Unit – 4: **Higher order derivatives And its application:**

Higher order derivatives: If $y = f(x)$ then the first derivative is $\frac{dy}{dx}$ or y'.

The second derivative is obtained by taking the derivative of y' and given by $\frac{d^2y}{dx^2}$ or y''.

The third derivative is obtained by taking the derivative of y'' and given by $\frac{d^3y}{dx^3}$ or y''' and so on.

The nth derivative of the function $y = f(x)$ is denoted by $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$.

Example 1: Find the second derivative of the following functions: (a) $y = 8x^3 - 9x^2 + 6x + 4$

Solution: $y = 8x^3 - 9x^2 + 6x + 4$, Differentiate w. r. t. to x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(8x^3 - 9x^2 + 6x + 4\right)$$

=> $y' = 12x^2 - 18x + 6$, Differentiate again w. r. t. to x.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(12x^2 - 18x + 6\right)$$

=> $\frac{d^2y}{dx^2} = 24x - 18$ *Ans.*

(b) $y = \frac{4x+2}{3x-1}$

Solution: $y = \frac{4x+2}{3x-1}$, Differentiate w. r. t. to x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{4x+2}{3x-1}\right)$$ use quotient rule

$$\Rightarrow \frac{dy}{dx} = \frac{(3x-1)\frac{d}{dx}(4x+2)-(4x+2)\frac{d}{dx}(3x-1)}{(3x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(3x-1)(4)-(4x+2)(3)}{(3x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{12x-4-12x-6}{(3x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{10}{(3x-1)^2}$$

Differentiate again w . r. t. to x.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[-10(3x-1)^{-2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -10(-2)(3x-1)^{-3}\frac{d}{dx}(3x-1)$$

$$= 20(3x-1)^{-3}(3)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{60}{(3x-1)^3} \quad Ans.$$

Note : If s is displacement then the rate of change of displacement is called velocity . i.e.

$$velocity = v = \frac{ds}{dt} \quad \text{......(i)}$$

The rate of change of velocity is called acceleration . i.e.

$$Acceleration = a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \quad \text{....(ii)}$$

Example 2 : An object is moving along a straight line with its position $s(t)$ (in feet) at time t (in seconds) :

$$s(t) = t^3 - 2t^2 - 7t + 9$$

(a) Find the velocity at any time t .

(b) Find the acceleration at any time t .

(c) The object stops when velocity is zero . For $t \geq 0$, when

does that occur.

Solution: $s(t) = t^3 - 2t^2 - 7t + 9$ .....(i)

(a) velocity : Differentiate (i) w.r.t. to t.

$$\frac{ds}{dt} = \frac{d}{dt}\left(t^3 - 2t^2 - 7t + 9\right)$$

$$\Rightarrow v = 3t^2 - 4t - 7 \quad ....(ii)$$

Eq(ii) shows the velocity at any time t.

(b) Acceleration : Differentiate (ii) w.r.t. to t.

$$\frac{dv}{dt} = \frac{d}{dt}\left(3t^2 - 4t - 7\right)$$

$$\Rightarrow a = 6t - 4 \quad ....(iii)$$

Eq(iii) shows the acceleration at any time t.

(c) Put $v = 0$ in (ii)

$$3t^2 - 4t - 7 = 0$$

$$\Rightarrow 3t^2 + 3t - 7t - 7 = 0$$

$$\Rightarrow 3t(t+1) - 7(t+1) = 0$$

$$\Rightarrow (t+1)(3t-7) = 0$$

$$\Rightarrow t + 1 = 0 \quad , \quad 3t - 7 = 0$$

$$\Rightarrow t = -1 \quad , \quad t = \frac{7}{3}$$

Since time is non-negative thus the object stops after $\frac{7}{3}$ Seconds.

1:nth derivative of $f(x) = (ax+b)^m$

As $f(x) = (ax+b)^m$

$\Rightarrow f'(x) = m(ax+b)^{m-1}.a$

$\Rightarrow f'(x) = am(ax+b)^{m-1}$

$\Rightarrow f''(x) = a^2m(m-1)(ax+b)^{m-2}$

$\Rightarrow f'''(x) = a^3m(m-1)(m-2)(ax+b)^{m-3}$

In General

$\Rightarrow f^n(x) = a^n m(m-1)(m-2).....(m-(n-1))(ax+b)^{m-n}$

$$= \left[\frac{a^n m(m-1).....(m-n+1)(m-n)(m-n-1)...2.1}{(m-n)(m-n-1).....2.1}\right](ax+b)^{m-n}$$

$$\Rightarrow \quad f^n(x) = a^n.\frac{m!}{(m-n)!}(ax+b)^{m-n} -----(i)$$

Eq (i) shows that nth derivative of f(x)

If $m = n$ **then put in (i)**

$$\Rightarrow \quad f^n(x) = a^n.\frac{n!}{(n-n)!}(ax+b)^{n-n}$$

$$\Rightarrow \quad f^n(x) = a^n.\frac{n!}{0!}(ax+b)^0 \qquad \text{As } 0! = 1$$

$$\Rightarrow \quad f^n(x) = a^n.n!$$

2: nth derivative of $y = \dfrac{1}{ax+b}$

As $\quad y = \dfrac{1}{ax+b}$

$$\Rightarrow \quad y = (ax+b)^{-1}$$

$$\Rightarrow \quad y_1 = (-1)a(ax+b)^{-2} \qquad \text{As } y_1 \text{ is the first derivative.}$$

$$\Rightarrow \quad y_2 = (-1)(-2)a^2(ax+b)^{-3}$$

$$\Rightarrow \quad y_3 = (-1)(-2)(-3)a^3(ax+b)^{-4}$$

$$\Rightarrow \quad y_3 = (-1)^3(1.2.3)a^3(ax+b)^{-4}$$

$$\Rightarrow \quad y_3 = (-1)^3.\frac{3!a^3}{(ax+b)^4}$$

In General:

$$\Rightarrow \quad y_n = \frac{(-1)^n.n!a^n}{(ax+b)^{n+1}} \qquad \textbf{Ans.}$$

3: If $y = \ln(ax+b)$, **Find the nth derivative** y_n

As $\quad y = \ln(ax+b)$

$$\Rightarrow \quad y_1 = \frac{a}{ax+b} = a(ax+b)^{-1}$$

$$\Rightarrow \quad y_2 = (-1)a^2(ax+b)^{-2}$$

$$\Rightarrow \quad y_3 = (-1)(-2)a^3(ax+b)^{-3}$$

$$\Rightarrow \quad y_4 = (-1)(-2)(-3)(a^4)(ax+b)^{-4}$$

$$\Rightarrow \quad y_4 = (-1)^3(1.2.3)(a^4)(ax+b)^{-4}$$

$$\Rightarrow \quad y_4 = \frac{(-1)^3.3!.a^4}{(ax+b)^4}$$

$$\Rightarrow \quad y_5 = \frac{(-1)^4.4!.a^5}{(ax+b)^5}$$

In General:

$$\Rightarrow \quad y_n = \frac{(-1)^{n-1}(n-1)!.a^n}{(ax+b)^n} \quad \textbf{Ans.}$$

4: **If** $y = \sin(ax+b)$,
Find the nth derivative y_n.

As $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$

$\frac{d}{dx}(\sin(ax+b))$
$= a\cos(ax+b)$

As $y = \sin(ax+b)$

$$\Rightarrow \quad y_1 = a\cos(ax+b)$$

$$\Rightarrow \quad y_1 = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$\Rightarrow \quad y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right)$$

$$\Rightarrow \quad y_2 = a^2\sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$\Rightarrow \quad y_2 = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$\Rightarrow \quad y_3 = a^3\cos\left(ax+b+\frac{2\pi}{2}\right)$$

$$\Rightarrow \quad y_3 = a^3\sin\left(ax+b+\frac{2\pi}{2}+\frac{\pi}{2}\right)$$

$$\Rightarrow \quad y_3 = a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$$

In General:

$$\Rightarrow \quad y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right) \quad \textbf{Ans.}$$

5: If $y=\cos(ax+b)$,

Find the nth derivative y_n.

$$-\sin\theta = \cos\left(\theta+\frac{\pi}{2}\right)$$

$$\frac{d}{dx}(\cos(ax+b)) = -a\sin(ax+b)$$

As $y=\cos(ax+b)$

$$\Rightarrow \quad y_1 = -a\sin(ax+b)$$

$$\Rightarrow \quad y_1 = a\cos\left(ax+b+\frac{\pi}{2}\right)$$

$$\Rightarrow \quad y_2 = -a^2\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$\Rightarrow \quad y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$\Rightarrow \quad y_2 = a^2\cos\left(ax+b+\frac{2\pi}{2}\right)$$

$$\Rightarrow \quad y_3 = -a^3\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$\Rightarrow \quad y_3 = a^3\cos\left(ax+b+\frac{2\pi}{2}+\frac{\pi}{2}\right)$$

$$\Rightarrow \quad y_3 = a^3\cos\left(ax+b+\frac{3\pi}{2}\right)$$

In General:

$$\Rightarrow \quad y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right) \quad \textbf{Ans.}$$

6: If $y=a^{mx}$ **find** y_n

Sol: As $y=a^{mx}$

$$\Rightarrow \quad y_1 = ma^{mx}.\ln a$$

$$\Rightarrow \quad y_2 = m^2a^{mx}(\ln a)^2$$

$$\Rightarrow \quad y_3 = m^3a^{mx}.(\ln a)^3$$

In General: $y_n = m^n a^{mx}.(\ln a)^n$

7: If $y = e^{mx}$

$\Rightarrow y_1 = me^{mx}$

$\Rightarrow y_2 = m^2 e^{mx}$

$\Rightarrow y_3 = m^3 e^{mx}$

In general $y_n = m^n e^{mx}$.

Example 3: Find the 5th derivative of the following functions. **(a)** $f(x) = (6x+4)^9$

Solution: As we know that if $f(x) = (ax+b)^n$

Then $f^n(x) = \dfrac{m!}{m-n!} a^m (ax+b)^{m-n}$ **....(i)**

Put $a=6$, $b=4$, $n=5$ *and* $m=9$ **in(i)**

$$f^5(x) = \frac{9!}{(9-5)!} 6^5 (6x+4)^{9-5}$$

$$\Rightarrow f^5(x) = \frac{9.8.7.6.5.4!}{4!} 6^5 (6x+4)^4$$

$$= 6^5 (15120)(6x+4)^4 \quad Ans.$$

(b) $f(x) = \dfrac{1}{4x+3}$

Solution: As we know that if $f(x) = \dfrac{1}{ax+b}$ **then**

$f^n(x) = \dfrac{(-1)^n n! a^n}{(ax+b)^{n+1}}$ **....(i)**

Put $a=4$, $b=3$. $n=5$ **in(i)**

$$f^5(x) = \frac{(-1)^5 5!(4)^5}{(4x+3)^6} = -\frac{5!\, 4^5}{(4x+3)^6} \quad Ans.$$

(c) $f(x) = \ln(4x+7)$

Solution: As we know that if $f(x) = \ln(ax+b)$ then

$$f^n(x)=\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n} \quad \ldots\ldots(i)$$

Put $a=4$, $b=7$ *and* $n=5$ **in (i)**

$$f^5(x)=\frac{(-1)^4.4!4^5}{(4x+7)^5}=\frac{4!4^5}{(4x+7)^5} \quad Ans.$$

(d) $f(x)=6^{4x}$

<u>Solution</u>: As we know that if $f(x)=a^{mx}$ **then**

$$f^n(x)=m^n a^{mx}(\ln a)^n \quad \ldots\ldots.(i)$$

Put $a=6$, $m=4$ *and* $n=5$ *in(i)*

$$f^5(x)=4^5.6^{4x}.(\ln 6)^5 \quad \textbf{Ans.}$$

(e) $f(x)=e^{4x}$

<u>Solution</u>: As we know that if $f(x)=e^{mx}$ **then**

$$f^n(x)=m^n e^{mx} \quad \ldots\ldots.(i)$$

Put $m=4$, $n=5$ **in(i)**

$$f^5(x)=4^5.e^{4x}$$

(f) $f(x)=\sin(5x+7)$

<u>Solution</u>: As we know that if $f(x)=\sin(ax+b)$ **,then**

$$f^n(x)=a^n\sin\left(ax+b+\frac{n\pi}{2}\right) \quad \ldots\ldots.(i)$$

Put $a=5$, $b=7$ *and* $n=5$ **in (i)**

$$f^5(x)=5^5\sin\left(5x+7+\frac{5\pi}{2}\right) \quad \textbf{Ans.}$$

<u>Derivative of parametric function</u> : If $x=f(t)$ **and** $y=g(t)$ **are parametric functions then**

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \textit{is the first derivative}$$

Now the second derivative is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right).\frac{dt}{dx} \quad \textit{use quotient rule}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left[\frac{\frac{dx}{dt}\frac{d}{dt}\left(\frac{dy}{dt}\right)-\frac{dy}{dt}.\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}\right].\frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}.\frac{d^2y}{dt^2}-\frac{dy}{dt}.\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}.\frac{1}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}.\frac{d^2y}{dt^2}-\frac{dy}{dt}.\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} \quad ......(A)$$

Eq (A) Shows the second derivative of parametric function .

<u>Example .6</u>: Find the second derivative $\frac{d^2y}{dx^2}$ **when**

$x(t)=1+t^2$ **and** $y(t)=t^3+2t^2+1$

<u>Solution</u>: Given $x=1+t^2$ **.....(i) and** $y=t^3+2t^2+1$ **.....(ii)**

As we know that

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}.\frac{d^2y}{dt^2}-\frac{dy}{dt}.\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} \quad ......(A)$$

$$\frac{dx}{dt}=\frac{d}{dt}\left(1+t^2\right)=2t \quad ....(a)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{d}{dt}(2t)$$

$$\Rightarrow \frac{d^2x}{dt^2}=2 \text{ ....}(b)$$

$$Now \quad \frac{dy}{dt}=\frac{d}{dt}\left(t^3+2t^2+1\right)=3t^2+4t \text{ ......}(c)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{d}{dt}\left(3t^2+4t\right)$$

$$\Rightarrow \frac{d^2y}{dt^2}=6t+4 \text{ .....}(d)$$

Put (a) , (b) , (c) and (d) in(A)

$$\frac{d^2y}{dx^2}=\frac{(2t)(6t+4)-(3t^2+4t)(2)}{(2t)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{12t^2+8t-6t^2-8t}{8t^3}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{6t^2}{8t^3}=\frac{3}{4t} \quad Ans.$$

OR

<u>By Alternate method:</u>

As we know that

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx} \text{ ... ...(B)}$$

Differentiate (i) and (ii) w . r. t. to t .

$$\frac{dx}{dt}=\frac{d}{dt}\left(1+t^2\right)=2t \text{ ......}(iii)$$

$$\Rightarrow \frac{dt}{dx}=\frac{1}{2t} \text{ ....}(iv)$$

$$Now \quad \frac{dy}{dt}=\frac{d}{dt}\left(t^3+2t^2+1\right)=3t^2+4t \text{ ......}(v)$$

Put (iv) and (v) in (B)

$$\frac{dy}{dx} = (3t^2 + 4t)\left(\frac{1}{2t}\right) = \frac{t(3t+4)}{2t}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3t+4}{2} \quad .....(vi)$$

Differentiate w .r.t. x .

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{3t+4}{2}\right) = \frac{d}{dx}\left(\frac{3}{2}t\right) + \frac{d}{dx}(2)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{3}{2}\frac{dt}{dx} + 0 \qquad put \quad from\ (iv)\ \frac{dt}{dx} = \frac{1}{2t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{3}{2}\left(\frac{1}{2t}\right) = \frac{3}{4t} \quad Ans.$$

<u>Example</u>. Use MAPLE command differentiate

$$f(x) = x^4 + x^2\sin x + x + 2 \text{ w.r.t. to } x.$$

<u>Solution</u>: Contenx Menu.

Open the MAPLE programme and new document in MAPLE and use keyboard keys and write

$diff(x\text{^}4 + x\text{^}2 * \sin(x) + x + 2, x, x);$

press enter key you get

Command:

$diff(x^4 + x^3 \cdot \sin(x) + x + 2, x, x)$;

$$12x^2 + 6x\sin(x) + 6x^2\cos(x) - x^3\sin(x)$$

<u>Exercise 4.1</u>:

<u>Q.1</u>: Find the indicated higher derivatives of the following functions.

(a) $f(x) = 3x^3 + 4x + 5 \quad , \ f''(x)$

<u>Solution</u>: $f(x) = 3x^3 + 4x + 5$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(3x^3 + 4x + 5)$$

$$\Rightarrow f'(x) = 9x^2 + 4$$

$$\Rightarrow \frac{d}{dx}(f'(x)) = \frac{d}{dx}(9x^2 + 4)$$

$$\Rightarrow \quad f''(x) = 18x \qquad Ans.$$

(b) $f(x) = x + \frac{1}{x} \quad , \quad f'''(x)$

Solution: $f(x) = x + \frac{1}{x}$

$$\frac{d}{dx}\left(f(x)\right) = \frac{d}{dx}\left(x + x^{-1}\right) = 1 - x^{-2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{d}{dx} f(x)\right) = \frac{d}{dx}\left(1 - x^{-2}\right)$$

$$\Rightarrow \quad f''(x) = -(-2)x^{-3} = 2x^{-3}$$

$$\Rightarrow \frac{d}{dx}\left(f''(x)\right) = \frac{d}{dx}\left(2x^{-3}\right)$$

$$\Rightarrow \quad f'''(x) = 2(-3)x^{-4} = -\frac{6}{x^4} \qquad Ans.$$

(c) $s(t) = \sqrt{5t+7} \quad , \quad s''(t)$

Solution: $s(t) = \sqrt{5t+7}$

$$\frac{d}{dt} s(t) = \frac{d}{dt}(5t+7)^{\frac{1}{2}}$$

$$\Rightarrow \quad s'(t) = \frac{1}{2}(5t+7)^{-\frac{1}{2}}(5) = \frac{5}{2}(5t+7)^{-\frac{1}{2}}$$

$$\Rightarrow \quad \frac{d}{dt} s'(t) = \frac{d}{dt}\left[\frac{5}{2}(5t+7)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \quad s''(t) = \left(\frac{5}{2}\right)\left(-\frac{1}{2}\right)(5t+7)^{-\frac{1}{2}-1}(5)$$

$$\Rightarrow \quad s''(t) = -\frac{25}{4}(5t+7)^{-\frac{3}{2}} \qquad Ans.$$

(d) $y = \frac{x+1}{x-1} \quad , \quad y''$

Solution : $y = \frac{x+1}{x-1}$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x+1}{x-1}\right)$$

$$\Rightarrow\ y'=\frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$\Rightarrow\ y'=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}$$

$$\Rightarrow\ y'=\frac{x-1-x-1}{(x-1)^2}$$

$$\Rightarrow\ y'=-\frac{2}{(x-1)^2}=-2(x-1)^{-2}$$

$$\frac{d}{dx}(y')=-2\frac{d}{dx}(x-1)^{-2}$$

$$\Rightarrow\ y''=(-2)(-2)(x-1)^{-3}=\frac{4}{(x-1)^3}\quad Ans.$$

Q.2: Use implicit rule to find out the second derivative of the following functions.

