightarrow f=-\frac{5}{2}$

Then the centre of circle (i) is $C(-g\ ,\ -f)=C\left(-\frac{1}{2}\ ,\ \frac{5}{2}\right)$

Let $D(x\ ,\ y)$ **is the mid point of chord AB .**

Then $D(x\ ,y)=D\left(\frac{-3+1}{2}\ ,\ \frac{4+5}{2}\right)=D\left(-1\ ,\ \frac{9}{2}\right)$

Now slope of AB is $m_1=\frac{y_2-y_1}{x_2-x_1}=\frac{5-4}{1-(-3)}=\frac{1}{1+3}=\frac{1}{4}$

The slope of CD is $m_2=\frac{y_2-y_1}{x_2-x_1}=\frac{\frac{9}{2}-\frac{5}{2}}{-1-\left(-\frac{1}{2}\right)}$

$$=> \quad m_2 = \frac{\frac{9-5}{2}}{\frac{-2+1}{2}} = \frac{4}{2} \times \frac{2}{-1} = -4$$

Now $m_1 . m_2 = \left(\frac{1}{4}\right)(-4) = -1$

Hence CD is perpendicular to chord AB.
As CD is the line from the centre of the circle to the mid point D of the chord AB and perpendicular to AB.
Hence (a), (b) and (c) are proved.

Example 16: **Show that the chords AB and DE are equidistance from the centre $C(0\ ,\ 0)$ of the circle $x^2 + y^2 = 4$. The coordinates of the end points of the two chords are $A(0\ ,2)$, $B(-2\ ,0)$, $D(0\ ,-2)$ and $E(2\ ,\ 0)$.**

Solution: The given circle $x^2 + y^2 = 4$ .....(i), with centre is $C(0\ ,\ 0)$.

Now the equation of the chord AB is

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1) \text{ ....(a) put values}$$

$$=> \quad y - 2 = \frac{(0-2)}{(-2-0)}(x-0) \quad => \quad y - 2 = \frac{-2}{-2}x$$

$$=> \quad y - 2 = x \quad or \quad 0 = x - y + 2 \quad => x - y + 2 = 0 \text{ ....}(ii)$$

Now the perpendicular distance of the chord AB from the centre C(0 , 0) is

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \text{ .....(b)}$$

From (ii) $a = 1$, $b = -1$, $c = 2$ and $x_1 = 0$, $y_1 = 0$ put in (b)

$$d_1 = \frac{|(1)(0) + (-1)(0) + 2|}{\sqrt{(1)^2 + (-1)^2}} = \frac{|2|}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2} \text{ .....}(c)$$

Now equation of DE : As $D(0\ ,-2)$, $E(2\ ,\ 0)$ put in (a)

$$y-(-2)=\frac{(0-(-2))}{(2-0)}(x-0) \Rightarrow y+2=\frac{2}{2}x$$

$$\Rightarrow 0=x-y-2 \qquad \Rightarrow x-y-2=0 \quad ......(iii)$$

The perpendicular distance of the chord DE from centre C (0 , 0) : From (iii)

Put $a=1$, $b=-1$, $c=-2$ **and** $x_1=0$, $y_1=0$ **put in (b)**

$$d_2=\frac{|(1)(0)+(-1)(0)-2|}{\sqrt{(1)^2+(-1)^2}}=\frac{|-2|}{\sqrt{1+1}}=\frac{2}{\sqrt{2}}=\sqrt{2} \quad .....(d)$$

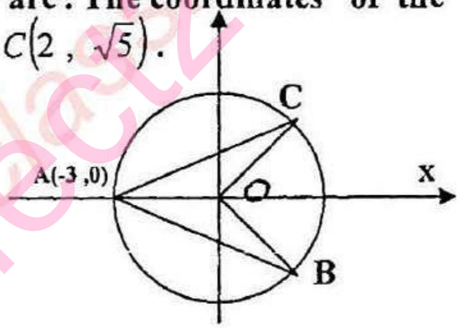
Hence from eq(c) and (d) both chords AB and CD have equal distance from the centre.

<u>Example 17</u>: Show that the angle subtended by the minor arc BC of the circle $x^2+y^2=9$ **is two times the angle subtended in the major arc. The coordinates of the minor arc are** $B(2\,,\,-\sqrt{5})$, $C(2\,,\,\sqrt{5})$.

<u>Solution</u> : $x^2+y^2=9$ **....(i)**

The centre of circle (i) is $O(0\,,\,0)$.

The end points of minor arc BC are



$B(2\,,\,-\sqrt{5})$, $C(2\,,\,\sqrt{5})$.

Note : Since $B(2\,,\,-\sqrt{5})$ **is a point on the circle and** $O(0\,,\,0)$ **is the centre of the circle. Then the radius of the circle is**

$$r=|OB|=\sqrt{(2-0)^2+(-\sqrt{5}-0)^2}=\sqrt{4+5}=\sqrt{9}=3$$

Also C lie in the first quadrant and B lie in 4^{th} **quadrant Then the major arc contains** 2^{nd} , 3^{rd} **and some parts of Ist and** 4^{th} **quadrant. Then take point A on major arc. Here we take A(- 3 , 0) because** $|OA|=3=r$

Let $A(-3\ ,\ 0)$ is a point on the major arc. Then join point A to B and C.