(a) $b^2x^2+a^2y^2=a^2b^2$

Solution: $b^2x^2+a^2y^2=a^2b^2$ .....(i)

$$\frac{d}{dx}(b^2x^2+a^2y^2)=\frac{d}{dx}(a^2b^2)$$

$$\Rightarrow\ b^2\frac{d}{dx}x^2+a^2\frac{d}{dx}y^2=0$$

$$\Rightarrow\ 2b^2x+2a^2y\frac{dy}{dx}=0$$

$$\Rightarrow\ 2a^2y\frac{dy}{dx}=-2b^2x$$

$$\Rightarrow\ \frac{dy}{dx}=-\frac{2b^2x}{2a^2y}$$

$$\Rightarrow\ y'=-\frac{b^2x}{a^2y}\ ......(ii)$$

$$\frac{d}{dx}y' = -\frac{b^2}{a^2}\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\Rightarrow\ y'' = -\frac{b^2}{a^2}\left(\frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2}\right) \quad \textit{put the value of } y' \textit{ from (ii)}$$

$$\Rightarrow\ y'' = -\frac{b^2}{a^2y^2}\left(y - x\left(-\frac{b^2x}{a^2y}\right)\right)$$

$$\Rightarrow\ y'' = -\frac{b^2}{a^2y^2}\left(y + \frac{b^2x^2}{a^2y}\right) = -\frac{b^2}{a^2y^2}\left(\frac{a^2y^2 + b^2x^2}{a^2y}\right) \ldots\text{(iii)}$$

From (i) put $b^2x^2 + a^2y^2 = a^2b^2$ **in (iii)**

$$\textit{Then } y'' = -\frac{b^2}{a^2y^2}\left(\frac{a^2b^2}{a^2y}\right) = -\frac{b^4}{a^2y^3}$$

(b) $x^2 + y^2 = r^2$

Solution: $x^2 + y^2 = r^2$ ......(i)

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}r^2$$

$$\Rightarrow\ 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow\ 2y\,y' = -2x$$

$$\Rightarrow\ y' = -\frac{2x}{2y} = -\frac{x}{y} \quad \ldots\ldots(ii)$$

$$\frac{d}{dx}y' = -\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\Rightarrow\ y'' = -\frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2} \quad \text{put the value of } y' \text{ from (ii)}$$

$$\Rightarrow y'' = -\frac{1}{y^2}\left(y - x\left(-\frac{x}{y}\right)\right)$$

$$= -\frac{1}{y^2}\left(y + \frac{x^2}{y}\right)$$

$$\Rightarrow y'' = -\frac{1}{y^2}\left(\frac{y^2 + x^2}{y}\right) \quad put \quad x^2 + y^2 = r^2$$

$$\Rightarrow y'' = -\frac{1}{y^2}\left(\frac{r^2}{y}\right) = -\frac{r^2}{y^3} \quad Ans.$$

(c) $y^2 - 2xy = 0$

Solution: $y^2 - 2xy = 0 \quad ......(i)$

$$\frac{d}{dx}\left(y^2 - 2xy\right) = \frac{d}{dx}(0)$$

$$\Rightarrow \frac{d}{dx}y^2 - 2\frac{d}{dx}(xy) = 0$$

$$\Rightarrow 2y\frac{dy}{dx} - 2\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0$$

$$\Rightarrow 2y\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y = 0$$

$$\Rightarrow 2(y - x)\frac{dy}{dx} = 2y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{y - x} \quad .....(ii)$$

$$Now \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y}{y - x}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(y - x)\frac{dy}{dx} - y\frac{d}{dx}(y - x)}{(y - x)^2}$$

$$\Rightarrow \ y'' = \frac{1}{(y-x)^2}\left[(y-x)\frac{dy}{dx} - y\left(\frac{dy}{dx}-1\right)\right] \text{....}(iii)$$

put (ii) in(iii)

$$\Rightarrow \ y'' = \frac{1}{(y-x)^2}\left[(y-x)\left(\frac{y}{y-x}\right) - y\left(\frac{y}{y-x}-1\right)\right]$$

$$\Rightarrow \ y'' = \frac{1}{(y-x)^2}\left(y - \frac{y^2}{y-x} + y\right)$$

$$\Rightarrow \ y'' = \frac{1}{(y-x)^2}\left(\frac{y^2 - yx - y^2 + y^2 - yx}{y-x}\right)$$

$$\Rightarrow \ y'' = \frac{1}{(y-x)^2}\left(\frac{y^2 - 2xy}{y-x}\right) \text{......}(iv)$$

Put (i) in (iv)

$$y'' = \frac{1}{(y-x)h2}\left(\frac{0}{y-x}\right) = 0 \quad Ans.$$

(d) $e^x + x = e^y + y$ **Solution:** $e^x + x = e^y + y$ **......(i)**

$$\frac{d}{dx}\left(e^x + x\right) = \frac{d}{dx}\left(e^y + y\right)$$

$$\Rightarrow \ e^x + 1 = e^y\frac{dy}{dx} + \frac{dy}{dx}$$

$$\Rightarrow \ \left(e^y + 1\right)\frac{dy}{dx} = e^x + 1$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{e^x+1}{e^y+1} \text{.....}(ii)$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{e^x+1}{e^y+1}\right)$$

$$\Rightarrow y'' = \frac{(e^y+1)\frac{d}{dx}(e^x+1)-(e^x+1)\frac{d}{dx}(e^y+1)}{(e^y+1)^2}$$

$$= \frac{(e^y+1)(e^x)-(e^x+1)(e^y)\frac{dy}{dx}}{(e^y+1)^2} \quad .....(iii)$$

Put (ii) iin (iii)

$$y'' = \frac{1}{(e^y+1)^2}\left[e^{x+y}+e^x-(e^{x+y}+e^y)\left(\frac{e^x+1}{e^y+1}\right)\right]$$

$$\Rightarrow y'' = \frac{1}{(e^y+1)^2}\left[\frac{(e^{x+y}+e^x)(e^y+1)-(e^{x+y}+e^y)(e^x+1)}{(e^y+1)}\right]$$

$$y'' = \frac{1}{(e^y+1)^3}\left(e^{x+y+y}+e^{x+y}+e^{x+y}+e^x-e^{x+y+x}-e^{x+y}-e^{x+y}-e^y\right)$$

$$\Rightarrow y'' = \frac{e^{x+2y}-e^{2x+y}+e^x-e^y}{(e^y+1)^3}$$

$$\Rightarrow y'' = \frac{e^x-e^y-e^{x+y}e^x+e^{x+y}e^y}{(e^y+1)^3}$$

$$\Rightarrow y'' = \frac{(e^x-e^y)-e^{x+y}(e^x-e^y)}{(e^y+1)^3}$$

$$\Rightarrow y'' = \frac{(e^x-e^y)(1-e^{x+y})}{(e^y+1)^3} \quad Ans.$$

<u>Q.3</u>: Use parametric differentiation to find out $\frac{d^2y}{dx^2}$ for the following parametric functions.

(a) $x = 4t^2+1$, $y = 6t^3+1$

<u>Solution</u>: $x = 4t^2+1$ **....(i)** , $y = 6t^3+1$ **......(ii)**

As we know that

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} \quad \text{......(A)}$$

Differentiate (i) w . r. t. to t

$$\frac{dx}{dt} = \frac{d}{dt}\left(4t^2+1\right) = 8t \quad .....(iii)$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(8t)$$

$$\Rightarrow \frac{d^2x}{dt^2} = 8 \quad ......(iv)$$

Differentiate (ii) w . r. t. to t .

$$\frac{dy}{dt} = \frac{d}{dt}\left(6t^3+1\right) = 18t^2 \quad .....(v)$$

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(18t^2\right)$$

$$\Rightarrow \frac{d^2y}{dt^2} = 36t \quad ......(vi)$$

Put (iii) , (iv) , (v) and (vi) in (A)

$$\frac{d^2y}{dx^2} = \frac{(8t)(36t)-\left(18t^2\right)(8)}{(8t)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{288t^2-144t^2}{512t^3} = \frac{144t^2}{512t^3}$$

$$\Rightarrow y'' = \frac{9}{32t}$$

(b) $x = 3at^2+2$, $y = 6t^4+9$

__**Solution**__ : $x = 3at^2+2$ **....(i)** , $y = 6t^4+9$ **.....(ii)**

As we know that

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} \quad \text{......(A)}$$

Differentiate (i) w . r. t. to t

$$\frac{dx}{dt} = \frac{d}{dt}\left(3at^2 + 2\right) = 6at \quad \text{........}(iii)$$

$$\frac{d}{d}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(6at)$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = 6a \text{ ....}(iv)$$

Differentiate (ii) w . r. t. to t ,

$$\frac{dy}{dt} = \frac{d}{dt}\left(6t^4 + 9\right) = 24t^3 \quad \text{.....(v)}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(24t^3\right)$$

$$\Rightarrow \quad \frac{d^2y}{dt^2} = 72t^2 \quad \text{........}(vi)$$

Put (iii) , (iv) , (v) and (vi) in (A)

$$\frac{d^2y}{dx^2} = \frac{(6at)(72t^2) - (24t^3)(6a)}{(6at)^3}$$

$$= \frac{432at^3 - 144at^3}{216a^3t^3} = \frac{288at^3}{216a^3t^3} = \frac{4}{3a^2} \quad \textbf{Ans.}$$

(c) $x = a\cos 2t$, $y = b\sin 2t$

Solution : $x = a\cos 2t$ **....(i)** , $y = b\sin 2t$ **.....(ii)**

As we know that

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} \quad \text{......(A)}$$

Differentiate (i) w . r. t. to t .

$$\frac{dx}{dt} = \frac{d}{dt}(a\cos 2t) = -2a\sin 2t \quad ......(iii)$$

$$=> \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(-2a\sin 2t)$$

$$=> \frac{d^2x}{dt^2} = -4a\cos 2t \quad ....(iv)$$

Differentiate (i) w . r. t. to t .

$$\frac{dy}{dt} = \frac{d}{dt}(b\sin 2t) = 2b\cos 2t \quad ....(v)$$

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(2b\cos 2t)$$

$$=> \quad \frac{d^2y}{dt^2} = -4b\sin 2t \quad ....(vi)$$

Put (iii) , (iv) , (v) and (vi) in (A)

$$\frac{d^2y}{dx^2} = \frac{(-2a\sin 2t)(-4b\sin 2t) - (2b\cos 2t)(-4a\cos 2t)}{(-2a\sin 2t)^3}$$

$$=> \quad y'' = \frac{8ab\sin^2 2t + 8ab\cos^2 2t}{-8a^3\sin^3 2t} = -\frac{8ab(\sin^2 2t + \cos^2 2t)}{8a^3\sin^3 2t}$$

$$=> \quad y'' = -\frac{b}{a^2\sin^3 2t} \quad Ans.$$

(d) $x = \dfrac{3at}{1+t^3} \quad , \quad y = \dfrac{3at^2}{1+t^3}$

<u>**Solution:**</u> $x = \dfrac{3at}{1+t^3}$ **....(i) ,** $y = \dfrac{3at^2}{1+t^3}$ **.....(ii)**

As we know that

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} \quad ...(A)$$

Differentiate (i) w . r. t. to t .

$$\frac{dx}{dt}=\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)=3a\left[\frac{(1+t^3)\frac{d}{dt}(t)-t\frac{d}{dt}(1+t^3)}{(1+t^3)^2}\right]$$

$$=> \frac{dx}{dt}=3a\left(\frac{1+t^3-t(3t^2)}{(1+t^3)^2}\right)=\frac{3a(1-2t^3)}{(1+t^3)^2} \quad .....(iii)$$

$$=> \frac{dt}{dx}=\frac{(1+t^3)^2}{3a(1-2t^3)} \quad ....(iv)$$

Differentiate (ii) w . r. t. to t .

$$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)=3a\left[\frac{(1+t^3)\frac{d}{dt}(t^2)-t^2\frac{d}{dt}(1+t^3)}{(1+t^3)^2}\right]$$

$$=> \frac{dy}{dt}=3a\left(\frac{(1+t^3)(2t)-t^2(3t^2)}{(1+t^3)^2}\right)$$

$$=> \frac{dy}{dt}=\frac{3a(2t+2t^4-3t^4)}{(1+t^3)^2}=\frac{3a(2t-t^4)}{(1+t^3)^2} \quad ....(v)$$

Put (iv) and (v) in (A)

$$\frac{dy}{dx}=\frac{3a(2t-t^4)}{(1+t^3)^2}\times\frac{(1+t^3)^2}{3a(1-2t^3)}=\frac{2t-t^4}{1-2t^3} \quad .....(vi)$$

Now eq(vi) differentiate w . r.t . to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{2t-t^4}{1-2t^3}\right)$$

$$=> \frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{2t-t^4}{1-2t^3}\right).\frac{dx}{dt}$$

$$=\left[\frac{\left(1-2t^3\right)\frac{d}{dt}\left(2t-t^4\right)-\left(2t-t^4\right)\frac{d}{dt}\left(1-2t^3\right)}{\left(1-2t^3\right)^2}\right].\frac{dx}{dt} \quad put\,(iii)$$

$$=\left[\frac{\left(1-2t^3\right)\left(2-4t^3\right)-\left(2t-t^4\right)\left(-6t^2\right)}{\left(1-2t^3\right)^2}\right]\times\left(\frac{3a\left(1-2t^3\right)}{\left(1+t^3\right)^2}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{3a\left(2-4t^3-4t^3+8t^6+12t^3-6t^6\right)}{\left(1-2t^3\right)\left(1+t^3\right)}$$

$$=\frac{3a\left(2t^6+4t^3+2\right)}{\left(1-2t^3\right)\left(1+t^3\right)}$$

Q.4: Find the indicated higher derivative of the following Functions : (a) $f(x)=\left(x^3+4x-5\right)^4,\ f^{iv}(x)$

Solution: $f(x)=\left(x^3+4x-5\right)^4$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(x^3+4x-5\right)^4$$

$$\Rightarrow f'(x)=4\left(x^3+4x-5\right)^3\frac{d}{dx}\left(x^3+4x-5\right)$$

$$=4\left(x^3+4x-5\right)^3\left(3x^2+4\right)$$

$$\frac{d}{dx}f'(x)=\frac{d}{dx}\left[4\left(x^3+4x-5\right)^3\left(3x^2+4\right)\right]$$

$$\Rightarrow f''(x)=4\left(x^3+4x-5\right)^3\frac{d}{dx}\left(3x^2+4\right)+4\left(3x^2+4\right)\frac{d}{dx}\left(x^3+4x-5\right)^3$$

$$\Rightarrow f''(x)=4\left(x^3+4x-5\right)^3(6x)+4\left(3x^2+4\right)3\left(x^3+4x-5\right)^2\left(3x^2+4\right)$$

$$\Rightarrow \quad f''(x) = 24x(x^3+4x-5)^3$$
$$+12(x^3+4x-5)^2(3x^2+4)^2$$

$$\Rightarrow \quad \frac{d}{dx} f''(x) = \frac{d}{dx}\left[24x(x^3+4x-5)^3\right]$$
$$+12\frac{d}{dx}\left[(x^3+4x-5)^2(3x^2+4)^2\right]$$

$$\Rightarrow f'''(x) = 24(x^3+4x-5)^3\frac{d}{dx}(x)+24x\frac{d}{dx}(x^3+4x-5)^3$$
$$+12(x^3+4x-5)^2\frac{d}{dx}(3x^2+4)^2+12(3x^2+4)^2\frac{d}{dx}(x^3+4x-5)^2$$

$$\Rightarrow f'''(x) = 24(x^3+4x-5)^3+72x(x^3+4x-5)^2(3x^2+4)$$
$$+24(x^3+4x-5)^2(3x^2+4)(6x)$$
$$+24(3x^2+4)^2(x^3+4x-5)(3x^2+4)$$

$$\Rightarrow f'''(x) = 24(x^3+4x-5)^3+72(x^3+4x-5)^2(3x^3+4x)$$
$$+144(x^3+4x-5)^2(3x^3+4x)+24(3x^2+4)^3(x^3+4x-5)$$

$$\frac{d}{dx} f'''(x) = 72(x^3+4x-5)^2(3x^2+4)+72(x^3+4x-5)^2(9x^2+4)$$
$$+144(x^3+4x-5)(3x^2+4)(3x^3+4x)+144(x^3+4x-5)^2(9x^2+4)$$
$$+288(x^3+4x-5)(3x^2+4)(3x^3+4x)+24(3x^2+4)^3(3x^2+4)$$
$$+72(3x^2+4)^2(6x)(x^3+4x-5)$$

$$f^{iv}(x) = 72(x^3+4x-5)^2(3x^2+4)+72(x^3+4x-5)^2(9x^2+4)$$
$$+144x(x^3+4x-5)(3x^2+4)^2+144(x^3+4x-5)^2(9x^2+4)$$
$$+288x(x^3+4x-5)(3x^2+4)^2+24(3x^2+4)^3(3x^2+4)$$
$$+432(3x^2+4)^2(x^3+4x-5)$$