Let $<BAC = \theta$ is the angle subtended by major arc.

Then slope of BA is $m_1 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0+\sqrt{5}}{-3-2} = -\dfrac{\sqrt{5}}{5} = -\dfrac{1}{\sqrt{5}}$

Then slope of CA is $m_2 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-\sqrt{5}}{-3-2} = \dfrac{\sqrt{5}}{5} = \dfrac{1}{\sqrt{5}}$

Then $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2} = \dfrac{-\dfrac{1}{\sqrt{5}} - \dfrac{1}{\sqrt{5}}}{1+\left(-\dfrac{1}{\sqrt{5}}\right)\left(\dfrac{1}{\sqrt{5}}\right)} = \dfrac{-\dfrac{2}{\sqrt{5}}}{1-\dfrac{1}{5}} = \dfrac{-\dfrac{2}{\sqrt{5}}}{\dfrac{5-1}{5}}$

=> $\tan\theta = -\dfrac{2}{\sqrt{5}} \times \dfrac{5}{4} = -\dfrac{\sqrt{5}}{2}$ .....(a)

Let $<BOC = 2\theta$.

Then slope of BO is $m_3 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0+\sqrt{5}}{0-2} = -\dfrac{\sqrt{5}}{2}$

Then slope of CO is $m_4 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-\sqrt{5}}{0-2} = \dfrac{\sqrt{5}}{2}$

Then $\tan 2\theta = \dfrac{m_4 - m_3}{1 + m_3 m_4} = \dfrac{\dfrac{\sqrt{5}}{2} - \left(-\dfrac{\sqrt{5}}{2}\right)}{1+\left(-\dfrac{\sqrt{5}}{2}\right)\left(\dfrac{\sqrt{5}}{2}\right)} = \dfrac{\dfrac{2\sqrt{5}}{2}}{1-\dfrac{5}{4}} = \dfrac{\sqrt{5}}{\dfrac{4-5}{4}}$

=> $\tan 2\theta = -\dfrac{4\sqrt{5}}{1} = -4\sqrt{5}$ .....(b)

Since $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ ....(c). Put (a) in (c)

$$\tan 2\theta = \frac{2\left(-\frac{\sqrt{5}}{2}\right)}{1-\left(-\frac{\sqrt{5}}{2}\right)^2} = \frac{-\sqrt{5}}{1-\frac{5}{4}} = \frac{-\sqrt{5}}{\frac{4-5}{4}} = -4\sqrt{5} \text{ which is (b)}$$

Thus proving (a) with result (b).

Hence $<BOC = 2 < BAC$.

Example 18: **Show that the angle in the semicircle of the circle** $(x-h)^2 + y^2 = a^2$ **is right angle.**

Solution: **The circle equation is** $(x-h)^2 + y^2 = a^2$ **...(i)**

The centre of (i) is $C(h\ ,\ 0)$ **and radius of circle is** a.

Since centre on the x- axis then AB is the diameter of the semi circle on the x – axis. Let $P(x_1\ ,\ y_1)$ **is any point on the semi circle.**

Since $C(h\ ,\ 0)$ **then** $OC = h$

And $AC = radius = a$

The coordinates of A is

$OA = OC - AC = h - a$

Then $A(h-a\ ,\ 0)$.

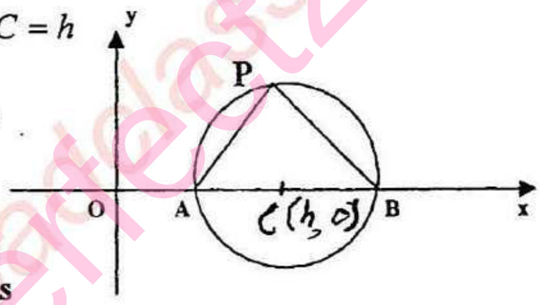
Also $AB = 2radius = 2a$

The coordinates of B is

$OB = OA + AB$

$= h - a + 2a = h + a$

Then $B(h+a\ ,\ 0)$.



Join P to A and B .. we will show that $<APB$ **is right angle.**

Now slope of AP is $m_1 = \frac{y_1 - 0}{x_1 - (h-a)} = \frac{y_1}{x_1 - h + a}$

Now slope of BP is $m_2 = \frac{y_1 - 0}{x_1 - (h+a)} = \frac{y_1}{x_1 - h - a}$

$$m_1.m_2 = \left(\frac{y_1}{x_1 - h + a}\right)\left(\frac{y_1}{x_1 - h - a}\right) = \frac{y_1^2}{(x_1 - h)^2 - a^2} \quad .....(ii)$$

Since point $P(x_1 \ , \ y_1)$ is the point on the circle (i)

Put $P(x_1 \ , \ y_1)$ in (i)

$$(x_1 - h)^2 + y_1^2 = a^2 \ \Rightarrow \ (x_1 - h)^2 - a^2 = -y_1^2 \quad ....(iii)$$

Put (iii) in (ii)

$$m_1.m_2 = \frac{y_1^2}{-y_1^2} = -1$$

Hence the product of slopes of AP and BP is equal to -1 .
Thus AP is perpendicular to BP .