(b) $f(x) = \tan^2 x \quad , \quad f''(x)$

Solution: $f(x) = \tan^2 x$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\tan^2 x \quad => \quad f'(x)=2\tan x\frac{d}{dx}\tan x$$

$$=> \quad f'(x)=2\tan x\sec^2 x \quad => \quad \frac{d}{dx}f'(x)=2\frac{d}{dx}(\tan x\sec^2 x)$$

$$=> \quad f''(x)=2\tan x\frac{d}{dx}(\sec^2 x)+2\sec^2 x\frac{d}{dx}(\tan x)$$

$$=> \quad f''(x)=2\tan x.\ 2\sec x.\sec x\tan x+2\sec^2 x.\sec^2 x$$

$$=> \quad f''(x)=4\tan^2 x.\ \sec^2 x+2\sec^4 x$$

$$=> \quad \frac{d}{dx}f''(x)=4\frac{d}{dx}(\tan^2 x.\ \sec^2 x)+2\frac{d}{dx}\sec^4 x$$

$$=> \quad f'''(x)=4\tan^2 x\frac{d}{dx}(.\ \sec^2 x)+4\ \sec^2 x\frac{d}{dx}(\tan^2 x)$$
$$+2(4)\sec^3 x\frac{d}{dx}\sec x$$

$$=> \quad f'''(x)=4\tan^2 x.\ \ 2\sec x.\sec x.\tan x+4\ \sec^2 x.2\tan x.\sec^2 x$$
$$+2(4)\sec^3 x.\sec x\tan x$$

$$=> \quad f'''(x)=8\tan^3 x\sec^2 x+8\ \sec^4 x\tan x$$
$$+8\sec^4 x.\tan x$$

$$=> \quad f'''(x)=8\tan^3 x\sec^2 x+16\ \sec^4 x\tan x$$

$$=> \quad f'''(x)=8\tan^3 x(1+\tan^2 x)+16\ (1+\tan^2 x)^2\tan x$$

(c) $f(x)=\frac{1}{2}e^{9x}$, $f''(x)$

Solution: $f(x)=\frac{1}{2}e^{9x}$

$$\frac{d}{dx}f(x)=\frac{1}{2}\frac{d}{dx}e^{9x} \quad => \quad f'(x)=\frac{1}{2}e^{9x}\frac{d}{dx}(9x)$$

$$=> \quad f'(x)=\frac{9}{2}e^{9x} \quad => \quad \frac{d}{dx}f'(x)=\frac{9}{2}\frac{d}{dx}e^{9x}$$

$$=> \quad f''(x)=\frac{9}{2}e^{9x}.(9) \quad => \quad f''(x)=\frac{9^2}{2}e^{9x}$$

$$\Rightarrow \frac{d}{dx}f''(x)=\frac{9^2}{2}\frac{d}{dx}e^{9x} \qquad \Rightarrow f'''(x)=\frac{9^3}{2}e^{9x}=\frac{729}{2}e^{3x}$$

(d) $f(x)=x^4.\ln(x^2)$, $f'''(x)$

Solution: $f(x)=x^4.\ln(x^2)$

$$\frac{d}{dx}f(x)=\frac{d}{dx}[x^4.\ln(x^2)]$$

$$\Rightarrow f'(x)=x^4\frac{d}{dx}\ln(x^2)+\ln(x^2)\frac{d}{dx}(x^4)$$

$$\Rightarrow f'(x)=x^4.\frac{1}{x^2}.\frac{d}{dx}(x^2)+\ln(x^2)4x^3$$

$$\Rightarrow f'(x)=x^4.\frac{1}{x^2}.2x+\ln(x^2)4x^3$$

$$\Rightarrow f'(x)=2x^3+4\ln(x^2)x^3$$

$$\Rightarrow \frac{d}{dx}f'(x)=2\frac{d}{dx}x^3+4\frac{d}{dx}[\ln(x^2)x^3]$$

$$\Rightarrow f''(x)=6x^2+4\ln(x^2)\frac{d}{dx}x^3+4x^3\frac{d}{dx}[\ln(x^2)]$$

$$\Rightarrow f''(x)=6x^2+4\ln(x^2)3x^2+4x^3.\frac{1}{x^2}.2x$$

$$\Rightarrow f''(x)=6x^2+12x^2\ln(x^2)+8x^2$$

$$\Rightarrow f''(x)=14x^2+12x^2\ln(x^2)$$

$$\Rightarrow \frac{d}{dx}f''(x)=14\frac{d}{dx}x^2+12\frac{d}{dx}[x^2\ln(x^2)]$$

$$\Rightarrow f'''(x)=14.(2x)+12x^2.\frac{d}{dx}[\ln(x^2))+12\ln(x^2)\frac{d}{dx}x^2$$

$$\Rightarrow f'''(x)=28x+12x^2.\frac{1}{x^2}.2x+12\ln(x^2)2x$$

$$\Rightarrow f'''(x)=28x+24x+24x\ln(x^2)$$

$$f'''(x)=52x+24x\ln(x^2)$$

Q.5: Use MAPLE command "diff" to find the indicated higher order derivative of the following functions.

(a) $f(x)=\sqrt{\sec 2x}$, $f''(x)$

Solution: Open MAPLE programme. Then in MAPPLE open new document and use keyboard keys and write

diff(sqrt(sec(2x)), x, x);

Command:

diff(sqrt(sec(2x)), x, x); **press enter key we get** $f''(x)$

$\sqrt{\sec(2x)}\tan(2x)^2+\sqrt{\sec(2x)}\left(2+2\tan(2x)^2\right)$

(b) $f(x)=\sin(\sin x)$, $f''(x)$

Solution: Open MAPLE programme. Then in MAPPLE open new document and use keyboard keys and write

diff(sin(sin(x)), x)

Command :

diff(sin(sin(x)), x) ; **Press enter key and get result**

cos(sin(x)) cos(x)

Q.6: **Find the indicated derivative of the following by using rule. (a)** $y=(3x+7)^{11}$, 7^{th} **derivative**

Solution: **The nth derivative of** $y=(ax+b)^m$ **is**

$$\Rightarrow y_n=a^n.\frac{m!}{(m-n)!}(ax+b)^{m-n} ------(i)$$

Put $a=3$, $b=7$, $m=11$, $n=7$ **in (i)**

$$\Rightarrow \quad y_7=\frac{11!}{(11-7)!}.3^7(3x+7)^{11-7}=\frac{11.10.9.8.7.6.5.4!}{4!}.3^7(3x+7)^4$$

$$=1663200.3^7(3x+4)^4$$

(b) $y=\ln(2x-4)$, 20^{th} **derivative**

Solution: $y=\ln(2x-4)=\ln(2x+(-4))$

The nth derivative of $\ln(ax+b)$ **is**

$$\Rightarrow \quad y_n=\frac{(-1)^{n-1}(n-1)!.a^n}{(ax+b)^n} \quad ......(i)$$

Put $a=2$, $b=-4$, $n=10$ **in (i)**

$$\Rightarrow \quad y_{10} = \frac{(-1)^{10-1}(10-1)!\,2^{10}}{(2x-4)^{10}}$$

$$\Rightarrow \quad y_{10} = \frac{(-1)^{9}.9!.2^{10}}{(2x-4)^{10}} = -9.8.7.6.5.4.3.2.2^{10}(2x-4)^{-10}$$

$$\Rightarrow \quad y_{10} = -2^{11}.(181440)(2x-4)^{-10} \quad Ans.$$

(c) $g(x) = 4\cos(3x+8)$ **, 6^{th} derivative**

<u>Solution</u>: The nth derivative of $y = c.\cos(ax+b)$ **is**

$$\Rightarrow \qquad y_n = c.a^n \cos\left(ax+b+\frac{n\pi}{2}\right) \text{......(i)}$$

Put $c=4$, , $a=3$, $b=8$, $n=6$ **in (i)**

$$y_6 = 4.3^6\cos\left(3x+8+\frac{6\pi}{2}\right) = 4.3^6\cos(3x+8+3\pi) \text{ Ans}$$

(d) $h(x) = 7e^{5x+4}$ **, 12^{th} derivative**

<u>Solution</u>: The nth derivative of $y = c.e^{ax+b}$ **is**

$$y_n = c.a^n e^{ax+b} \text{......(i)}$$

Put $c=7$, $a=5$, $b=4$, $n=12$ **in(i)**

$$\Rightarrow \qquad y_{12} = 7.5^{12}e^{5x+4} \quad \text{Ans.}$$

<u>Taylor series</u>: If $f(x)$ be function and has n – derivatives at $x = x_o$ Then

$$f(x) = f(x_o) + (x-x_o)f'(x_o) + \frac{(x-x_o)^2}{2!}f''(x_o) + \frac{(x-x_o)^3}{3!}f'''(x_o) + \text{.......................} + \frac{(x-x_o)^n}{n!}f^n(x_o) \quad \text{.......}(A)$$

OR

$$f(x) = f(x_o+h) = f(x_o) + hf'(x_o) + \frac{h^2}{2!}f''(x_o) + \frac{h^3}{3!}f'''(x_o) + \text{.......................} + \frac{h^n}{n!}f^n(x_o) \quad \text{.......}(B)$$

Where $x = x_o + h \quad \Rightarrow h = x - x_o$.

Eq (A) and (B) is called Taylor Series.

Example 8: The function $f(x)=e^x$ and its derivatives evaluated at $x_o=0$ are known by $f(0)=1$, $f'(0)=1$, $f''(0)=1$, $f'''(0)=1$, $f^{iv}(0)=1$. Use fourth order Taylor polynomial about $x_o=0$ to estimate $f(0.2)$ at $x=0.2$

Solution: As by Taylor series

$$f(x)=f(x_o)+(x-x_o)f'(x_o)+\frac{(x-x_o)^2}{2!}f''(x_o)+\frac{(x-x_o)^3}{3!}f'''(x_o)$$

$$+\frac{(x-x_o)^3}{3!}f'''(x_o)+\dots\dots\dots\dots(A)$$

Put $x_o=0$ in (A)

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)\dots\text{(B)}$$

Put $f(0)=1$, $f'(0)=1$, $f''(0)=1$, $f'''(0)=1$, $f^{iv}(0)=1$ in(B)

$$f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\dots\dots\dots(C)$$, put $x=0.2$ in(C)

$$f(0.2)=1+0.2+\frac{(0.2)^2}{2!}+\frac{(0.2)^3}{3!}+\frac{(0.2)^4}{4!}$$

$$=1+0.2+0.02+0.00133+0.00007=1.2214$$

Maclaurin,s series: Put $x_o=0$ in (A)

$$f(x)=f(0+h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(0)+\frac{h^3}{3!}f'''(0)$$

$$+\dots\dots\dots\dots+\frac{h^n}{n!}f^n(0)\quad\dots\dots(C)$$

OR

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)$$

$$+\dots\dots\dots\dots+\frac{x^n}{n!}f^n(0)\quad\dots\dots(D)$$

Eq (C) and eq (D) shows the macularian series.

Example 9: Use Taylor,s series to approximate the value of a function $f(x)=e^x$ at a point $x_o=2$

Solution: $f(x)=e^x$, $x_o=2$

Put $x=2$ => $f(2)=e^2=7.3891$

$f'(x)=e^x$ => $f'(2)=e^2=7.891$

$f''(x)=e^x$ => $f''(2)=e^2=7.891$

$f'''(x)=e^x$ => $f'''(2)=e^2=7.891$

And so on.

As by taylor series at $x_o=2$ **.**

$$f(x)=f(2)+(x-2)f'(2)+\frac{(x-2)^2}{2!}f''(2)+\frac{(x-3)^3}{3!}f'''(2)+....$$

Put values

$$e^x=7.3891+(x-2)(7.3891)+\frac{(x-2)^2}{2!}(7.3891)$$

$$+\frac{(x-3)^3}{3!}(7.3891)+.........$$

Example 10: **Use maclaurin series to approximate the value of a function** $f(x)=a^x$ **at a point** $x_o=0$

Solution: $f(x)=a^x$ => $f(0)=a^0=1$

$f'(x)=a^x\ln a$ => $f'(0)=a^0\ln a=\ln a$

$f''(x)=a^x(\ln a)^2$ => $f''(0)=a^0(\ln a)^2=(\ln a)^2$

$f'''(x)=a^x(\ln a)^3$ => $f'''(0)=a^0(\ln a)^3=(\ln a)^3$

And so on.

As by maclaurin,s series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+...... \quad ......(A)$$

Put values

$$=> a^x=1+x\ln a+\frac{x^2}{2!}(\ln a)^2+\frac{x^3}{3!}(\ln a)^3+...............$$

Example 11: **Use Maclaurin,s series to approximate the value of a function** $f(x)=e^x$ **at a point** $x_o=0$

Solution : $f(x)=e^x \quad => \quad f(0)=e^0=1$

$f'(x)=e^x \quad => \quad f'(0)=e^0=1$

$f''(x)=e^x \quad => \quad f''(0)=1$

$f'''(x)=e^x \quad => \quad f'''(0)=1$

And so on.

As by maclaurin, s series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots\dots \quad \dots\dots\textbf{(A)}$$

Put values

$$e^x=1+(x)(1)+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(1)+\dots\dots\dots$$

$$=> \quad e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\dots\dots \quad \textbf{Ans.}$$

Example 12: Use maclaurin series to approximate the value of a function $f(x)=\sin x$ at a point $x_o=0$

Solution: $f(x)=\sin x \quad => \quad f(0)=\sin 0=0$

$f'(x)=\cos x \quad => \quad f'(0)=\cos 0=1$

$f''(x)=-\sin x \quad => \quad f''(0)=-\sin 0=0$

$f'''(x)=-\cos x \quad => \quad f'''(0)=-\cos 0=-1$

And so on.

As by maclaurin, s series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots\dots \quad \dots\dots\textbf{(A)}$$

Put values

$$\sin x=0+(x)(1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(-1)+\dots\dots$$

$$=> \quad \sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\dots\dots$$

Example 13: Use maclaurin series to approximate the value of a function $f(x)=\cos x$ at a point $x_o=0$

Solution: $f(x)=\cos x \quad => \quad f(0)=\cos 0=1$

$$f'(x) = -\sin x \quad => \quad f'(0) = -\sin 0 = 0$$

$$f''(x) = -\cos x \quad => \quad f''(0) = -\cos 0 = -1$$

$$f'''(x) = \sin x \quad => \quad f'''(0) = \sin 0 = 0$$

$$f^{(iv)}(x) = \cos x \quad => \quad f^{(iv)}(0) = \cos 0 = 1$$

And so on .

As by maclaurin ,s series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots \quad \ldots\ldots\text{(A)}$$

Put values

$$\cos x = 1 + (x)(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \ldots\ldots$$

$$=> \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots$$

<u>Example 14</u>: Use maclaurin series to approximate the value of a function $f(x) = \tan x$ **at a point** $x_o = 0$

<u>Solution</u>: $f(x) = \tan x \quad => \quad f(0) = \tan 0 = 0$

$$f'(x) = \sec^2 x \quad => \quad f'(0) = \sec^2(0) = 1$$

$$f''(x) = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$$

$$=> \quad f''(0) = 2\sec^2(0)\tan 0 = 2(1)(0) = 0$$

$$f'''(x) = 2\sec^2 x.\sec^2 x + 2\sec x(\sec x \tan x)\tan x$$

$$= 2\sec^4 x + 2\sec^2 x \tan^2 x$$

$$=> \quad f'''(0) = 2\sec^4(0) + 2\sec^2(0)\tan^2(0)$$

$$= 2(1) + 2(1)(0) = 2$$

And so on

As by maclaurin , s series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots \quad \ldots\ldots\text{(A)}$$

Put values

$$\tan x = 0 + (x)(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + ...$$

$$=> \quad \tan x = x + \frac{2x^3}{3!} + ...$$

Example 15: **Use maclaurin series to approximate the value of a function $f(x) = \log_a(1+x)$ at a point $x_o = 0$**

Solution: $f(x) = \log_a(1+x)$

$$f(0) = \log_a(1+0) = \log_a 1$$

$$f'(x) = \frac{1}{1+x}\log_a e = (1+x)^{-1}\log_e a$$

$$f'(0) = \frac{1}{1+0}\log_a e = \log_a e$$

$$f''(x) = -(1+x)^{-2}\log_a e = -\frac{1}{(1+x)^2}\log_a e$$

$$f''(0) = -\frac{1}{(1+0)^2}\log_a e = -\log_a e$$

$$f'''(x) = (-1)(-2)(1+x)^{-3}\log_a e = \frac{2!}{(1+x)^3}\log_a e$$

$$f'''(0) = \frac{2!}{(1+0)^3}\log_a e = 2!\log_a e$$

And so on.

As by maclaurin, s series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... \quad ......(A)$$

Put values

$$\log_a(1+x) = \log_a 1 + x\log_a e + \frac{x^2}{2!}(-\log_a e) + \frac{x^3}{3!}2!\log_a e + ...$$

$$\log_a(1+x) = \log_a 1 + x\log_a e - \frac{x^2}{2}\log_a e + \frac{x^3}{3}\log_a e - ....$$

Example 16: **Use maclaurin series to approximate the value**

of a function $f(x)=\ln(1+x)$ at a point $x_o=0$

Solution: $f(x)=\ln(1+x) \Rightarrow f(0)=\ln(1+0)=0$

$$f'(x)=\frac{1}{1+x}=(1+x)^{-1} \Rightarrow f'(0)=\frac{1}{1+0}=1$$

$$f''(x)=-(1+x)^{-2}=-\frac{1}{(1+x)^2} \Rightarrow f''(0)=-\frac{1}{(1+0)^2}=-1$$

$$f'''(x)=(-1)(-2)(1+x)^{-3}=\frac{2!}{(1+x)^2}$$

$$\Rightarrow f'''(0)=\frac{2!}{(1+0)^2}=2!$$

And so on

As by maclaurin,s series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots \quad \ldots\ldots(A)$$

Put values

$$\ln(1+x)=0+(x)(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2!)+\ldots\ldots$$

$$\Rightarrow \ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots$$

Example.17: Use MAPLE command taylar for the function

(a) $f(x)=e^x$ by taylar series expansion to first four terms

(b) $f(x)=\sin x$ by taylar series expansion to first five terms

Solution: (a) Open MAPLE programme. Then in MAPPLE open new document and use keyboard keys and write $taylor(e\wedge x, x=0,4)$; then press enter kkey

Command :

$taylor(e^x, x=0,4);$

$$1+\ln(e)\,x+\frac{1}{2}\ln(e)^2x^2+\frac{1}{6}\ln(e)^3x^3+O(x^4)$$

(b) Open MAPLE programme. Then in MAPPLE open new document and use keyboard keys and write $taylor(\sin(x), x=0,5);$ press enter key

Command:

taylor(sin(*x*), *x* = 0, 5);

$x - \frac{1}{6}x^3 + O(x^5)$

Equation of the tangent line and normal line to the curve at given point :

The equation of the tangent line at the point $P(x_o, y_o)$ is

$y - y_o = f'(x_o)(x - x_o)$ **, where $m = f'(x_o)$ is the slope of the tangent line at the point $P(x_o, y_o)$ and $f'(x_o)$ is the derivative of $f(x)$ at point x_o .**

The equation of the normal line at the point $P(x_o, y_o)$ is

$$y - y_o = -\frac{1}{f'(x_o)}(x - x_o)$$

Example. 18: Find the equations of the tangent and normal lines on a curve $y = x^2$ at a point $P(2, 4)$.

Solution: $y = x^2$ **.....(i)**

Differentiate (i) w . r. t. to x .

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x$$

=> $f'(x) = 2x$ **Put** $x = 2$

=> $f'(2) = 2(2) = 4$ **, Thus** $m = 4$

Now equation of tangent line is

$y - y_o = m(x - x_o)$ **, put** $x_o = 2$, $y_o = 4$

=> $y - 4 = 4(x - 2)$

=> $y - 4 = 4x - 8$

=> $0 = 4x - y - 8 + 4$

=> $4x - y - 4 = 0$ *Ans.*

Now equation of normal line is

$y - y_o = -\frac{1}{m}(x - x_o)$ **put values**

$$\Rightarrow \quad y-4=-\frac{1}{4}(x-2)$$

$$\Rightarrow \quad 4y-16=-x+2$$

$$\Rightarrow \quad x+4y-16-2=0$$

$$\Rightarrow \quad x+4y-18=0 \quad Ans.$$

Example 19: Find the equations of the tangent and normal lines on a curve $y=9-x^2$ at a point, when y crosses the x – axis.

Solution: $y=9-x^2$ ......(i)

As y crosses x –axis then on the x – axis $y=0$

Put $y=0$ in (i)

$$0=9-x^2$$

$$\Rightarrow \quad x^2=9 \quad \Rightarrow \quad x=\pm 3$$

Thus we have points $A(3\ ,\ 0)$ *and* $B(-3\ ,0)$

Now differentiate (i) w . r. t. to x,

$$\frac{dy}{dx}=\frac{d}{dx}(9-x^2)$$

$$\Rightarrow \quad f'(x)=-2x \text{ ....}(ii)$$

At point $A(3\ ,\ 0)$: Put $x=3$ in (ii)

$$m=f'(3)=-2(3)=-6$$

Now equation of tangent line is

$$y-y_o=m(x-x_o)\ \text{, put } x_o=3\ ,\ y_o=0$$

$$y-0=-6(x-3)$$

$$\Rightarrow \quad y=-6x+18$$

$$\Rightarrow \quad 6x+y-18=0$$

Now equation of normal line is

$$y-y_o=-\frac{1}{m}(x-x_o) \quad \text{put values}$$

$\Rightarrow \quad y-0=-\frac{1}{-6}(x-3)$

$\Rightarrow \quad y=\frac{1}{6}(x-3)$

$\Rightarrow \quad 6y=x-3 \qquad \Rightarrow \quad 0=x-6y-3$

Or $\quad x-6y-3=0 \quad$ *Ans.*

At point $A(-3\ ,\ 0)$: **Put** $x=-3$ **in (ii)**

$m=f'(-3)=-2(-3)=6$

Now equation of tangent line is

$y-y_o=m(x-x_o)$ **, put** $x_o=-3\ ,\ y_o=0$

$y-0=6(x-(-3))$

$\Rightarrow \quad y=6x+18$

$\Rightarrow \quad 0=6x-y+18 \quad or \quad 6x-y+18=0 \quad$ *Ans.*

Now equation of normal line is

$y-y_o=-\frac{1}{m}(x-x_o)$ **put values**

$\Rightarrow \quad y-0=-\frac{1}{6}(x-(-3))$

$\Rightarrow \quad 6y=-(x+3)$

$\Rightarrow \quad 6y=-x-3$

$\Rightarrow \quad x+6y+3=0 \quad$ *Ans.*

The Angle of intersection of the two curves : If m_1 is the slope of the first curve and m_2 is the slope of the second curve then the angle of intersection between the two curves is $\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$ **.**

Example 20: Find the angle of intersection in between the curves $y=x^3-2x+1$ and $y=x^2+1$ at the point of intersection $(2\ ,\ 5)$.

Solution: $y=x^3-2x+1$ **....(i) and** $y=x^2+1$ **......(ii)**

Put (i) in (ii)

$$x^3 - 2x + 1 = x^2 + 1$$

$$\Rightarrow \quad x^3 - x^2 - 2x + 1 - 1 = 0$$

$$\Rightarrow \quad x^3 - x^2 - 2x = 0$$

$$\Rightarrow \quad x(x^2 - x - 2) = 0$$

$$\Rightarrow x = 0 \quad and \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad x^2 + x - 2x - 2 = 0$$

$$\Rightarrow \quad x(x+1) - 2(x+1) = 0$$

$$\Rightarrow \quad (x+1)(x-2) = 0$$

$$\Rightarrow \quad x + 1 = 0 \quad and \quad x - 2 = 0$$

$$\Rightarrow \quad x = -1 \quad and \quad x = 2$$

Thus $x = 0\ ,\ -1\ ,\ 2$.

Put $x = 0$ *in(i)* $\Rightarrow$ $y = 0 - 2(0) + 1 = 1$

Put $x = -1$ **in(i)** $\Rightarrow$ $y = (-1)^3 - 2(-1) + 1 = -1 + 2 + 1 = 2$

Put $x = 2$ **in (i)** $\Rightarrow$ $y = (2)^3 - 2(2) + 1 = 8 - 4 + 1 = 5$

Thus we have points of intersection $A(0\ ,\ 1)$, $B(-1\ ,\ 2)$ **and** $C(2\ ,\ 5)$.

Now we find tangent line equation at point $C(2\ ,\ 5)$.

Differentiate (i) w . r. t. to x .

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 2x + 1)$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 2 \quad ....(iii)$$

Put $x = 2$ **in (iii)**

$$m_1 = \left(\frac{dy}{dx}\right)_{(2\ ,\ 5)} = 3(2)^2 - 2 = 12 - 2 = 10$$

Now differentiate (ii) w . r . t. to x .

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 1) = 2x \quad ....(iv)$$

Put $x = 2$ in (iv)

$$m_2 = \left(\frac{dy}{dx}\right)_{(2,\ 5)} = 2(2) = 4$$

The angle of intersection between the two curves is

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \quad \text{Put } m_1 = 10,\quad m_2 = 4$$

$$\Rightarrow \tan\theta = \frac{10-4}{1+(10)(4)}$$

$$= \frac{6}{1+40} = \frac{6}{41}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{6}{41}\right) = 0.1453 \quad \textit{Ans.}$$

Example.21: Find all the points on the curve $y = 2x^3 + 4x^2$ where tangent line is parallel to the line $y = 8x - 4$.

Solution: Since the given line $y = 8x - 4$ .....(i)

Slope of the given line is $\frac{dy}{dx} = 8$

Given curve $y = 2x^3 + 4x^2$ ......(ii)

Differentiate (ii) w.r.t. to x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^3 + 4x^2\right)$$

$$\frac{dy}{dx} = 6x^2 + 8x \text{ ....(iii)}$$

Put $\frac{dy}{dx} = 8$ in (iii) $\Rightarrow 6x^2 + 8x = 8$

$\Rightarrow 6x^2 + 8x - 8 = 0$

$\Rightarrow 2\left(3x^2 + 4x - 4\right) = 0$ both sides divided by 2

$\Rightarrow 3x^2 + 4x - 4 = 0 \Rightarrow 3x^2 - 3x + 4x - 4 = 0$

$\Rightarrow 3x(x-1) + 4(x-1) = 0 \Rightarrow (x-1)(3x-1) = 0$

$\Rightarrow x - 1 = 0$ or $3x - 1 = 0$

$\Rightarrow x = 1 \quad or \quad 3x = 1$

$\Rightarrow x = 1 \quad or \quad x = \frac{1}{3}$

Put $x = 1$ in (ii) $\Rightarrow y = 2(1)^3 + 4(1)^2 = 2 + 4 = 6$

Put $x = \frac{1}{3}$ in (ii) $\Rightarrow y = 2\left(\frac{1}{3}\right)^3 + 4\left(\frac{1}{3}\right)^2$

$\Rightarrow y = \frac{2}{27} + \frac{4}{9} = \frac{6+4}{27} = \frac{10}{27}$

Thus the tangent line to the curve $y = 2x^3 + 4x^2$ is parallel to the line $y = 8x - 4$ at the points $(1, \; 6)$ *and* $\left(\frac{1}{3}, \frac{10}{27}\right)$

Exercise 4.2:

Q.1: In each case, find the equation of the tangent line to the curve at the indicated value of x:

(a) $y = \sqrt{x+1}$; $x = 3$

Solution: $y = \sqrt{x+1}$ .....(i) put $x = 3$ in(i)

$y = \sqrt{3+1} = \sqrt{4} = 2$

Thus the point is $P(3, \; 4)$.

Differentiate (i) w.r.t. to x,

$$\frac{dy}{dx} = \frac{d}{dx}(x+1)^{\frac{1}{2}}$$

$$\Rightarrow f'(x) = \frac{1}{2}(x+1)^{1-\frac{1}{2}} = \frac{1}{2}(x+1)^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x+1}} \text{........}(ii)$$

Put $x = 3$ in (ii) $f'(3) = \frac{1}{2\sqrt{3+1}} = \frac{1}{2\sqrt{4}} = \frac{1}{(2)(2)} = \frac{1}{4}$

Thus $m = \frac{1}{4}$

Now equation of tangent line is

$$y - y_o = m(x - x_o) \text{ , put } \quad x_o = 3 \text{ , } y_o = 2$$

$$y - 2 = \frac{1}{4}(x-3) \quad => \quad y = \frac{1}{4}(x-3) + 2$$

$$=> y = \frac{x-3+8}{4} \quad => y = \frac{x+5}{4} \quad Ans.$$

(b) $y = \sin(2x + \pi)$, $x = 0$

Solution: $y = \sin(2x+\pi)$ **.......(i)**

Put $x = 0$ **in (i)**

$$y = \sin(0+\pi) = \sin\pi = 0$$

Then the point is $P(0\ ,\ 0)$

Now differentiate (i) w . r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}\sin(2x+\pi)$$

$$=> f'(x) = 2\cos(2x+\pi) \qquad put \quad x = 0$$

$$=> m = f'(0) = 2\cos(0+\pi)$$

$$=> m = 2\cos\pi = 2(-1)$$

$$=> m = -2$$

Now equation of tangent line is

$$y - y_o = m(x - x_o) \text{ , put } x_o = 0 \text{ , } y_o = 0$$

$$=> y - 0 = -2(x-0)$$

$$=> y = -2x \quad Ans.$$

(c) $y = x^2e^{-x}$, $x = 1$

Solution: $y = x^2e^{-x}$ **....(i)** **put** $x = 1$ **in (i)**

$$y = (1)^2e^{-1} = \frac{1}{e}$$

Thus the point is $A\left(1\ ,\ \frac{1}{e}\right)$ **. Differentiate (i) w .r. t. to x .**

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2e^{-x}\right)$$

$$\Rightarrow f'(x)=x^2\frac{d}{dx}\left(e^{-x}\right)+e^{-x}\frac{d}{dx}\left(x^2\right)$$

$$\Rightarrow f'(x)=-x^2e^{-x}+2xe^{-x}\quad \text{....(ii)} \qquad put\quad x=1\ in(ii)$$

$$m=f'(1)=-(1)^2e^{-1}+2(1)e^{-1}$$

$$\Rightarrow m=-\frac{1}{e}+\frac{2}{e}=\frac{1}{e}$$

Now equation of tangent line is

$$y-y_o=m(x-x_o)\ \text{, put }\ x_o=1\ ,\ y_o=\frac{1}{e}$$

$$y-\frac{1}{e}=\frac{1}{e}(x-1)$$

$$\Rightarrow y=\frac{x}{e}-\frac{1}{e}+\frac{1}{e}$$

$$\Rightarrow y=\frac{x}{e}\quad Ans.$$

(d) $y=\dfrac{x}{x^2+1}$, $x=1$

Solution: $y=\dfrac{x}{x^2+1}$ ......(i) **put** $x=1$ **in (i)**

$$y=\frac{1}{(1)^2+1}=\frac{1}{1+1}=\frac{1}{2}$$

Thus the point is $P\left(1,\frac{1}{2}\right)$ **. Now**

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$$

$$\Rightarrow f'(x)=\frac{\left(x^2+1\right)\frac{d}{dx}(x)-x\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}$$

$$\Rightarrow \quad f'(x) = \frac{x^2+1-(x)(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$\Rightarrow \quad f'(x) = \frac{1-x^2}{(x^2+1)^2} \quad ....(ii) \qquad put \ x=1 \ \ in(i)$$

$$m = f'(1) = \frac{1-1}{(1+1)^2} = \frac{0}{4}$$

$$\Rightarrow \quad m = 0$$

Now equation of tangent line is

$$y - y_o = m(x - x_o) \text{ , put } \ x_o = 1 \ , \ y_o = \frac{1}{2}$$

$$\Rightarrow \quad y - \frac{1}{2} = 0(x-1)$$

$$\Rightarrow \quad y - \frac{1}{2} = 0$$

$$\Rightarrow \quad y = \frac{1}{2} \quad or \quad y = 0.5 \quad Ans.$$

<u>Q.2</u>: In each case, find the equation to the curve at the indicated value of x :

(a) $y = xe^x \ , \quad x = 1$

<u>Solution</u> : $y = xe^x$ **....(i) , put** $x = 1$ **in(i)**

$$y = 1.e^1 = e$$

Then the point is $P(1 \ , \ e)$. **Differentiate (i) w . r. t. to x ,**

$$\frac{dy}{dx} = \frac{d}{dx}(xe^x)$$

$$\Rightarrow \quad f'(x) = x\frac{d}{dx}e^x + e^x\frac{d}{dx}(x)$$

$$\Rightarrow \quad f'(x) = xe^x + e^x \qquad put \quad x = 1$$

$$m = f'(1) = 1.e^1 + e^1 \qquad \Rightarrow m = e + e = 2e$$

Now equation of normal line is

$$y-y_o=-\frac{1}{m}(x-x_o)\ .....\text{(ii)} \qquad \text{put } x_o=1\ ,\ y_o=e\ ,\ m=2e$$

$$y-e=-\frac{1}{2e}(x-1)$$

$$\Rightarrow\ y=\frac{-x+1}{2e}+e \quad \Rightarrow\ y=\frac{-x+1+2e^2}{2e} \quad Ans.$$

(b) $y=(2x+1)^6$, $x=0$

Solution : $y=(2x+1)^6$ ......(i) , $x=0$

Put $x=0$ **in (i)**

$$y=(2(0)+1)^5=(1)^5=1$$

Then the point is $P(0\ ,\ 1)$.

Differentiate w . r. t. to x,

$$\frac{dy}{dx}=\frac{d}{dx}(2x+1)^6=6(2x+1)^5\frac{d}{dx}(2x+1)$$

$$\Rightarrow\ f'(x)=12(2x+1)^5 \qquad put\ \ x=0$$

$$\Rightarrow\ m=f'(0)=12(0+1)^5=12$$

Now equation of normal line is

$$y-y_o=-\frac{1}{m}(x-x_o)\ .....\text{(ii)}$$

Put $x_o=0$, $y_o=1$ *and* $m=12$ **in (ii)**

$$y-1=-\frac{1}{12}(x-0)$$

$$\Rightarrow\ y=-\frac{x}{12}+1 \quad Ans.$$

(c) $y=\cos(x-\pi)$, $x=\frac{\pi}{2}$.

Solution: $y=\cos(x-\pi)$ .....(i) **put** $x=\frac{\pi}{2}$ **in (i)**

$$y = \cos\left(\frac{\pi}{2} - \pi\right) = \cos\left(-\frac{\pi}{2}\right) = 0$$

Then the point is $P\left(\frac{\pi}{2}\ ,\ 0\right)$. **Differentiate(i) w . r. t. to x .**

$$\frac{dy}{dx} = \frac{d}{dx}\cos(x - \pi)$$

$$\Rightarrow\ f'(x) = -\sin(x - \pi) \qquad put\ \ x = \frac{\pi}{2}$$

$$m = f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \pi\right)$$

$$\Rightarrow\ m = -\sin\left(-\frac{\pi}{2}\right) = -(-1)$$

$$\Rightarrow\ m = 1$$

Now equation of normal line is

$$y - y_o = -\frac{1}{m}(x - x_o)\ \ .....(ii)$$

Put $x_o = \frac{\pi}{2}$, $y_o = 0$ *and* $m = 1$ **in (ii)**

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$

$$\Rightarrow\ y = -x + \frac{\pi}{2} \quad Ans.$$

(d) $y = x^3 \ln x$, $x = 1$

<u>Solution</u>: $y = x^3 \ln x$ ......(i) **put** $x = 1$ **in (i)**

$$y = (1)^3 \ln 1 = 0$$

Then the point is $P(1\ ,\ 0)$. **Differentiate (i) w .r. t. to x .**

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^3\ln x\right)=x^3\frac{d}{dx}\ln x+\ln x\frac{d}{dx}\left(x^3\right)$$

$=>\ f'(x)=x^3.\frac{1}{x}+(\ln x)\left(3x^2\right)=x^2+3x^2\ln x \quad put\ x=1$

$=>\ m=f'(1)=(1)^2+3(1)^2\ln 1$

$=>\ m=1+3(0)=1$

Now equation of normal line is

$$y-y_o=-\frac{1}{m}\left(x-x_o\right)\ \text{.....(ii)}$$

Put $x_o=1$, $y_o=0$ *and* $m=1$ **in (ii)**

$y-0=-1(x-1)$

$=>\ y=-x+1 \quad$ *Ans.*

Q.3: (a) Find an equation of the tangent line to the curve $x^2+y^2=13$ at the point $(-2\ ,\ 3)$.

(b) Find an equation of the tangent line to the curve $\sin(x-y)=xy$ at $(0\ ,\ \pi)$.

(c) Find an equation of the normal line to the curve $x^2+2xy=y^3$ at $(1\ ,-1)$.

(d) Find an equation of the normal line to the curve $x^2\sqrt{y-2}=y^2-3x-1$ at $(1,\ 2)$

Solution: (a) : $x^2+y^2=13$ **......(i) at** $(-2\ ,\ 3)$.

Differentiate (i) w . r. t. to x

$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(13)$$

$=>\ 2x+2y\frac{dy}{dx}=0$

$=>\ 2y\frac{dy}{dx}=-2x$

$=>\ \frac{dy}{dx}=-\frac{x}{y}$ **put** $x=-2$, $y=3$

$$m = f'(-2) = -\frac{-2}{3} \qquad => \; m = \frac{2}{3}$$

Now equation of the tangent line is

$$y - y_o = m(x - x_o) \; \text{....(ii)}$$

Put $x_o = -2$, $y_o = 3$, $m = \frac{2}{3}$ **in (ii)**

$$y - 3 = \frac{2}{3}(x - (-2))$$

$$=> \; y + 3 = \frac{2}{3}(x + 2) \quad Ans.$$

(b) $\sin(x - y) = xy$ .....(i) **at** $(0 \,,\, \pi)$.