Then $\angle APB = 90^O$ **.**

<u>Example 19</u>: Show that the perpendicular at the outer end point $P(1, \ 1)$ of the radial segment is tangent to the circle $x^2 + y^2 - 13x - 5y + 16 = 0$.

<u>Solution</u>: $x^2 + y^2 - 13x - 5y + 16 = 0 \quad ......(i)$

Here $2g = -13 \ \Rightarrow \ g = -\frac{13}{2}$

$2f = -5 \ \Rightarrow \ f = -\frac{5}{2}$

Then the centre of circle (i) is $C(-g \ , -f) = C\left(\frac{13}{2} , \frac{5}{2}\right)$.

The equation of tangent line at point $P(x_1 , \ y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ **is**

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0 \quad ....(ii)$$

Put $x_1 = 1 \ , \ y_1 = 1 \ , \ g = -\frac{13}{2} \ , f = -\frac{5}{2} \ , \ c = 16$ **in (ii)**

$$x + y - \frac{13}{2}(x + 1) - \frac{5}{2}(y + 1) + 16 = 0 \quad \textit{multiplied by } 2$$

$$\Rightarrow \ 2x + 2y - 13x - 13 - 5y - 5 + 32 = 0$$

=> $-11x - 3y + 14 = 0$

=> $11x + 3y - 14 = 0$ ....(iii)

Eq(iii) is the equation of the tangent line to the circle (i) at Point (1 , 1).

From (iii) => $3y = -11x + 14$

=> $y = -\frac{11}{3}x + \frac{14}{3}$

Then the slope of (iii) is $m_1 = -\frac{11}{3}$

Since CP is the radial segment . Then slope of CP is

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{5}{2} - 1}{\frac{13}{2} - 1} = \frac{\frac{5-2}{2}}{\frac{13-2}{2}} = \frac{3}{2} \times \frac{2}{11} = \frac{3}{11}$$

Now $m_1.m_2 = \left(-\frac{11}{3}\right)\left(\frac{3}{11}\right) = -1$

Then the product of slopes of tangent to the circle and radial segment CP is equal to -1.

Thus the perpendicular at the outer end point $P(1, 1)$ **of the radial segment is tangent to the given circle at point P.**

Exercise 8.3:

Q.1: If $A(2, 2)$ **and** $B(3, 1)$ **are the end points of the chord AB of the circle** $x^2 + y^2 - 4x - 2y + 4 = 0$ **, then show that**

(a) The line from the centre of the circle is perpendicular to AB , also bisects the chord AB .

(b) The line from the centre of the circle to the mid point of the chord AB is perpendicular to the chord AB .

Solution: $x^2 + y^2 - 4x - 2y + 4 = 0$ .....(i)

The general equation of the circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$ ....(ii)

Comparing (i) and (ii), then

$2g = -4 \Rightarrow g = -2$

$2f = -2 \Rightarrow f = -1$

Then the centre of circle (i) is $C(-g\ ,\ -f) = C(2\ ,\ 1)$

Let $D(x\ ,\ y)$ is the mid point of chord AB.

Then $D(x\ ,\ y) = D\left(\frac{2+3}{2}\ ,\ \frac{2+1}{2}\right) = D\left(\frac{5}{2}\ ,\ \frac{3}{2}\right)$

Now slope of AB is $m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-2}{3-2} = -\frac{1}{1} = -1$

The slope of CD is $m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{3}{2} - 1}{\frac{5}{2} - 2} = \frac{\frac{3-2}{2}}{\frac{5-4}{2}} = \frac{1}{2} \times \frac{2}{1} = 1$

Now $m_1 . m_2 = (-1)(1) = -1$

Hence CD is perpendicular to chord AB.

As CD is the line from the centre of the circle to the mid point D of the chord AB and perpendicular to AB.

Hence (a) , (b) are proved.

Q.2: If $A(0,\ 0)$ and $B(0\ ,\ 5)$ are the end points of the chord AB of the circle $x^2 + y^2 + 4x - 5y = 0$, then show that

(a) The line from the centre of the circle is perpendicular to AB , also bisects the chord AB.

(b) The line from the centre of the circle to the mid point of the chord AB is perpendicular to the chord AB.

Solution: $x^2 + y^2 + 4x - 5y = 0$ .....(i)

The general equation of the circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$ ....(ii)

Comparing (i) and (ii), then

$2g = 4 \Rightarrow g = 2$

$2f = -5 \Rightarrow f = -\frac{5}{2}$

Then the centre of circle (i) is $C(-g\ ,\ -f)=C\left(-2\ ,\ \frac{5}{2}\right)$

Let $D(x\ ,\ y)$ is the mid point of chord AB.