Differentiate (i) w. r. t. to x .

$$\frac{d}{dx}\sin(x - y) = \frac{d}{dx}(xy)$$

$$=> \; \cos(x - y)\frac{d}{dx}(x - y) = x\frac{dy}{dx} + y\frac{dx}{dx}$$

$$=> \; \cos(x - y)\left(\frac{dx}{dx} - \frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$=> \; \cos(x - y)\left(1 - \frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$=> \; \cos(x - y) - \cos(x - y)\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$=> \; \cos(x - y) - y = x\frac{dy}{dx} + \cos(x - y)\frac{dy}{dx}$$

$$or \quad (x + \cos(x - y))\frac{dy}{dx} = \cos(x - y) - y$$

$$\frac{dy}{dx} = \frac{\cos(x - y) - y}{x + \cos(x - y)} \quad \textbf{put} \;\; x = 0 \;\; and \;\; y = \pi$$

$$m = \left(\frac{dy}{dx}\right)_{(0\,,\,\pi)} = \frac{\cos(0 - \pi) - \pi}{0 + \cos(0 - \pi)}$$

$$\Rightarrow \quad m = \frac{\cos(-\pi) - \pi}{\cos(-\pi)} = \frac{-1-\pi}{-1} = 1+\pi$$

Now equation of the tangent line is

$$y - y_o = m(x - x_o) \quad \text{....(ii)}$$

Put $x_o = 0$, $y_o = \pi$, $m = 1+\pi$ **in (ii)**

$$y - \pi = (1+\pi)(x-0)$$

$$\Rightarrow \quad y - \pi = x(1+\pi) \quad \textit{Ans.}$$

(c) $x^2 + 2xy = y^3$ **at** $(1\ ,-1)$ **.**

$$\frac{d}{dx}\left(x^2 + 2xy\right) = \frac{d}{dx}\left(y^3\right)$$

$$\Rightarrow \quad \frac{d}{dx}\left(x^2\right) + 2\frac{d}{dx}(xy) = 3y^2\frac{dy}{dx}$$

$$\Rightarrow \quad 2x + 2\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 3y^2\frac{dy}{dx}$$

$$\Rightarrow \quad 2x + 2x\frac{dy}{dx} + 2y = 3y^2\frac{dy}{dx}$$

$$\Rightarrow \quad 2x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -2x - 2y$$

$$\Rightarrow \quad \left(2x - 3y^2\right)\frac{dy}{dx} = -(2x+2y)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x+2y}{2x-3y^2} \qquad \textit{put} \quad x = 1\ , \quad y = -1$$

$$m = \left(\frac{dy}{dx}\right)_{(1\ ,-1)} = -\frac{2(1)+2(-1)}{2(1)-3(-1)} = -\frac{2-2}{2+3}$$

$$\Rightarrow \quad m = 0$$

Now equation of normal line is

$$y - y_o = -\frac{1}{m}(x - x_o) \quad \text{.....(ii)}$$

Put $x_o = 1$, $y_o = -1$ *and* $m = \frac{1}{0}$ **in (ii)**

$$y-(-1)=-\frac{1}{0}(x-1)$$

$$\Rightarrow \quad 0(y+1)=-(x-1)$$

$$\Rightarrow \quad 0=-x+1$$

$$or \quad x=1 \qquad Ans.$$

(d) $x^2\sqrt{y-2}=y^2-3x-1$ **at** $(1,\ 2)$

Differentiate w.r.t. to x

$$\frac{d}{dx}\left(x^2(y-2)^{\frac{1}{2}}\right)=\frac{d}{dx}\left(y^2-3x-1\right)$$

$$\Rightarrow \quad x^2\frac{d}{dx}(y-2)^{\frac{1}{2}}+(y-2)^{\frac{1}{2}}\frac{d}{dx}x^2=\frac{d}{dx}y^2-3\frac{d}{dx}(x)-\frac{d}{dx}(1)$$

$$\Rightarrow \quad x^2.\frac{1}{2}(y-2)^{\frac{1}{2}-1}.\frac{d}{dx}(y-2)+(y-2)^{\frac{1}{2}}.2x=2y\frac{dy}{dx}-3$$

$$\Rightarrow \frac{1}{2}x^2(y-2)^{-\frac{1}{2}}.\frac{dy}{dx}+2x(y-2)^{\frac{1}{2}}=2y\frac{dy}{dx}-3$$

$$\Rightarrow \quad \frac{x^2}{2(y-2)^{\frac{1}{2}}}.\frac{dy}{dx}+2x(y-2)^{\frac{1}{2}}=2y\frac{dy}{dx}-3$$

$$\Rightarrow \quad \frac{x^2}{2(y-2)^{\frac{1}{2}}}.\frac{dy}{dx}-2y\frac{dy}{dx}=-2x(y-2)^{\frac{1}{2}}-3$$

$$\Rightarrow \quad \left(\frac{x^2}{2(y-2)^{\frac{1}{2}}}-2y\right)\frac{dy}{dx}=-2x(y-2)^{\frac{1}{2}}-3$$

$$\Rightarrow \quad \left(\frac{x^2-4y(y-2)^{\frac{1}{2}}}{2(y-2)^{\frac{1}{2}}}\right)\frac{dy}{dx}=-2x(y-2)^{\frac{1}{2}}-3$$

$$\Rightarrow \frac{dy}{dx}=\frac{2(y-2)^{\frac{1}{2}}\left(-2x(y-2)^{\frac{1}{2}}-3\right)}{x^2-4y(y-2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\text{(i)}$$

Put $x=1$ *and* $y=2$ **in (i)**

$$\frac{dy}{dx}=\frac{2(2-2)^{\frac{1}{2}}\left[-2(1)(2-2)^{\frac{1}{2}}-3\right]}{1-4(2)(2-2)^{\frac{1}{2}}}=\frac{2(0)(0-3)}{1-0}=\frac{0}{1}$$

$$\Rightarrow \frac{dy}{dx}=m=0$$

Now equation of normal line is

$$y-y_o=-\frac{1}{m}(x-x_o) \text{ .....(ii)}$$

Put $x_o=1$, $y_o=2$ *and* $m=\frac{1}{0}$ **in (ii)**

$$y-2=-\frac{1}{0}(x-1)$$

$$\Rightarrow 0(y-2)=-(x-1)$$

$$\Rightarrow 0=-x+1$$

$$or \quad x=1 \quad Ans.$$

Q.4: (a) Show that the first four terms in the taylor series expansion of $f(x)=\tan x$ about $x=\frac{\pi}{4}$ are :

$$1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\frac{\pi}{4}\right)^3$$

Solution: $f(x)=\tan x$

Put $x=\frac{\pi}{4} \Rightarrow f\left(\frac{\pi}{4}\right)=\tan\frac{\pi}{4}=1$

$$f'(x)=\frac{d}{dx}\tan x=\sec^2 x$$

Put $x=\frac{\pi}{4} \Rightarrow f'\left(\frac{\pi}{4}\right)=\sec^2\left(\frac{\pi}{4}\right)=\left(\sqrt{2}\right)^2=2$

$$f''(x)=2\sec x\sec x\tan x=2\sec^2 x\tan x$$

$$put \quad x=\frac{\pi}{4} \quad => \quad f''\left(\frac{\pi}{4}\right)=2\sec^2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)$$

$$=> \quad f''\left(\frac{\pi}{4}\right)=2\left(\sqrt{2}\right)^2(1)=4$$

$$f'''(x)=\frac{d}{dx}\left(2\sec^2 x\tan x\right)$$

$$=2\sec^2 x\frac{d}{dx}\tan x+2\tan x\frac{d}{dx}\sec^2 x$$

$$=> \quad f'''(x)=2\sec^2 x.\sec^2 x+2\tan x.(2)\sec x.\ \sec x\tan x$$

$$=> \quad f'''(x)=2\sec^4 x+4\tan^2 x\sec^2 x \qquad put \quad x=\frac{\pi}{4}$$

$$=> \quad f'''\left(\frac{\pi}{4}\right)=2\sec^4\left(\frac{\pi}{4}\right)+4\tan^2\left(\frac{\pi}{4}\right)\sec^2\left(\frac{\pi}{4}\right)$$

$$=2\left(\sqrt{2}\right)^2+4(1)^2\left(\sqrt{2}\right)^2=2(2)+4(2)$$

$$=> \quad f'''\left(\frac{\pi}{4}\right)=4+8=12$$

The first four terms by taylor series at $x=\frac{\pi}{4}$ is given by

$$f(x)=f\left(\frac{\pi}{4}\right)+\left(x-\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)$$

$$+\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right)\ ....(A)$$

Put values in (A)

$$\tan x=1+\left(x-\frac{\pi}{4}\right)(2)+\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}(4)+\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}(12)$$

$$=1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\frac{\pi}{4}\right)^3 \quad \text{Proved.}$$

(b) Show that the first four terms in the Taylor series expansion of $f(x)=\sqrt{x}$ about $x=4$ are

$$2+\frac{1}{2}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3$$

Solution: $f(x)=\sqrt{x}=x^{\frac{1}{2}}$

Put $x=4 \quad => \quad f(4)=\sqrt{4}=2$

$$f'(x)=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$$

Put $x=4 \quad => \quad f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{2.2}=\frac{1}{4}$

$$f''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}=-\frac{1}{4}x^{-\frac{3}{2}}=-\frac{1}{4x^{\frac{3}{2}}}$$

Put $x=4 \quad => \quad f''(4)=-\frac{1}{4(4)^{\frac{3}{2}}}=-\frac{1}{4(2)^3}=-\frac{1}{4.8}=-\frac{1}{32}$

$$f'''(x)=\left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1}=\frac{3}{8}x^{-\frac{5}{2}}=\frac{3}{8x^{\frac{5}{2}}}$$

Put $x=4 \quad => \quad f'''(4)=\frac{3}{8(4)^{\frac{5}{2}}}=\frac{3}{(8)(2)^5}=\frac{3}{256}$

The first four terms by taylor series at $x=4$ is given by

$$f(x)=f(4)+(x-4)f'(4)+\frac{(x-4)^2}{2!}f''(4)$$

$$+\frac{(x-4)^3}{3!}f'''(4)\;.........(A)$$

Put values in (A)

$$\sqrt{x}=2+(x-4)\left(\frac{1}{4}\right)+\frac{(x-4)^2}{2!}\left(-\frac{1}{32}\right)+\frac{(x-4)^3}{3!}\left(\frac{3}{256}\right)$$

$$=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3$$

(c) Show that the first four terms in the Taylor series expansion of $f(x)=x+e^x$ about $x=1$ are

$$(1+e)x+e\left[\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\frac{(x-1)^4}{4!}+\ldots\right]$$

Solution: $f(x)=x+e^x$

Put $f(1)=1+e^1=1+e$

$f'(x)=1+e^x \quad => \quad f'(1)=1+e^1=1+e$

$f''(x)=e^x \quad => \quad f''(1)=e^1=e$

$f'''(x)=e^x \quad => \quad f'''(1)=e^1=e$

$f^{iv}(x)=e^x \quad => \quad f^{iv}(1)=e^1=e$

The first four terms by taylor series at $x=1$ is given by

$$f(x)=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)$$

$$+\frac{(x-1)^3}{3!}f'''(1)+\frac{(x-1)^4}{4!}f^{iv}(x)\ \ldots \qquad \ldots\ldots\ldots(A)$$

Put values in (A)

$$f(x)=1+e+(x-1)(1+e)+\frac{(x-1)^2}{2!}(e)+\frac{(x-1)^3}{3!}(e)+\frac{(x-1)^4}{4!}(e)$$

$$=1+e+x+xe-1-e+\frac{(x-1)^2}{2!}(e)+\frac{(x-1)^3}{3!}(e)+\frac{(x-1)^4}{4!}(e)$$

$$=(1+e)x+e\left[\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\frac{(x-1)^4}{4!}+\ldots\right]$$

Q.5: Find the Maclaurin series expansion for the following functions :

(a) $f(x)=\dfrac{1}{1+x}$

Solution: $f(x)=\dfrac{1}{1+x} \quad \Rightarrow \quad f(0)=\dfrac{1}{1+0}=1$

$$f'(x)=\frac{d}{dx}(1+x)^{-1}=-(1+x)^{-2}=-\frac{1}{(1+x)^2}$$

Put $x=0 \quad \Rightarrow \quad f'(0)=-\dfrac{1}{(1+0)^2}=-\dfrac{1}{1}=-1$

$$f''(x)=(-1)(-2)(1+x)^{-3}=\frac{2}{(1+x)^3}$$

Put $x=0 \quad \Rightarrow f''(0)=\dfrac{2}{(1+0)^3}=2$

$$f'''(x)=2(-3)(1+x)^{-4}=-\frac{6}{(1+x)^4}$$

Put $x=0 \quad \Rightarrow \quad f'''(0)=-\dfrac{6}{(1+0)^4}=-6$

And so on.

Now by maclaurin series is given by

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots \quad \ldots\ldots.(A)$$

Put values in (A)

$$\frac{1}{1+x}=1+(x)(-1)+\frac{x^2}{2!}(2)+\frac{x^3}{3!}(-6)+\ldots\ldots..$$

$$=1-x+x^2-x^3+\ldots\ldots\ldots..$$

(b) $f(x)=\sin^2 x$

Solution: $f(x)=\sin^2 x$

Put $x=0 \quad \Rightarrow \quad f(0)=\sin^2 0=0$

$$f'(x)=2\sin x\cos x=\sin 2x$$

Put $x=0 \quad \Rightarrow \quad f'(0)=\sin 0=0$

$$f''(x)=2\cos 2x$$

Put $x=0$ => $f''(0)=2\cos 0=2(1)=2$

$f'''(x)=-4\sin 2x$

Put $x=0$ => $f'''(0)=-4\sin 0=-4(0)=0$

$f^{iv}(x)=-8\cos 2x$

Put $x=0$ => $f^{iv}(0)=-8\cos 0=-8$

And so on

Now by maclaurin series is given by

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots \quad \ldots\ldots\text{(A)}$$

Put values in (A)

$$\sin^2 x=0+(0)(x)+\frac{x^2}{2!}(2)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(-8)+\ldots\ldots$$

$$=x^2-\frac{x^4}{4.3.2.1}8+\ldots\ldots$$

$$=x^2-\frac{1}{3}x^4+\ldots\ldots$$

(c) $f(x)=\cosh x$

Solution: $f(x)=\cosh x$

Put $x=0$ => $f(0)=\cosh 0=1$

$f'(x)=\sinh x$

Put $x=0$ => $f'(0)=\sinh 0=0$

$f''(x)=\cosh x$

Put $x=0$ => $f''(0)=\cosh 0=1$

$f'''(x)=\sinh x$

Put $x=0$ => $f'''(0)=\sinh 0=0$

$f^{iv}(x)=\cosh x$

Put $x=0$ => $f^{iv}(0)=\cosh 0=1$

And so on.

Now by maclaurin series is given by

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots \quad \textbf{.......(A)}$$

Put values in (A)

$$\cosh x = 1+(x)(0)+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(1)+\ldots.$$

$$=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots\ldots..$$

(d) $f(x)=\ln(1-4x)$

<u>Solution</u> : $f(x)=\ln(1-4x)$

Put $x=0 \quad => \quad f(0)=\ln(1-0)=\ln 1=0$

$$f'(x)=\frac{-4}{1-4x}=-4(1-4x)^{-1}$$

put $x=0 \quad => \quad f'(0)=-\frac{4}{1-0}=-4$

$$f''(x)=-4(-1)(1-4x)^{-2}(-4)=-\frac{16}{(1-4x)^2}$$

put $x=0 \quad => \quad f''(0)=-\frac{16}{(1-0)^2}=-16$

$$f'''(x)=-16(-2)(1-4x)^{-3}(-4)=-\frac{128}{(1-4x)^3}$$

put $x=0 \quad => \quad f'''(0)=-\frac{64}{(1-0)^3}=-64$

And so on .

Now by maclaurin series is given by

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots \quad \textbf{.......(A)}$$

Put values in (A)

$$\ln(1-4x)=0+(x)(-4)+\frac{x^2}{2!}(-16)+\frac{x^3}{3!}(-128)+\ldots\ldots$$

$$=-4x-8x^2-\frac{64}{3}x^3-\ldots\ldots\ldots..$$

Q.6: (a) Use Maclaurin,s series for e^x to show that the sum of the infinite series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+......is\ \ e$

Solution : As $f(x)=e^x \Rightarrow f(0)=e^0=1$

$f'(x)=e^x \Rightarrow f'(0)=e^0=1$

$f''(x)=e^x \Rightarrow f''(0)=e^0=1$

$f'''(x)=e^x \Rightarrow f'''(0)=1$

And so on.

As by maclaurin , s series

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+...... \quad ......(A)$$

Put values

$$e^x=1+\frac{x}{1!}(1)+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(1)+..........$$

$$\Rightarrow e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.......... \quad ....(B)$$

Put $x=0$ in (B)

$$e^1=1+\frac{1}{1!}+\frac{(1)^2}{2!}+\frac{(1)^3}{3!}+.......$$

$$\Rightarrow e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.......... \quad (C)$$

(b) Use part (a) to find out the value of e that must be accurate to 4 decimal places.

From (C) $e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+..........$

$$\Rightarrow e=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}+\frac{1}{5040}...$$

$$\Rightarrow e=2+0.5+0.16666667+0.04166667+0.00833333$$
$$+0.00138889+0.00019841+....$$

$$\Rightarrow e=2.7183 \quad Ans$$

Q.7: Find the angle of intersection between the following curves.

(a) $x^2 - y^2 = a^2$, $x^2 + y^2 = a^2\sqrt{2}$

Solution: $x^2 - y^2 = a^2$ ....(i) and $x^2 + y^2 = a^2\sqrt{2}$ ......(ii)

First we find the point of intersection .

Adding (i) and (ii)

$$x^2 - y^2 = a^2$$

$$\frac{x^2 + y^2 = a^2\sqrt{2}}{2x^2 = \left(\sqrt{2}+1\right)a^2}$$

$$\Rightarrow \quad x^2 = \frac{\sqrt{2}+1}{2}a^2 \quad \Rightarrow \quad x = \sqrt{\frac{\sqrt{2}+1}{2}}a$$

Put $x^2 = \left(\frac{\sqrt{2}+1}{2}\right)a^2$ **in(i)**

$$\left(\frac{\sqrt{2}+1}{2}\right)a^2 - y^2 = a^2 \quad \Rightarrow \quad y^2 = \left(\frac{\sqrt{2}+1}{2} - 1\right)a^2$$

$$\Rightarrow \quad y^2 = \left(\frac{\sqrt{2}-1}{2}\right)a^2 \quad \Rightarrow \quad y = \sqrt{\frac{\sqrt{2}-1}{2}}a$$

Thus the points of intersection are $\left(\sqrt{\frac{\sqrt{2}+1}{2}}a \,,\, \sqrt{\frac{\sqrt{2}-1}{2}}a\right)$

Differentiate (i) w . r. t. to x ,

$$\frac{d}{dx}\left(x^2 - y^2\right) = \frac{d}{dx}a^2$$

$$\Rightarrow \quad 2x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad -2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{y} \quad ....(iii)$$

Then slope $m_1 = \frac{x}{y}$ **put values of** x *and* y

$$m_1 = \frac{\sqrt{\frac{\sqrt{2}+1}{2}}a}{\sqrt{\frac{\sqrt{2}-1}{2}}a} = \frac{\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}-1}}$$

$$m_1 = \sqrt{\frac{(\sqrt{2}+1)}{(\sqrt{2}-1)} \times \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}} = \sqrt{\frac{(\sqrt{2}+1)^2}{(\sqrt{2})^2-1}} = \sqrt{\frac{(\sqrt{2}+1)^2}{2-1}}$$

$$\Rightarrow m_1 = \sqrt{(\sqrt{2}+1)^2} = \sqrt{2}+1$$

Differentiate (ii) w.r.t. to x ,

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}a^2\sqrt{2}$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \qquad \Rightarrow \quad 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \quad ....(iv)$$

Now slope $m_2 = -\frac{x}{y} = -\frac{\sqrt{\frac{\sqrt{2}+1}{2}}a}{\sqrt{\frac{\sqrt{2}-1}{2}}a} = -\frac{\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}-1}}$

$$m_2 = -\sqrt{\frac{(\sqrt{2}+1)}{(\sqrt{2}-1)} \times \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}} = \sqrt{\frac{(\sqrt{2}+1)^2}{(\sqrt{2})^2-1}} = \sqrt{\frac{(\sqrt{2}+1)^2}{2-1}}$$

$$\Rightarrow m_2 = -\sqrt{(\sqrt{2}+1)^2} = -(\sqrt{2}+1)$$

Let θ is the angle between two lines then

$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$ **put values**

$$\tan\theta = \frac{-(\sqrt{2}+1)-(\sqrt{2}+1)}{1+((\sqrt{2}+1)(-(\sqrt{2}+1)))} = \frac{-2(\sqrt{2}+1)}{1-(\sqrt{2}+1)^2}$$

$$\Rightarrow \tan\theta = \frac{-2\left(\sqrt{2}+1\right)}{1-2-2\sqrt{2}-1} = \frac{-2\left(\sqrt{2}+1\right)}{-2\left(1+\sqrt{2}\right)} = 1$$

$$\Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$$

(b) $y^2 = ax \;,\quad x^3 + y^3 = 3axy$

Solution: $y^2 = ax$ **....(i) and** $x^3 + y^3 = 3axy$ **....(ii)**

From (i) $x = \dfrac{y^2}{a}$ **....(iii)**

Put (iii) in (ii)

$$\left(\frac{y^2}{a}\right)^3 + y^3 = 3a\left(\frac{y^2}{a}\right)y \Rightarrow \frac{y^6}{a^3} + y^3 = 3y^3$$

$$\Rightarrow \frac{y^6}{a^3} + y^3 - 3y^3 = 0 \quad \Rightarrow \frac{y^6}{a^3} - 2y^3 = 0$$

$$\Rightarrow y^6 - 2a^3y^3 = 0 \quad \Rightarrow y^3\left(y^3 - 2a^3\right) = 0$$

$$\Rightarrow y^3 = 0 \quad , \quad y^3 - 2a^3 = 0$$

$$\Rightarrow y = 0 \quad , \quad y^3 = 2a^3$$

$$\Rightarrow y = \left(2a^3\right)^{\frac{1}{3}} = 2^{\frac{1}{3}}a$$

$$\Rightarrow y = \sqrt[3]{2}\, a$$

Put $y = 0$ **in (iii)** $\Rightarrow \quad x = \dfrac{0}{a} = 0$

Put $y = (2)^{\frac{1}{3}}a$ **in (iii)**

$$x = \frac{\left(2^{\frac{1}{3}}a\right)^2}{a} = 2^{\frac{2}{3}}a$$

Thus the point of intersection are $(0\;,\;0)$ **and** $\left(2^{\frac{2}{3}}a\;,2^{\frac{1}{3}}a\right)$

Now differentiate (i) and (ii) w . r. t. x ,

$$\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}(ax)$$

$$\Rightarrow \quad 2y\frac{dy}{dx}=a \qquad \Rightarrow \quad \frac{dy}{dx}=\frac{a}{2y} \quad \text{......}(iv)$$

Now slpoe $m_1=\dfrac{a}{2y}$ **put** $y=2^{\frac{1}{3}}a$

$$\Rightarrow \quad m_1=\frac{a}{2\left(2^{\frac{1}{3}}a\right)}=\frac{1}{2^{\frac{1}{3}+1}}=\frac{1}{2^{\frac{4}{3}}}=2^{-\frac{4}{3}}$$

Now $\dfrac{d}{dx}\left(x^3+y^3\right)=\dfrac{d}{dx}(3axy)$

$$\Rightarrow \quad 3x^2+3y^2\frac{dy}{dx}=3a\left(x\frac{dy}{dx}+y\frac{dx}{dx}\right)$$

$$\Rightarrow \quad 3x^2+3y^2\frac{dy}{dx}=3ax\frac{dy}{dx}+3ay$$

$$\Rightarrow \quad 3y^2\frac{dy}{dx}-3ax\frac{dy}{dx}=-3x^2+3ay$$

$$\Rightarrow \quad 3\left(y^2-ax\right)\frac{dy}{dx}=3\left(ay-x^2\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{ay-x^2}{y^2-ax} \quad \text{.....(iv)}$$

Then slope $m_2=\dfrac{ay-x^2}{y^2-ax}$ **put values of x and y.**

$$m_2=\frac{a\left(2^{\frac{1}{3}}a\right)-\left(2^{\frac{2}{3}}a\right)^2}{\left(2^{\frac{1}{3}}a\right)^2-a\left(2^{\frac{2}{3}}a\right)}=\frac{2^{\frac{1}{3}}a^2-2^{\frac{4}{3}}a^2}{2^{\frac{2}{3}}a^2-2^{\frac{2}{3}}a^2}=\frac{2^{\frac{1}{3}}a^2-2^{\frac{4}{3}}a^2}{0}$$

Let θ **is the angle between two lines then**

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad \textbf{put values}$$

$$\Rightarrow \tan\theta = \frac{\left(\dfrac{2^{\frac{1}{3}}a^2 - 2^{\frac{4}{3}}a^2}{0}\right) - 2^{-\frac{4}{3}}}{1 + \left(\dfrac{2^{\frac{1}{3}}a^2 - 2^{\frac{4}{3}}a^2}{0}\right)\left(2^{-\frac{4}{3}}\right)} = \frac{\dfrac{\left(2^{\frac{1}{3}}a^2 - 2^{\frac{4}{3}}a^2\right) - (0)\left(2^{-\frac{4}{3}}\right)}{0}}{\dfrac{0 + \left(2^{\frac{1}{3}}a^2 - 2^{\frac{4}{3}}a^2\right)\left(2^{-\frac{4}{3}}\right)}{0}}$$

$$\Rightarrow \tan\theta = \frac{\left(2^{\frac{1}{3}}a^2 - 2^{\frac{4}{3}}a^2\right)}{\left(2^{\frac{1}{3}}a^2 - 2^{\frac{4}{3}}a^2\right)\left(2^{-\frac{4}{3}}\right)} = 2^{\frac{4}{3}}$$

$$\Rightarrow \tan\theta = \left(2^4\right)^{\frac{1}{3}} = \sqrt[3]{16}$$

$$\Rightarrow \theta = \tan^{-1}\left(\sqrt[3]{16}\right) \qquad \textit{Ans.}$$

Q.8: Find the points on the curvr $y = 5x^3 - 4x^2$, where tangent line is parallel to the line $y = 5x - 3$.

Solution: Since the given line $y = 5x - 3$ …..(i)

Slope of the given line is $\dfrac{dy}{dx} = 5$

Given curve $y = 5x^3 - 4x^2$ ……(ii)

Differentiate (ii) w.r.t. to x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(5x^3 - 4x^2\right)$$

$$\frac{dy}{dx} = 15x^2 - 8x \text{ …….(iii)}$$

Put $\dfrac{dy}{dx} = 5$ **in (iii)** $\Rightarrow 15x^2 - 8x = 5$

$\Rightarrow 15x^2 - 8x - 5 = 0$. **Then by quadratic formula**

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{a^2} \quad \text{put} \quad a=15\ ,\ b=-8,\ c=-5$$

$$\Rightarrow x=\frac{-(-8)\pm\sqrt{(-8)^2-4(15)(-5)}}{2(15)}=\frac{8\pm\sqrt{64+300}}{30}$$

$$\Rightarrow \quad x=\frac{8\pm\sqrt{364}}{30}=\frac{8\pm\sqrt{4\times 91}}{30}$$

$$\Rightarrow \quad x=\frac{8\pm 2\sqrt{91}}{30}=\frac{4\pm\sqrt{91}}{15}$$

hus $x=\dfrac{4-\sqrt{91}}{15}$ or $x=\dfrac{4+\sqrt{91}}{15}$

<u>Definition</u>: If $f(x)$ be a continuous function in an interval $(a,\ b)$ then

(i) $f(x)$ is strictly increasing in the interval if
$f(x_1)<f(x_2)$ **, for** $x_1<x_2$.

(ii) $f(x)$ is strictly decreasing in the interval if
$f(x_1)>f(x_2)$ **, for** $x_1<x_2$.

<u>Definition</u>: If a function $f(x)$ is differential in an interval $(a,\ b)$, Then

(i) $f(x)$ is strictly increasing in the interval if
$f'(x)>0$ **for all** $x\in(a\ ,b)$.

(ii) $f(x)$ is strictly decreasing in the interval if
$f'(x)<0$ **for all** $x\in(a\ ,b)$.

(iii) The function is neither increasing nor decreasing at the point $(c\ ,f(c))$ if $f'(c)=0$. Where $c\in(a\ ,b)$.

<u>Example.22:</u> Find the interval at which the function $f(x)=x^2$ is increasing or dcreasing.

<u>Solution</u>: $f(x)=x^2$ **, differentiate w.r.t. x**

$$\frac{d}{dx}(f(x))=\frac{d}{dx}(x^2)$$

=> $f'(x) = 2x$ .....(i)

Since $f(x)$ is increasing if $f'(x) > 0$ .Then from (i)

$f'(x) = 2x > 0$ if $x > 0$.

Thus $f(x)$ is increasing if $x > 0$ or $x = (0 \ ,\infty)$

Since $f(x)$ is decreasing if $f'(x) < 0$ .Then from (i)

$f'(x) = 2x < 0$ if $x < 0$.

Thus $f(x)$ is decreasing if $x < 0$ or $x = (-\infty \ , \ 0)$.

<u>Example 23</u>: Determine the value of x at which the function $f(x) = x^2 + 2x - 3$ is increasing or decreasing. Also find the point at which the given point is neither increasing nor decreasing.

<u>Solution</u>: $f(x) = x^2 + 2x - 3$ ......(i)

Differentiate (i) w .r. t. to x,

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(x^2 + 2x - 3\right)$$

=> $f'(x) = 2x + 2$

$= 2(x+1)$ ....(ii)

(i) we know that $f(x)$ is increasing if $f'(x) > 0$

Then $2(x+1) > 0$ => $x + 1 > 0$

=> $x > -1$

Thus $f(x)$ is increasing if $x > -1$.

Thus $f(x)$ is increasing in $(-1 \ , \ \infty)$

(ii) we know that $f(x)$ is decreasing if $f'(x) < 0$

Then $2(x+1) < 0$ => $x + 1 < 0$

=> $x < -1$

Thus $f(x)$ is decreasing if $x < -1$.

Thus $f(x)$ is decreasing in $(-\infty \ , \ -1)$

(iii)To find the point where the function is neither increasing nor decreasing then $f'(x) = 0$

$$2(x+1)=0 \quad => \quad x+1=0$$

$$=> \quad x=-1$$

Put $x=-1$ **in (i)**

$$f(-1)=(-1)^2+2(-1)-3=1-2-3$$

$$=> \quad f(-1)=-4$$

Thus the function $f(x)$ is neither increasing nor decreasing at the point $(-1\ ,-4)$.

<u>Relative maximum</u>: A function $f(x)$ is called relative maximum at a number c *if* $f(c)\geq f(x)$ for all x in open interval containing c.

<u>Relative maximum</u>: A function $f(x)$ is called relative minimum at a number d *if* $f(d)\leq f(x)$ for all x in open interval containing d.

The relative maxima or relative minima is called relative extrema.

<u>Critical Values and Critical point</u>: Let $f(x)$ be a function defined at a number c and either $f'(c)=0$ or $f'(c)$ does not exist. Then the number c is called a critical value of $f(x)$ and the point $P(c\ ,\ f(c))$ is called Critical point.

<u>Note</u>: If $f(c)$ does not defined then c is not a critical value.

(i) If there is relative maximum at c, then the functional value $f(c)$ at that point is the maximum value.

(ii) If there is relative minimum at c, then the functional value $f(c)$ at that point is the minimum value.

<u>Example 24</u>: Find the critical values for the following functions.

(a) $f(x)=4x^3-5x^2-8x+20$

<u>Solution</u>: $f(x)=4x^3-5x^2-8x+20$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(4x^3-5x^2-8x+20\right)$$

$$=> \quad f'(x)=12x^2-10x-8 \quad ....(i)$$

Since $f'(x)$ is defined for all x. To obtained the critical value then $f'(x)=0$

$$\Rightarrow 12x^2-10x-8=0 \quad \Rightarrow 12x^2+6x-16x^2-8=0$$
$$\Rightarrow 6x(2x+1)-8(2x+1)=0$$
$$\Rightarrow (2x+1)(6x-8)=0$$
$$\Rightarrow 2x+1=0 \quad , \quad 6x-8=0$$
$$\Rightarrow 2x=-1 \quad , \quad 6x=8$$
$$\Rightarrow x=-\frac{1}{2} \quad , \quad x=\frac{8}{6}=\frac{4}{3}$$

Thus the critical points are $x=-\frac{1}{2}, \frac{4}{3}$.

(b) $f(x)=\frac{x^2}{x-2}$

<u>Solution:</u> $f(x)=\frac{x^2}{x-2}$ ......(i) $\quad \frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^2}{x-2}\right]$

$$\Rightarrow f'(x)=\frac{(x-2)\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(x-2)}{(x-2)^2}$$
$$\Rightarrow f'(x)=\frac{(x-2)(2x)-x^2}{(x-2)^2}$$
$$\Rightarrow f'(x)=\frac{2x^2-4x-x^2}{(x-2)^2} \Rightarrow f'(x)=\frac{x^2-4x}{(x-2)^2} \quad ....(i)$$

Since $f'(x)$ does not defined for $x=2$, but $f(x)$ also does not defined at $x=2$.
Thus $x=2$ is not critical value. For critical value $f'(x)=0$

$$\Rightarrow \frac{x^2-4x}{(x-2)^2}=0 \quad \Rightarrow x^2-4x=0$$

$\Rightarrow \quad x(x-4)=0$

$\Rightarrow \quad x=0 \quad , \; x-4=0$

$\Rightarrow \quad x=4$

Thus the critical value is $x=0, \quad 4.$

(c) $f(x)=12x^{\frac{1}{2}}-2x^{\frac{3}{2}}$

<u>Solution</u>: $f(x)=12x^{\frac{1}{2}}-2x^{\frac{3}{2}}$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(12x^{\frac{1}{2}}-2x^{\frac{3}{2}}\right)$$

$$\Rightarrow \quad f'(x)=(12)\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}-2\left(\frac{3}{2}\right)x^{\frac{3}{2}-1}$$

$$\Rightarrow \quad f'(x)=6x^{-\frac{1}{2}}-3x^{\frac{1}{2}}$$

$$\Rightarrow \quad f'(x)=\frac{6}{\sqrt{x}}-3\sqrt{x} \quad \text{....(i)}$$

Since $f'(x)$ **does not exist for** $x=0$ **, but** $f(x)$ **is defined for** $x=0$ **. Thus** $x=0$ **is the critical value .**

Now for other critical value $f'(x)=0$

$$\frac{6}{\sqrt{x}}-3\sqrt{x}=0 \qquad \Rightarrow \frac{6-3x}{\sqrt{x}}=0$$

$\Rightarrow \quad 6-3x=0$

$\Rightarrow \quad 6=3x \qquad or \quad x=2$

Thus the critical values are $x=0 \quad , \quad 2.$

(d) $f(x)=3x^{\frac{4}{3}}-12x^{\frac{1}{3}}$

<u>Solution</u>: $f(x)=3x^{\frac{4}{3}}-12x^{\frac{1}{3}}$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(3x^{\frac{4}{3}}-12x^{\frac{1}{3}}\right)$$

$$\Rightarrow \quad f'(x)=3\left(\frac{4}{3}\right)x^{\frac{4}{3}-1}-12\left(\frac{1}{3}\right)x^{\frac{1}{3}-1}$$

$$\Rightarrow \quad f'(x)=4x^{\frac{1}{3}}-4x^{-\frac{2}{3}}$$

$$\Rightarrow \quad f'(x)=4x^{\frac{1}{3}}-\frac{4}{x^{\frac{2}{3}}} \quad \ldots.(i)$$

Since $f'(x)$ does not exist for $x=0$, but $f(x)$ is defined for $x=0$. Thus $x=0$ is the critical value.

Now for other critical value $f'(x)=0$

$$4x^{\frac{1}{3}}-\frac{4}{x^{\frac{2}{3}}}=0 \quad \Rightarrow \quad \frac{4x^{\frac{1}{3}}.x^{\frac{2}{3}}-4}{4x^{\frac{2}{3}}}=0$$

$$\Rightarrow \quad 4x^{\frac{1}{3}+\frac{2}{3}}-4=0 \quad \Rightarrow \quad 4x-4=0$$

$$\Rightarrow \quad 4x=4 \qquad or \quad x=1$$

Thus the critical values are $x=0\ ,\ 1.$

(e) $f(x)=6x^{\frac{2}{3}}-4x$

Solution: $f(x)=6x^{\frac{2}{3}}-4x$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(6x^{\frac{2}{3}}-4x\right)$$

$$\Rightarrow \quad f'(x)=6\left(\frac{2}{3}\right)x^{\frac{2}{3}-1}-4$$

$$\Rightarrow \quad f'(x)=4x^{-\frac{1}{3}}-4$$

=> $f'(x) = \frac{4}{x^{\frac{1}{3}}} - 4$ ....(i)

Since $f'(x)$ does not exist for $x = 0$, but $f(x)$ is defined for $x = 0$. Thus $x = 0$ is the critical value.