Then $D(x\ ,\ y)=D\left(\frac{0+0}{2}\ ,\ \frac{0+5}{2}\right)=D\left(0\ ,\ \frac{5}{2}\right)$

Now slope of AB is $m_1=\frac{y_2-y_1}{x_2-x_1}=\frac{5-0}{0-0}=\frac{5}{0}=\infty$

Then AB lie along y- axis .

The slope of CD is $m_2=\frac{y_2-y_1}{x_2-x_1}=\frac{\frac{5}{2}-\frac{5}{2}}{0-(-2)}=\frac{0}{2}=0$

Then CD parallel to $x-$axis .

Hence CD is perpendicular to chord AB.

As CD is the line from the centre of the circle to the mid point D of the chord AB and perpendicular to AB.

Hence (a) , (b) are proved.

Q.3: (a) Show that the chords AB and DE are equidistance from the centre $C(0\ ,\ 0)$ of the circle $x^2+y^2=4$. The coordinates of the end points of the two chords are $A(-2\ ,0)$, $B(0\ ,2)$, , $D(0\ ,2)$ and $E(2\ ,\ 0)$.

Solution: The given circle $x^2+y^2=4$ .....(i) , with centre is $C(0\ ,\ 0)$.

Now the equation of the chord AB is

$$y-y_1=\frac{(y_2-y_1)}{(x_2-x_1)}(x-x_1)\ \text{.......(a) put values}$$

$$\Rightarrow\ y-0=\frac{(2-0)}{(0-(-2))}(x-(-2))\quad \Rightarrow\ y=\frac{2}{2}(x+2)$$

$$\Rightarrow\ y=x+2\quad or\quad 0=x-y+2$$

$$\Rightarrow\ x-y+2=0\ \text{.........}(ii)$$

Now the perpendicular distance of the chord AB from the centre C(0 , 0) is

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \quad .....(b)$$

From (ii) $a = 1$, $b = -1$, $c = 2$ **and** $x_1 = 0$, $y_1 = 0$ **put in (b)**

$$d_1 = \frac{|(1)(0) + (-1)(0) + 2|}{\sqrt{(1)^2 + (-1)^2}} = \frac{|2|}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2} \quad .....(c)$$

Now equation of DE : As $D(0\ ,\ 2)$, $E(2\ ,\ 0)$ **put in (a)**

$$y - 2 = \frac{(0-2)}{(2-0)}(x - 0) \quad => \quad y - 2 = -\frac{2}{2}x$$

$$=> \quad y - 2 = -x \qquad => \quad x + y - 2 = 0 \quad ......(iii)$$

The perpendicular distance of the chord DE from

centre C (0 , 0) : From (iii)

Put $a = 1$, $b = 1$, $c = -2$ **and** $x_1 = 0$, $y_1 = 0$ **put in** (b)

$$d_2 = \frac{|(1)(0) + (1)(0) - 2|}{\sqrt{(1)^2 + (-1)^2}} = \frac{|-2|}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2} \quad .....(d)$$

Hence from eq(c) and (d) both chords AB and CD have equal distance from the centre .

(b) $x^2 + y^2 = 16$ **. The coordinates of the end points of the two chords are** $A(-4\ ,\ 0)$, $B(0\ ,\ 4)$, , $D(0\ ,\ 4)$ **and** $E(4\ ,\ 0)$**.**

<u>Solution</u>: The given circle $x^2 + y^2 = 16$ **.....(i) , with centre is** $C(0\ ,\ 0)$ **.**

Now the equation of the chord AB is

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1) \quad ....(a) \text{ put values}$$

$$=> \quad y - 0 = \frac{(4-0)}{(0-(-4))}(x - (-4)) \quad => \quad y = \frac{4}{4}(x+4)$$

$$=> \quad y = x + 4 \quad or \quad 0 = x - y + 4$$

$$=> \quad x - y + 4 = 0 \quad ....(ii)$$

Now the perpendicular distance of the chord AB from the

centre C(0 , 0) is

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \quad .....(b)$$

From (ii) $a = 1$, $b = -1$, $c = 4$ and $x_1 = 0$, $y_1 = 0$ put in (b)

$$d_1 = \frac{|(1)(0) + (-1)(0) + 4|}{\sqrt{(1)^2 + (-1)^2}} = \frac{|4|}{\sqrt{1+1}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \quad .....(c)$$

Now equation of DE : As $D(0\ ,\ 4)$, $E(4\ ,\ 0)$ put in (a)

$$y - 4 = \frac{(0-4)}{(4-0)}(x - 0) \quad => \quad y - 4 = -\frac{4}{4}x$$

$$=> \quad y - 4 = -x \qquad => \quad x + y - 4 = 0 \quad ......(iii)$$

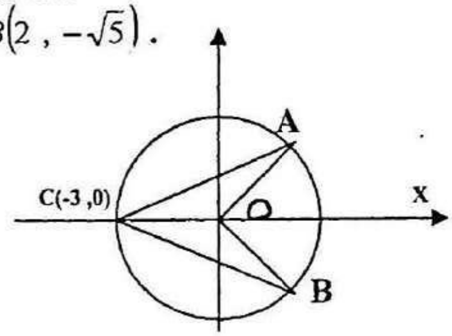
The perpendicular distance of the chord DE from centre C (0 , 0) : From (iii)

Put $a = 1$, $b = 1$, $c = -4$ and $x_1 = 0$, $y_1 = 0$ put in (b)

$$d_2 = \frac{|(1)(0) + (1)(0) - 4|}{\sqrt{(1)^2 + (1)^2}} = \frac{|-4|}{\sqrt{1+1}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \quad .....(d)$$

Hence from eq(c) and (d) both chords AB and CD have equal distance from the centre .

Q.4: Show that the angle subtended by the minor arc AB of the circle (a) $x^2 + y^2 = 9$ is two times the angle subtended in the major arc . The coordinates of the minor arc are $A(2\ ,\ \sqrt{5})$, $B(2\ ,\ -\sqrt{5})$.



Solution : $x^2 + y^2 = 9$ ....(i)

The centre of circle (i) is $O(0\ ,\ 0)$.

The end points of minor arc AB are $A(2\ ,\ \sqrt{5})$, $B(2\ ,\ -\sqrt{5})$.

Note : Since $B(2\ ,\ -\sqrt{5})$ is a point on the circle and $O(0\ ,\ 0)$ is the centre of the circle. Then the radius of the circle is

$$r = |OB| = \sqrt{(2-0)^2 + (-\sqrt{5}-0)^2} = \sqrt{4+5} = \sqrt{9} = 3$$

Also A lie in the first quadrant and B lie in 4^{th} quadrant Then the major arc contains 2^{nd} , 3^{rd} and some parts of Ist and 4^{th} quadrant. Then take point C on major arc. Here we take C(- 3 , 0) because $|OC| = 3 = r$

Let $C(-3\ ,\ 0)$ is a point on the major arc. Then join point C to points A and B.

Let $\angle ACB = \theta$ is the angle subtended by major arc.

Then slope of AC is $m_1 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-\sqrt{5}}{-3-2} = \dfrac{\sqrt{5}}{5} = \dfrac{1}{\sqrt{5}}$

Then slope of BC is $m_2 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-(-\sqrt{5})}{-3-2} = \dfrac{\sqrt{5}}{-5} = -\dfrac{1}{\sqrt{5}}$

Then $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2} = \dfrac{\dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{5}}}{1 + \left(\dfrac{1}{\sqrt{5}}\right)\left(-\dfrac{1}{\sqrt{5}}\right)} = \dfrac{\dfrac{2}{\sqrt{5}}}{1 - \dfrac{1}{5}} = \dfrac{\dfrac{2}{\sqrt{5}}}{\dfrac{5-1}{5}}$

=> $\tan\theta = \dfrac{2}{\sqrt{5}} \times \dfrac{5}{4} = \dfrac{\sqrt{5}}{2}$ .....(a)

Let $\angle AOB = 2\theta$.

Then slope of AO is $m_3 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-\sqrt{5}}{0-2} = \dfrac{\sqrt{5}}{2}$

Then slope of BO is $m_4 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0+\sqrt{5}}{0-2} = -\dfrac{\sqrt{5}}{2}$

Then

$$\tan 2\theta = \frac{m_3 - m_4}{1 + m_3 m_4} = \frac{\frac{\sqrt{5}}{2} - \left(-\frac{\sqrt{5}}{2}\right)}{1 + \left(\frac{\sqrt{5}}{2}\right)\left(-\frac{\sqrt{5}}{2}\right)} = \frac{\frac{2\sqrt{5}}{2}}{1 - \frac{5}{4}} = \frac{\sqrt{5}}{\frac{4-5}{4}}$$

$$\Rightarrow \tan 2\theta = -\frac{4\sqrt{5}}{1} = -4\sqrt{5} \quad .....(b)$$

Since $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$ ....(c). **Put (a) in (c)**

$$\tan 2\theta = \frac{2\left(\frac{\sqrt{5}}{2}\right)}{1 - \left(\frac{\sqrt{5}}{2}\right)^2} = \frac{\sqrt{5}}{1 - \frac{5}{4}} = \frac{\sqrt{5}}{\frac{4-5}{4}} = \frac{4\sqrt{5}}{-1} = -4\sqrt{5}$$

which is eq(b)

Thus proving (a) with result (b).

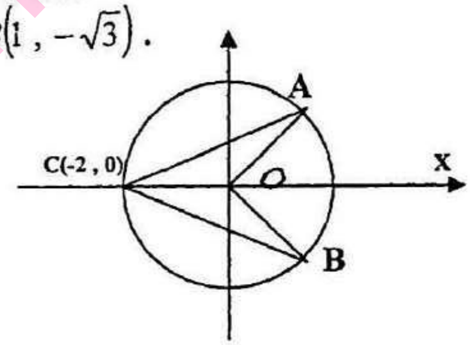
Hence $< AOB = 2 < ACB$.