Now for other critical value $f'(x) = 0$

$\frac{4}{x^{\frac{1}{3}}} - 4 = 0$ => $\frac{4 - 4x^{\frac{1}{3}}}{x^{\frac{1}{3}}} = 0$

=> $4 - 4x^{\frac{1}{3}} = 0$ => $4x^{\frac{1}{3}} = 4$

=> $x^{\frac{1}{3}} = 1$ *or* $x = 1$

Thus the critical values are $x = 0$, 1.

Theorem: If a continuous function $f(x)$ has a relative extremum at c, then c must be a critical value of $f(x)$.

Example 25: The function $f(x)$ is defined by $f(x) = x^3 - 3x^2 - 9x + 1$. Determine the intervals at which the function $f(x)$ is strictly increasing or decreasing.

Solution: $f(x) = x^3 - 3x^2 - 9x + 1$ .....(i)

Differentiate (i) w.r.t. to x.

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(x^3 - 3x^2 - 9x + 1\right)$$

=> $f'(x) = 3x^2 - 6x - 9$

=> $f'(x) = 3\left(x^2 - 2x - 3\right) = 3\left[x^2 + x - 3x - 3\right]$

$= 3\left[x(x+1) - 3(x+1)\right]$

=> $f'(x) = 3(x+1)(x-3)$ ....(ii)

For critical value put $f'(x) = 0$

$3(x+1)(x-3) = 0$

=> $x + 1 = 0$, $x - 3 = 0$

$\Rightarrow x = -1 \quad , \; x = 3$

Thus the critical value is $x = -1$, 3

To find the interval where $f(x)$ is increasing or decreasing we take some points left to and right to critical values.

i.e.

We take a point left to -1 and a point right to -1.

We also take a point left to 3 and a point right to 3.

We take -2 left to -1 and 0 right to -1.

Put $x = -2$ in (ii)

$$f'(-2) = 3(-2+1)(-2-3) = 3(-1)(-5) = 15 > 0$$

Since derivative of $f(x)$ is positive to the left of -1.

Thus $f(x)$ is increasing when $x < -1$

Put $x = 0$ in (ii)

$$f'(0) = 3(0+1)(0-3) = -9 < 0$$

Now we take 2 to the left of 3.

Put $x = 2$ in (ii)

$$f'(2) = 3(2+1)(2-3) = 3(3)(-1) = -9 < 0$$

Thus right to -1 and left to 3 the derivative of $f(x)$ is negative. Thus $f(x)$ is decreasing in the interval $-1 < x < 3$.

Now we take 4 to the right of 3.

Put $x = 3$ in (ii)

$$f'(4) = 3(4+1)(4-3) = 3(5)(1) = 15 > 0$$

Since derivative of $f(x)$ is positive to the right of 3.

Thus $f(x)$ is increasing when $x > 3$

Thus $f(x)$ is increasing in the intervals for $x < -1$ and for $x > 3$.

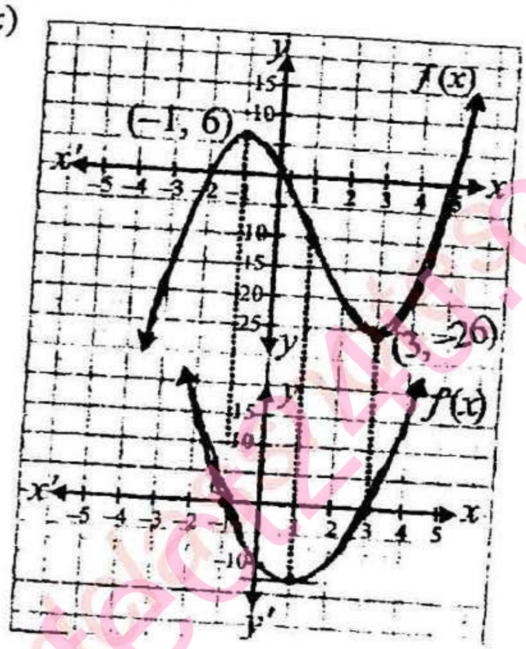
$f(x)$ is decreasing in the interval for $-1 < x < 3$.

<u>Example 26</u>: Draw the function $f(x) = x^3 - 3x^2 - 9x + 1$ and its derivatives $f'(x) = 3x^2 - 6x - 9$. Use these graphs to tell about the following questions.

(a) When $f'(x)$ is positive. What does that mean in terms of the graph of $f(x)$?.

(b) When the graph of $f(x)$ is decreasing . what does that mean in terms of the graph of $f(x)$?.

Solution: The graph of $f(x)$ and $f'(x)$ are shown in figure .



(a) If $f'(x) > 0$ then $f(x)$ is increasing function .

(b) $f(x)$ is decreasing when $f'(x) < 0$.

First derivative test :

The first derivative of a function can be used to determine whether the function is increasing or decreasing in a given interval . The main steps are following .

(i) Find those values of c in which $f'(c) = 0$ of $f'(c)$ does not exist but $f(c)$ is defined . i.e. Find all the critical values of $f(x)$.

(ii) The point $(c, \ f(c))$ is a relative maximum if $f'(x) > 0$ for all x in the open interval $(a \ , c)$ to the left of c , and $f'(x) < 0$ for all x in the open interval $(c \ , \ b)$ to the right of c.

(iii) The point $(c,\ f(c))$ is a relative minimum if $f'(x)<0$ for all x in the open interval $(a\ ,c)$ to the left of c, and $f'(x)>0$ for all x in the open interval $(c\ ,\ b)$ to the right of c.

(iv) The point $(c,\ f(c))$ is not a extremum if $f'(x)$ has the same sign in both open interval $(a\ ,c)$ and $(c\ ,\ b)$.

Example 27: Examine the function $f(x)=2x^3+3x^2-12x-5$ for the relative extrema using first –derivative test.

Solution: $f(x)=2x^3+3x^2-12x-5$ ……(i)

Differentiate (i) w .r. t. to x .

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(2x^3+3x^2-12x-5\right)$$

$$\Rightarrow\quad f'(x)=6x^2+6x-12$$

$$\Rightarrow\quad f'(x)=6\left(x^2+x-2\right)$$

$$=6\left(x^2-x+2x-2\right)$$

$$\Rightarrow\quad f'(x)=6[x(x-1)+2(x-1)]$$

$$\Rightarrow\quad f'(x)=6(x-1)(x+2)\quad\text{………(ii)}$$

For critical value put $f'(x)=0$

$$6(x-1)(x+2)=0$$

$$x-1=0\quad,\quad x+2=0$$

$$\Rightarrow x=1\quad,\ x=-2$$

Thus the critical value is $x=1\ ,-2$.

To find the interval where $f(x)$ is increasing or decreasing we take some points left to and right to critical values. i.e.

We take a point left to -2 and a point right to -2.

We also take a point left to 1 and a point right to 1.

We take -3 left to -2 and 0 right to -2.

Put $x=-3$ in (ii)

$$f'(-3)=6(-3+2)(-3-1)=6(-1)(-4)=24>0$$

Put $x=0$ in (ii)

$f'(0) = 6(0+2)(0-1) = -12 < 0$

Thus the value of $f'(x)$ is positive to the left of -2 and negative to the right of -2.

Thus $x = -2$ gives a relative maximum point.

Put $x = -2$ in(i)

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) - 5$$
$$= -16 + 12 + 24 - 5$$
$$= 15$$

Now we take 0 left to 1 and 2 right to 1.

Put $x = 0$ in (ii)

$$f'(0) = 6(0+2)(0-1) = -12 < 0$$

Put $x = 2$ in (ii)

$$f'(2) = 6(2+2)(2-1) = 24 > 0$$

Thus the value of $f'(x)$ is negative to the left of 1 and positive to the right of 1.

Thus $x = 1$ gives a relative minimum point.

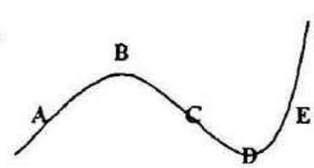
Put $x = 1$ in (i)

$$f(1) = 2(1)^3 + 3(1)^2 - 12(1) - 5$$
$$= 2 + 3 - 12 - 5 = -12$$

Thus $f(x)$ has a relative maximum at $(-2, 15)$ and a relative minimum at $(1, -12)$.

<u>Definition: Concavity</u>: A portion of a graph that is cupped upward is called concave up and a portion of a graph that is cupped downward is called concave down.

In figure A to C is concave down and from C to E concave up.



<u>Note</u>: The graph of a function $f(x)$ is concave up in the

open interval (a, b) if $f''(x) > 0$ for all $x \in (a, b)$ and it is concave down ward if $f''(x) < 0$ for all $x \in (a, b)$.

Inflection point: The point $P(c, f(c))$ on the graph of function $y = f(x)$ at which the concavity changes from upward to downward or downward to upward is called point of inflection.

If $y = f(x)$ is continuous on (a, b) and $x = c$ is an inflection point then either $f''(c) = 0$ or $f''(c)$ does not exist.

Second derivative test: Let $f(x)$ be a function and has first and second derivatives of $f(x)$ exist on an open interval (a, b). If $c \in (a, b)$ such that $f'(c) = 0$, then

(i) if $f''(c) < 0$, then there is relative maximum at $x = c$ and the graph of $f(x)$ is concave downward in the neighbourhood of $P(c, f(c))$.

(ii) if $f''(c) > 0$, then there is relative minimum at $x = c$ and the graph of $f(x)$ is concave upward in the neighbourhood of $P(c, f(c))$.

(iii) If $f''(c) = 0$ then the second derivative test is fails.

Note: The second derivative test only works for those critical values for which $f'(c) = 0$. It does not work for those critical values for which $f'(c)$ does not exist or $f''(c) = 0$. If $f'(c)$ does not exist or $f''(c) = 0$ then find the relative extrema use first derivative test.

Example 28: Use the second derivative test to determine whether each critical value of the function $f(x) = 3x^5 - 5x^3 + 2$ corresponds to a relative maximum, a relative minimum, or neither.

Solution: $f(x) = 3x^5 - 5x^3 + 2$ ....(i)

The first derivative of $f(x)$ is given by

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(3x^5-5x^3+2\right)$$

$$\Rightarrow f'(x)=15x^4-15x^2 \quad ....(ii)$$

The 2^{nd} derivative of $f(x)$ is given by

$$\frac{d}{dx}f'(x)=\frac{d}{dx}\left(15x^4-15x^2\right)$$

$$\Rightarrow f''(x)=60x^3-30x \quad ......(iii)$$

For critical values put $f'(x)=0$ in (ii)

$$15x^4-15x^2=0$$

$$\Rightarrow 15x^2\left(x^2-1\right)=0$$

$$\Rightarrow x^2=0 \ , \ x^2-1=0$$

$$\Rightarrow x=0 \ , \ x^2=1$$

$$\Rightarrow x=\pm1$$

Thus critical values is $x=0$, -1 , 1

Put $x=0$ in (iii)

$$f''(0)=(60)(0)-(30)(0)=0$$

Thus the second derivative test is failed at $x=0$

Put $x=-1$ in (iii)

$$f''(-1)=60(-1)^3-30(-1)=-60+30=-30<0$$

Thus the relative maximum at $x=-1$

Put $x=-1$ in (i)

$$f(-1)=3(-1)^5-5(-1)^3+2=-3+5+2=4$$

Thus the relative maximum value of $f(x)=4$ at $x=-1$

Now put $x=1$ in (iii)

$$f''(1)=60(1)^3-30(1)=60-30=30>0$$

Thus the relative minimum at $x=1$

Put $x=1$ in (i)

$$f(1)=3(1)^5-5(1)^3+2=3-5+2=0$$

Thus the relative minimum value of $f(x)=0$ at $x=1$.

Example.30: Use MAPLE command to compute to maximize and minimize value of the following functions.

(a) $f(x)=\cos x$ (b) $f(x)=x^4-2x^2+3$ in interval $[-1,\ 2]$

Solution: (a) Open the MAPLE programme and then open new document. Use keyboard keys and write

$maximize(\cos(x));$ and then press enter key

Command:

$> \ maximize(\cos(x));$ 1

For minimize Use keyboard keys and write

$minimize(\cos(x));$ and then press enter key

Command:

$> \ minimize(\cos(x));$ -1

(b) Open the MAPLE programme and then open new document. Use keyboard keys and write

$\max imize(x\char`^4-2*x\char`^2+3,x=1..2);$ press enter key.

Command:

$> \ maximize(x^4-2\cdot x^2+3,x=-1..2);$ 11

For minimize Use keyboard keys and write

$\min imize(x\char`^4-2*x\char`^2+3,x=1..2);$ press enter key.

Command:

$> \ minimize(x^4-2\cdot x^2+3,x=-1..2);$ 2

Exercise 4.3:

Q.1: Find the critical values of the given functions and show where the functions is increasing and where it is decreasing.

(a) $f(x)=x^3+3x^2+1$

Solution: $f(x)=x^3+3x^2+1$ ....(i)

Differentiate w. r. t. to x.

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(x^3+3x^2+1\right)$$

$=> \ f'(x)=3x^2+6x$ ......(ii)

For critical value :

put $f'(x) = 0$ **in (ii)**

$3x^2 + 6x = 0$

$\Rightarrow 3x(x+2) = 0$

$\Rightarrow x = 0 \quad , \quad x + 2 = 0$

$\Rightarrow \quad x = -2$

Thus the critical value is $x = 0 \, , \, -2$.

To find the interval where $f(x)$ is increasing or decreasing we take some points left to and right to critical values . i.e.

We take a point left to 0 and a point right to 0 .

We also take a point left to -2 and a point right to -2.

We take - 1 to the left of 0 .

Put $x = -1$ in (ii)

$f'(-1) = 3(-1)^2 + 6(-1) = 3 - 6 = -3 < 0$

Thus the function is decreasing in the interval $x < 0$.

We take 1 to the right of 0 .

Put $x = 1$ in (ii)

$f'(1) = 3(1)^2 + 6(1) = 9 > 0$

Thus the function is increasing in the interval $x > 0$.

We take - 3 left to -2 and -1 right to -2 .

Put $x = -3$ in (ii)

$f'(-3) = 3(-3)^2 + 6(-3) = 27 - 18 = 9 > 0$

Thus $f(x)$ is increasing in the interval $x < -2$ to the left of -2 .

We take - 1 to the right of -2 .

Put $x = -1$ in (ii)

$f'(-1) = 3(-1)^2 + 6(-1) = 3 - 6 = -3 < 0$

Thus $f(x)$ is decreasing to the left of 0 and to right to -2.

i.e. $f(x)$ is decreasing in the interval $(-2 \;\; , 0)$

and $f(x)$ is increasing when $x < -2$ and $x > 0$ or

$(-\infty \;\; -2) U (0 \;,\; \infty)$.

(b) $f(x) = x^3 + 35x^2 - 125x + 9375$

Solution :

$f(x) = x^3 + 35x^2 - 125x + 9375 \ldots\ldots\text{(i)}$

Differentiate w . r. t. to x .

$\frac{d}{dx}f(x) = \frac{d}{dx}\left(x^3 + 35x^2 - 125x + 9375\right)$

$\Rightarrow f'(x) = 3x^2 + 70x - 125 \quad \ldots\ldots\text{(ii)}$

For critical value

put $f'(x) = 0$ **in (ii)**

$3x^2 + 70x - 125 = 0$

$\Rightarrow 3x^2 + 75x - 5x - 125 = 0$

$\Rightarrow 3x(x+25) - 5(x+25) = 0$

$\Rightarrow (x+25)(3x-5) = 0$

$\Rightarrow x+25 = 0 \;, 3x-5 = 0$

$\Rightarrow x = -25 \quad , x = \frac{5}{3}$

Thus the critical value is $x = -25 \;, \frac{5}{3}$.

To find the interval where $f(x)$ is increasing or decreasing we take some points left to and right to critical values . i.e.

We take a point left to -25 and a point right to -25 .

We also take a point left to $\frac{5}{3}$ and a point right to $\frac{5}{3}$.

We take 1 to the left of $\frac{5}{3}$.

Put $x = 1$ in (ii)

$f'(1) = 3(1)^2 + 70(1) - 125 = 3 + 70 - 125 = -52 < 0$

Thus the function is decreasing in the interval $x < \frac{5}{3}$.

We take 2 to the right of $\frac{5}{3}$.

Put $x = 2$ in (ii)

$f'(2) = 3(2)^2 + 70(2) - 125 = 12 + 140 - 125 = 27 > 0$

Thus the function is increasing in the interval $x > \frac{3}{5}$.

We take -26 left to -25 and -24 right to -25.

Put $x = -26$ in (ii)

$$f'(-26) = 3(-26)^2 + 70(-26) - 125$$
$$= 2028 - 1820 - 125 = 83 > 0$$

Thus $f(x)$ is increasing in the interval $x < -25$ to the left of -25.

We take -24 to the right of -25.

Put $x = -24$ in (ii)

$$f'(-24) = 3(-24)^2 + 70(-24) - 125$$
$$= 1728 - 1680 - 125 = -77 < 0$$

Thus $f(x)$ is decreasing in the interval $x > -25$

From above $f(x)$ is decreasing in the interval $x < \frac{5}{3}$

and $x > -25$. i.e. $\left(-25 \ , \frac{5}{3}\right)$.

and $f(x)$ is increasing when $x > \frac{5}{3}$ and $x < -25$ or

$(-\infty \ -25) U \left(\frac{5}{3} \ , \infty\right)$.

<u>Q.2</u>: Find the critical values of the following functions.

(a) $f(x) = 2x^3 - 3x^2 - 72x + 15$

<u>Solution</u>: $f(x) = 2x^3 - 3x^2 - 72x + 15$ ....(i)

Differentiate (i) w. r. t. to x.

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(2x^3 - 3x^2 - 72x + 15\right)$$

=> $f'(x) = 6x^2 - 6x - 72$ .....(ii)

For critical values put $f'(x) = 0$

$\Rightarrow\ 6x^2 - 6x - 72 = 0 \quad \Rightarrow\ 6(x^2 - x - 12) = 0$

$\Rightarrow\ x^2 - x - 12 = 0 \quad \Rightarrow\ x^2 + 3x - 4x - 12 = 0$

$\Rightarrow\ x(x+3) - 4(x+3) = 0 \quad \Rightarrow\ (x+3)(x-4) = 0$

$\Rightarrow\ x + 3 = 0 \quad , \quad x - 4 = 0$

$\Rightarrow\ x = -3 \quad , \quad x = 4$

Thus critical values $x = -3$, 4.

(b) $f(x) = \frac{1}{3}x^3 - x^2 - 15x + 6\ldots$

Solution : $f(x) = \frac{1}{3}x^3 - x^2 - 15x + 6$ ....(i)

Differentiate w . r. t. to x .