(b) $x^2 + y^2 = 4$ **is two times the angle subtended in the major arc. The coordinates of the minor arc are** $A(1, \sqrt{3})$, $B(1, -\sqrt{3})$.

Solution: $x^2 + y^2 = 4$ ....(i)

The centre of circle (i) is $O(0, 0)$.

The end points of minor arc AB are $A(1, \sqrt{3})$, $B(1, -\sqrt{3})$.



Note : Since $B(1, -\sqrt{3})$ is a point on the circle and $O(0, 0)$ is the centre of the circle. Then the radius of the circle is

$$r = |OB| = \sqrt{(1-0)^2 + (-\sqrt{3}-0)^2} = \sqrt{1+3} = \sqrt{4} = 2$$

Also A lie in the first quadrant and B lie in 4^{th} quadrant Then the major arc contains 2^{nd}, 3^{rd} and some parts of Ist and 4^{th} quadrant. Then take point C on major arc. Here we take C(-2, 0) because $|OC| = 2 = r$

Let $C(-2, 0)$ is a point on the major arc. Then join point C to A and B.

Let $< ACB = \theta$ is the angle subtended by major arc.

Then slope of AC is $m_1 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-\sqrt{3}}{-2-1} = -\dfrac{\sqrt{3}}{-3} = \dfrac{1}{\sqrt{3}}$

Then slope of BC is $m_2 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-(-\sqrt{3})}{-2-1} = \dfrac{\sqrt{3}}{-3} = -\dfrac{1}{\sqrt{3}}$

Then $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2} = \dfrac{\frac{1}{\sqrt{3}} - \left(-\frac{1}{\sqrt{3}}\right)}{1 + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)} = \dfrac{\frac{2}{\sqrt{3}}}{1 - \frac{1}{3}} = \dfrac{\frac{2}{\sqrt{3}}}{\frac{3-1}{3}}$

=> $\tan\theta = \dfrac{2}{\sqrt{3}} \times \dfrac{3}{2} = \sqrt{3}$ .....(a)

Let $< AOB = 2\theta$.

Then slope of AO is $m_3 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-\sqrt{3}}{0-1} = \sqrt{3}$

Then slope of BO is $m_4 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0+\sqrt{3}}{0-1} = -\sqrt{3}$

Then $\tan 2\theta = \dfrac{m_3 - m_4}{1 + m_3 m_4} = \dfrac{\sqrt{3} - (-\sqrt{3})}{1 + (\sqrt{3})(-\sqrt{3})} = \dfrac{2\sqrt{3}}{1-3} = \dfrac{2\sqrt{3}}{-2}$

=> $\tan 2\theta = -\sqrt{3}$ .....(b)

Since $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ ....(c). Put (a) in (c)

$$\tan 2\theta = \frac{2(\sqrt{3})}{1-(\sqrt{3})^2} = \frac{2\sqrt{3}}{1-3} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$ which is (b).

Thus proving (a) with result (b).

Hence $< AOB = 2 < ACB$.

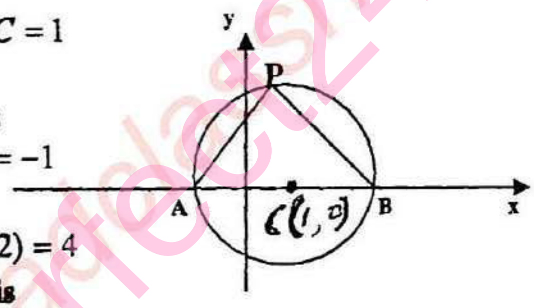
Q.5: Show that the angle in the semicircle of the circle

(a) $(x-h)^2 + y^2 = a^2$, $h = 1$, $a = 2$ is right angle.

Solution: The circle equation is $(x-1)^2 + y^2 = (2)^2$ ...(i)

The centre of (i) is $C(1 , 0)$ and radius of circle is 2.

Since centre on the x- axis then AB is the diameter of the semi circle on the x – axis. Let $P(x_1 , y_1)$ is any point on the semi circle.



Since $C(1 , 0)$ then $OC = 1$

And $AC = radius = 2$

The coordinates of A is

$OA = OC - AC = 1 - 2 = -1$

Then $A(-1 , 0)$.

Also $AB = 2radius = 2(2) = 4$

The coordinates of B is

$OB = OA + AB$

$= -1 + 4 = 3$

Then $B(3 , 0)$.

Join P to A and B. we will show that $< APB$ is right angle.

Now slope of AP is $m_1 = \frac{y_1 - 0}{x_1 - (-1)} = \frac{y_1}{x_1 + 1}$

Now slope of BP is $m_2 = \frac{y_1 - 0}{x_1 - 3} = \frac{y_1}{x_1 - 3}$

$$m_1.m_2 = \left(\frac{y_1}{x_1+1}\right)\left(\frac{y_1}{x_1-3}\right) = \frac{y_1^2}{x_1^2 - 2x_1 - 2} \quad .....(ii)$$

Since point $P(x_1 , y_1)$ is the point on the circle (i)

Put $P(x_1 , y_1)$ in (i)

$$(x_1-1)^2+y_1^2=4 \Rightarrow x_1^2-2x_1+1-4=-y_1^2$$

$$\Rightarrow x_1^2-2x_1-3=-y_1^2 \text{ ....(iii)}$$

Put (iii) in (ii)

$$m_1.m_2=\frac{y_1^2}{-y_1^2}=-1$$

Hence the product of slopes of AP and BP is equal to -1.
Thus AP is perpendicular to BP.
Then $<APB=90^o$.

(b) $(x-h)^2+y^2=a^2$, $h=3$, $a=4$ is right angle.

Solution: The circle equation is $(x-3)^2+y^2=(4)^2$ ...(i)

The centre of (i) is $C(3 , 0)$ and radius of circle is 4.
Since centre on the x- axis then AB is the diameter of the semi circle on the x – axis. Let $P(x_1 , y_1)$ is any point on the semi circle.

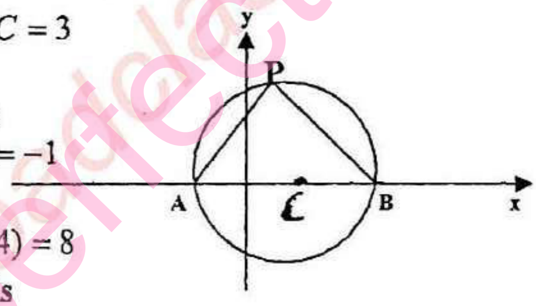
Since $C(3 , 0)$ then $OC=3$

And $AC=radius=4$

The coordinates of A is

$OA=OC-AC=3-4=-1$

Then $A(-1 , 0)$.



Also $AB=2radius=2(4)=8$

The coordinates of B is

$OB=OA+AB=-1+8=7$

Then $B(7 , 0)$.

Join P to A and B. we will show that $<APB$ is right angle.

Now slope of AP is $m_1=\dfrac{y_1-0}{x_1-(-1)}=\dfrac{y_1}{x_1+1}$

Now slope of BP is $m_2=\dfrac{y_1-0}{x_1-7}=\dfrac{y_1}{x_1-7}$

$$m_1.m_2 = \left(\frac{y_1}{x_1+1}\right)\left(\frac{y_1}{x_1-7}\right) = \frac{y_1^2}{x_1^2-6x_1-7} \quad .....(ii)$$

Since point $P(x_1\ ,\ y_1)$ is the point on the circle (i)

Put $P(x_1\ ,\ y_1)$ in (i)

$$(x_1-3)^2 + y_1^2 = 16 \quad => \quad x_1^2 - 6x_1 + 9 - 16 = -y_1^2$$

$$=> \quad x_1^2 - 6x_1 - 7 = -y_1^2 \quad ....(iii)$$

Put (iii) in (ii)

$$m_1.m_2 = \frac{y_1^2}{-y_1^2} = -1$$

Hence the product of slopes of AP and BP is equal to -1 .
Thus AP is perpendicular to BP .

Then $< APB = 90^O$.

Q.6: Show that the perpendicular at the outer end point
(a) $P(1,\ 5)$ **of the radial segment is tangent to the circle**

$$x^2 + y^2 + x - 5y - 2 = 0 .$$

Solution: $x^2 + y^2 + x - 5y - 2 = 0 \quad ......(i)$

Here $2g = 1 \quad => \quad g = \frac{1}{2}$

$2f = -5 \quad => \quad f = -\frac{5}{2}$

Then the centre of circle (i) is $C(-g\ ,\ -f) = C\left(-\frac{1}{2}\ ,\ \frac{5}{2}\right)$.

The equation of tangent line at point $P(x_1\ ,\ y_1)$ to the circle
$x^2 + y^2 + 2gx + 2fy + c = 0$ **is**

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0 \quad ....(ii)$$

Put $x_1 = 1\ ,\ y_1 = 5\ ,\ g = \frac{1}{2}\ ,\ f = -\frac{5}{2}\ ,\ c = -2$ **in (ii)**

$$x + 5y + \frac{1}{2}(x+1) - \frac{5}{2}(y+5) - 2 = 0 \quad \textit{multiplied by } 2$$

$$=> \quad 2x + 10y + x + 1 - 5y - 25 - 2 = 0$$

$=> \ 3x+5y-26=0 \ ....(iii)$

Eq(iii) is the equation of the tangent line to the circle (i) at Point (1 , 5).

From (iii) $=> \ 5y=-3x+26$

$$=> \ y=-\frac{3}{5}x+\frac{26}{5}$$

Then the slope of (iii) is $m_1=-\frac{3}{5}$

Since CP is the radial segment . Then slope of CP is

$$m_2=\frac{y_2-y_1}{x_2-x_1}=\frac{\frac{5}{2}-5}{-\frac{1}{2}-1}=\frac{\frac{5-10}{2}}{-\frac{2+1}{2}}=-\frac{5}{2}\times\frac{2}{-3}=\frac{5}{3}$$

Now $m_1.m_2=\left(-\frac{3}{5}\right)\left(\frac{5}{3}\right)=-1$

Then the product of slopes of tangent to the circle and radial segment CP is equal to - 1 .

Thus the perpendicular at the outer end point $P(1,\ 1)$ **of the radial segment is tangent to the given circle at point P .**

(b) $P(5,\ 6)$ **of the radial segment is tangent to the circle**

$$x^2+y^2-22x-4y+25=0.$$

<u>Solution</u>: $x^2+y^2-22x-4y+25=0$ **......(i)**

Here $2g=-22 \ => \ g=-11$

$2f=-4 \ => \ f=-2$

Then the centre of circle (i) is $C(-g\ ,-f)=C(11,\ 2)$.

The equation of tangent line at point $P(x_1,\ y_1)$ **to the circle**

$x^2+y^2+2gx+2fy+c=0$ **is**

$x_1x+y_1y+g(x+x_1)+f(y+y_1)+c=0$ **....(ii)**

Put $x_1=5\ ,\ y_1=6\ ,\ g=-11\ ,\ f=-2\ ,\ c=25$ **in (ii)**

$5x+6y-11(x+5)-2(y+6)+25=0$

$=> \ 5x+6y-11x-55-2y-12+25=0$

=> $-6x+4y-42=0$ *divided by* -2

=> $3x-2y+21=0$ ....(iii)

Eq(iii) is the equation of the tangent line to the circle (i) at Point (5, 6).

From (iii) => $-2y=-3x-21$

=> $y=\frac{3}{2}x+\frac{21}{2}$

Then the slope of (iii) is $m_1=\frac{3}{2}$

Since CP is the radial segment . Then slope of CP is

$$m_2=\frac{y_2-y_1}{x_2-x_1}=\frac{2-6}{11-5}=-\frac{4}{6}=-\frac{2}{3}$$

Now $m_1.m_2=\left(\frac{3}{2}\right)\left(-\frac{2}{3}\right)=-1$

Then the product of slopes of tangent to the circle and radial segment CP is equal to - 1 .

Thus the perpendicular at the outer end point $P(1,\ 1)$ **of the radial segment is tangent to the given circle at point P .**

<u>Review Exercise</u>. 8:

<u>Q.1</u>: Choose the correct option.

(i) If radius of a circle is 2 cm then its equation is

(a) $x^2+y^2=2$ **(b)** $x^2+y^2=\sqrt{2}$

(c) $x^2+y^2=8$ **(d)** $x^2+y^2=4$

(ii) In the general equation of circle the coordinates of the circle are

(a) $(x,\ y)$ **(b)** $(-x,-y)$ **(c)** $(f,\ g)$ **(d)** $(-f,-g)$

(iii) For the general equation of circle the radius can be calculated by $r=$

(a) $\sqrt{x^2+y^2-c^2}$ **(b)** $\sqrt{(-x)^2+(-y)^2-c^2}$

(c) $\sqrt{(-f)^2+(-g)^2-c^2}$ **(d)** $\sqrt{(-f)^2+(-g)^2+c^2}$

(iv) A point of the circle at which a tangent meets the circle is called

(a) Point of contact (b) normal
(c) center point (d) non of these

(v) If the dimensional of the quadratic equations $m^2\left(x_1^2-a_1\right)-2mx_1y_1+\left(y_1^2-a_2\right)=0$ is greater than zero tham tangent are

(a) real and different (b) imaginary
(c) real and concident (d) equal

(vi) The lenth of the tangent drawn from the point $P(3\ ,\ 4)$ on the circle $x^2+y^2-9=0$ is

(a) 1 (b) 2 (c) 3 (d) 4

(vii) The angle subtended can be calculated by using the trigonometric identity .

(a) $\tan 2\theta=\dfrac{m_1-m_2}{1+m_1m_2}$ (b) $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

(c) $\tan 2\theta=\dfrac{2\tan^2\theta}{1+\tan^2\theta}$ (d) (b) $\tan 2\theta=\dfrac{\tan^2\theta}{1-2\tan^2\theta}$

(viii) What is the contact point for the lines $x=7$ touches the $x^2+y^2-4x-6y-12=0$

(a) $(2\ ,8)$ (b) $(8\ ,2)$ (c) $(3,\ 7)$ (d) $(7\ ,\ 3)$

(xi) For the condition of tangent line $y=mx+c$ should touch the circle $x^2+y^2=r^2$ if

(a) $c=r\sqrt{1+m^2}$ (b) $c=-r\sqrt{1+m^2}$
(c) both option (a) and (b) (d) not option (b) and (c)

(x) A line perpendicular to the contac point to a circle called

(a) Tangent (b) Normal (c) Chord (d) Diameter

<u>Answers</u>: (i) d (ii) d (iii) c (iv) a (v) a
(vi) d (vii) b (viii) d (ix) c (x) b