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{3}x^3 - x^2 - 15x + 6\right)$$

$\Rightarrow\ f'(x) = x^2 - 2x - 15$ ......(ii)

For critical value put $f'(x) = 0$ **in (ii)**

$x^2 - 2x - 15 = 0 \quad \Rightarrow\ x^2 + 3x - 5x - 15 = 0$

$\Rightarrow\ x(x+3) - 5(x+3) = 0 \quad \Rightarrow\ (x+3)(x-5) = 0$

$\Rightarrow\ x + 3 = 0 \quad , \quad x - 5 = 0$

$\Rightarrow\ x = -3 \quad , \quad x = 5$

Thus the critical value is $x = -3$, 5.

(c) $f(x) = 6x^{\frac{2}{3}} - 4x$

Solution: $f(x) = 6x^{\frac{2}{3}} - 4x$

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(6x^{\frac{2}{3}} - 4x\right) \Rightarrow\ f'(x) = 6\left(\frac{2}{3}\right)x^{\frac{2}{3}-1} - 4$$

$\Rightarrow\ f'(x) = 4x^{-\frac{1}{3}} - 4$ ......(i)

For critical value put $f'(x) = 0$ **in (i)**

$$4x^{-\frac{1}{3}} - 4 = 0 \quad \Rightarrow \quad 4x^{-\frac{1}{3}} = 4$$

$$\Rightarrow x^{-\frac{1}{3}} = 1 \quad \Rightarrow \frac{1}{x^{\frac{1}{3}}} = 1$$

$$\Rightarrow 1 = x^{\frac{1}{3}} \quad \Rightarrow x = (1)^{\frac{1}{3}} = 1$$

Also from (i) at $x = 0$ then $f'(0)$ does not exist
But $f(0)$ is defined. Then 0 is also the critical value.
Thus the critical value is $x = 0\ ,\ 1$.

(d) $f(x) = 3x^{\frac{4}{3}} - 12x^{\frac{1}{3}}$

<u>Solution</u>: $f(x) = 3x^{\frac{4}{3}} - 12x^{\frac{1}{3}}$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(3x^{\frac{4}{3}} - 12x^{\frac{1}{3}}\right)$$

$$\Rightarrow \quad f'(x) = 3\left(\frac{4}{3}\right)x^{\frac{4}{3}-1} - 12\left(\frac{1}{3}\right)x^{\frac{1}{3}-1}$$

$$\Rightarrow \quad f'(x) = 4x^{\frac{1}{3}} - 4x^{-\frac{2}{3}} \quad \ldots\ldots(i)$$

For critical value put $f'(x) = 0$ in (i)

$$4x^{\frac{1}{3}} - 4x^{-\frac{2}{3}} = 0$$

$$\Rightarrow \quad 4x^{\frac{1}{3}} - \frac{4}{x^{\frac{2}{3}}} = 0 \quad \Rightarrow \quad \frac{4x - 4}{x^{\frac{2}{3}}} = 0$$

$$\Rightarrow \quad 4x - 4 = 0 \quad \Rightarrow \quad 4x = 4$$

$$\Rightarrow \quad x = 1$$

Also from (i) at $x = 0$ then $f'(0)$ does not exist
But $f(0)$ is defined.
Then 0 is also the critical value.
Thus critical values is $x = 0\ ,\ 1$.

Q.3: **Determine whether the given function has a relative maximum, a relative minimum or neither at the given critical values for the following problems.**

(a) $f(x) = (x^3 - 3x + 1)^7$, at $x = 1$, $x = -1$

Solution: $f(x) = (x^3 - 3x + 1)^7$

$$\frac{d}{dx} f(x) = \frac{d}{dx}(x^3 - 3x + 1)^7$$

$$=> f'(x) = 7(x^3 - 3x + 1)^6(3x^2 - 3)$$

$$= (7)(3)(x^3 - 3x + 1)^6(x^2 - 1)$$

$$=> f'(x) = 21(x^3 - 3x + 1)^6(x^2 - 1) \quad \text{........(i)}$$

$$\frac{d}{dx} f'(x) = 21\frac{d}{dx}\left[(x^3 - 3x + 1)^6(x^2 - 1)\right]$$

$$=> f''(x) = (21)(x^3 - 3x + 1)^6 \frac{d}{dx}(x^2 - 1) + 21(x^2 - 1)\frac{d}{dx}(x^3 - 3x + 1)^6$$

$$=> f''(x) = 21(x^3 - 3x + 1)^6(2x) + 21(x^2 - 1)(6)(x^3 - 3x + 1)^5(2x - 3)$$

$$=> f''(x) = (x^3 - 3x + 1)^5[42x + 126(x^2 - 1)(2x - 3)] \text{......}(ii)$$

Put $x = 1$ in (ii)

$$f''(1) = (1 - 3 + 1)^5[42 + 126(1 - 1)(2 - 3)]$$

$$=> f''(1) = (-1)^5(42 + 0) = -42 < 0$$

So $f(x)$ has relative maximum value at $x = 1$

Put $x = -1$ in (ii)

$$f''(-1) = (-1 + 3 + 1)^5[42 + 126(1 - 1)(-2 - 3)]$$

$$=> f''(-1) = (3)^5(42 + 0) = 3^5.42 > 0$$

Thus $f(x)$ has relative minimum value at $x = -1$.

(b) $f(x) = (x^4 - 4x + 2)^5$ at $x = 1$

Solution: $f(x) = \left(x^4 - 4x + 2\right)^5$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(x^4 - 4x + 2\right)^5$$

$$\Rightarrow \quad f'(x) = 5\left(x^4 - 4x + 2\right)^4\left(4x^3 - 4\right)$$

$$\Rightarrow \quad f'(x) = 20\left(x^4 - 4x + 2\right)^4\left(x^3 - 1\right) \quad .....\textbf{(i)}$$

$$\frac{d}{dx} f'(x) = 20\frac{d}{dx}\left(x^4 - 4x + 2\right)^4\left(x^3 - 1\right)$$

$$f''(x) = 20\left(x^4 - 4x + 2\right)^4\left(3x^2\right) + (20)(4)\left(x^4 - 4x + 2\right)^3\left(4x^3 - 4\right)$$

$$\Rightarrow f'(x) = \left(x^4 - 4x + 2\right)^3\left[60x^2\left(x^4 - 4x + 2\right) + 80\left(4x^3 - 4\right)\right] ...\textbf{(ii)}$$

Put $x = 1$ **in (ii)**

$$f''(1) = (1 - 4 + 2)^3\left[60(1 - 4 + 2) + 80(4 - 4)\right]$$

$$= (-1)^3(-60 + 0) = 60 > 0$$

Thus $f(x)$ **has relative minimum at** $x = 1$ **.**

(c) $f(x) = \left(x^2 - 4\right)^4\left(x^2 - 1\right)^3$ **, at** $x = 1$, $x = 2$ **.**

Solution: $f(x) = \left(x^2 - 4\right)^4\left(x^2 - 1\right)^3$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[\left(x^2 - 4\right)^4\left(x^2 - 1\right)^3\right] \quad \textbf{use product rule}$$

$$\Rightarrow f'(x) = \left(x^2 - 4\right)^4\frac{d}{dx}\left(x^2 - 1\right)^3 + \left(x^2 - 1\right)^3\frac{d}{dx}\left(x^2 - 4\right)^4$$

$$\Rightarrow f'(x) = \left(x^2 - 4\right)^4(3)\left(x^2 - 1\right)^2(2x) + \left(x^2 - 1\right)^3(4)\left(x^2 - 4\right)^3(2x)$$

$$= \left(x^2 - 4\right)^3\left(x^2 - 1\right)^2\left[6x\left(x^2 - 4\right) + 8x\left(x^2 - 1\right)\right]$$

$$= \left(x^2 - 4\right)^3\left(x^2 - 1\right)^2\left(6x^3 - 24x + 8x^3 - 8x\right)$$

$$= \left(x^2 - 4\right)^3\left(x^2 - 1\right)^2\left(14x^3 - 32x\right)$$

$$\frac{d}{dx} f'(x) = \frac{d}{dx}\left[\left(x^2 - 4\right)^3\left(x^2 - 1\right)^2\left(14x^3 - 32x\right)\right] \quad \textbf{use product rule}$$

$$f''(x) = (x^2-4)^3(x^2-1)^2(42x^2-32)$$
$$+(x^2-4)^3(14x^3-32x)(2)(x^2-1)(2x)$$
$$+(x^2-1)^2(14x^3-32x)(3)(x^2-4)^2(2x)$$

$$\Rightarrow f''(x) = (x^2-4)^3(x^2-1)^2(42x^2-32)$$
$$+4x(x^2-4)^3(14x^3-32x)(x^2-1)$$
$$+6x(x^2-1)(14x^3-32x)(x^2-4) \quad \text{............}(ii)$$

Put $x=1$ in (ii)

$$f''(1) = (1-4)^3(1-1)^2(42-32)+4(1-4)^3(14-32)(1-1)$$
$$+6(1-1)(14-32)(1-4)$$

$$\Rightarrow f''(1) = 0$$

Also put $x=2$ in (ii)

We get $f''(2)=0$

Thus the given function has no relative maximum nor relative minimum at $x=1$, $x=2$. so $f(x)$ is neither.

(d) $f(x)=\sqrt[3]{x^3-48}$ **, at** $x=4$

<u>**Solution:**</u> $f(x)=\sqrt[3]{x^3-48}=(x^3-48)^{\frac{1}{3}}$

$$\frac{d}{dx}f(x)=\frac{d}{dx}(x^3-48)^{\frac{1}{3}}$$

$$\Rightarrow f'(x)=\frac{1}{3}(x^3-48)^{\frac{1}{3}-1}(3x^2)$$

$$= x^2(x^3-48)^{-\frac{2}{3}} \quad \text{........}(i)$$

$$f''(x)=x^2\left(-\frac{2}{3}\right)(x^3-48)^{-\frac{2}{3}-1}(3x^2)+2x(x^3-48)^{-\frac{2}{3}}$$

$$=2x(x^3-48)^{-\frac{2}{3}}\left[-x^3(x^3-48)^{-1}+1\right] \quad \text{.....}(ii)$$

Put $x=4$ in (ii)

$$f''(4) = 2(4)(64-48)^{-\frac{2}{3}}\left(-64(64-48)^{-1}+1\right)$$

$$= \frac{8}{(16)^{\frac{2}{3}}}\left(-\frac{64}{16}+1\right) = \frac{8}{(16)^{\frac{2}{3}}}(-3)$$

$$\Rightarrow \quad f''(4) = -\frac{24}{\sqrt[3]{256}} < 0$$

Thus $f(x)$ has relative maximum at $x = 4$.

Q.4: Find all the critical values of the functions in the following problems and determine where the graph of the functions is rising , falling , concave up or concave down . sketch the graph .

(a): $f(x) = 2(x+20)^2 - 8(x+20) + 7$ **....(i)**

Solution: $f(x) = 2(x+20)^2 - 8(x+20) + 7$

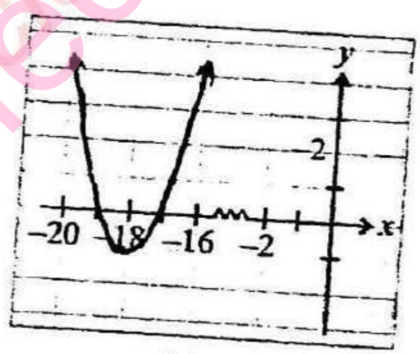
Differentiate w.r.t. to x ,

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(2(x+20)^2 - 8(x+20) + 7\right)$$

$\Rightarrow f'(x) = 2(2)(x+20) - 8$

$\Rightarrow f'(x) = 4x + 80 - 8$

$\Rightarrow f'(x) = 4x + 72$ **....(ii) .**



(a)

For critical value $f'(x) = 0$

Then from (ii) $\Rightarrow 4x = -72$

$\Rightarrow \quad x = -18$

The critical value is $x = -18$ **.**

Now the critical point :

put $x = -18$ **in (i)**

$$f(-18) = 2(-18+20)^2 - 8(-18+20) + 7$$

$$= 2(2)^2 - 8(2) + 7 = 8 - 16 + 7 = -1$$

Thus the critical point is $(x, \quad f(x)) = (-18 \, , -1)$**.**

The function $f(x)$ **is risinging(relative minimum) in the interval if** $f'(x) < 0$ **in that interval .**

The From (ii) => $f'(x) = 4x + 72 < 0$

=> $4x < -72$ => $x < -\dfrac{72}{4}$

=> $x < -18$.

Thus the function $f(x)$ is risinging in $(-\infty, -18)$.

The function $f(x)$ is falling(relative maximum) in the interval if $f'(x) > 0$ in that interval.

The From (ii) => $f'(x) = 4x + 72 > 0$

=> $4x > -72$ => $x > -\dfrac{72}{4}$

=> $x > -18$.

Thus the function $f(x)$ is falling in $(-18, \infty)$.

The graph is concave up.

(b) $f(x) = \dfrac{1}{3}x^3 - 9x + 2$

Solution: $f(x) = \dfrac{1}{3}x^3 - 9x + 2$ ...(i)

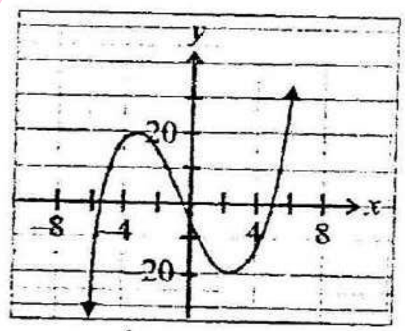
Differentiate w.r.t. to

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{1}{3}x^3 - 9x + 2\right)$$

=> $f'(x) = \dfrac{3}{3}x^2 - 9$

$= x^2 - 9$

$= (x-3)(x+3)$ ...(ii)



(b)

For critical value $f'(x) = 0$

From (ii) => $(x-3)(x+3) = 0$

=> $x - 3 = 0$ or $x + 3 = 0$

=> $x = 3$ or $x = -3$

Thus the critical values are $x = 3$ or $x = -3$

Critical point: put $x = 3$ **in (i)**

$$f(3)=\frac{1}{3}(3)^3-9(3)+2=9-27+2=-16$$

put $x=-3$ in (i)

$$f(-3)=\frac{1}{3}(-3)^3-9(-3)+2=-9+27+2=20$$

Thus critical points are $(3,-16)$ *and* $(-3,\ 20)$

To find the interval where $f(x)$ is increasing or decreasing we take some points left to and right to critical values . i.e.

We take a point left to 3 and a point right to 3 .

We also take a point left to -3 and a point right to -3.

We take 2 to the left of 3 .

Put $x=2$ in (ii)

$$f'(2)=(2-3)(2+3)=-1(5)=-5<0$$

Thus the function is decreasing in the interval $x<3$.

We take 4 to the right of 3 .

Put $x=4$ in (ii)

$$f'(4)=(4-3)(4+3)=1(7)=7>0$$

Thus the function is increasing in the interval $x>3$.

We take - 4 left to - 3 and -2 right to -3 .

Put $x=-4$ in (ii)

$$f'(-4)=(-4-3)(-4+3)=(-7)(-1)=7>0$$

Thus $f(x)$ is increasing in the interval $x<-3$

We take - 2 to the right of -3 .

Put $x=-2$ in (ii)

$$f'(-2)=(-2-3)(-2+3)=(-5)(1)=-5<0$$

Thus $f(x)$ is increasing in the interval $x>-3$.

Thus $f(x)$ is decreasing to the left of 3 and to right to -3.

i.e. $f(x)$ is decreasing (falling) in the interval $(-3\ ,3)$

and $f(x)$ is increasing (rising) when $x<-3$ and $x>3$

or $(-\infty\ -3)U(3\ ,\ \infty)$.

Now we find relative maximum and relative minimum :

As derivative of $f(x)$ is negative to the left of 3

and positive to the right of 3 .
Then the critical value 3 leads to relative minimum point. $x=3$ in (i)

$$f(3)=\frac{1}{3}(3)^3-9(3)+2=9-27+2=-16$$

Thus the relative minimum $(3\ ,-16)$.
As derivative of $f(x)$ is positive to the left of -3 and negative to the right of -3.
Then the critical value -3 leads to relative maximum point. Put $x=-3$ in (i)

$$f(-3)=\frac{1}{3}(-3)^3-9(-3)+2=-9+27+2=20$$

Then relative maximum is $(-3\ ,20)$
The graph is concave up $(0\ ,\infty)$ and concave down in $(-\infty,\ 0)$.

Q.5: Find all relative extrema of the following functions.

(a) $f(x)=x^3-3x^2+1$

Solution: $f(x)=x^3-3x^2+1$ ......(i)

The first derivative of $f(x)$ is

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(x^3-3x^2+1\right)$$

$$\Rightarrow f'(x)=3x^2-6x \quad .....(ii)$$

For critical vulue put $f'(x)=0$ in (ii)

$$3x^2-6x=0 \quad \Rightarrow 3x(x-2)=0$$

$$\Rightarrow x=0 \quad , x-2=0$$

$$\Rightarrow x=2$$

Then the critical values are $x=0\ ,\ 2$

The second derivative of $f(x)$ is

$$\frac{d}{dx}f'(x)=\frac{d}{dx}\left(3x^2-6x\right)$$

$$\Rightarrow f''(x)=6x-6 \quad ......(iii)$$

Put $x=0$ in (iii)

$$f''(0)=6(0)-6=-6<0$$

Then $f(x)$ has relative maximum at $x=0$

Put $x=0$ in (i)

$$f(0)=0-3(0)+1=1$$

Thus the relative maximum of $f(0)=1$ at $x=0$.

Put $x=2$ in (iii)

$$f''(2)=6(2)-6=12-6=6>0$$

Then $f(x)$ has relative minimum at $x=2$.

Put $x=2$ in (i)

$$f(2)=(2)^3-3(2)^2+1=8-12+1=-3$$

Thus the relative minimum of $f(2)=-3$ $x=2$.

(b) $f(x)=x^3+6x^2+9x+2$

Solution: $f(x)=x^3+6x^2+9x+2$ ......(i)

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(x^3+6x^2+9x+2\right)$$

$$\Rightarrow f'(x)=3x^2+12x+9 \quad .....(ii)$$

$$\frac{d}{dx}f'(x)=\frac{d}{dx}\left(3x^2+12x+9\right)$$

$$\Rightarrow f''(x)=6x+12 \quad .....(iii)$$

For critical values put $f'(x)=0$ in (ii)

$3x^2+12x+9=0$ then by quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad ......(A)$$

Put $a=3$, $b=12$ *and* $c=9$ in (A)

$$x=\frac{-12\pm\sqrt{(12)^2-4(3)(9)}}{2(3)}=\frac{-12\pm\sqrt{144-108}}{6}$$

$$\Rightarrow x=\frac{-12\pm\sqrt{36}}{6}=\frac{-12\pm 6}{6}$$

$$\Rightarrow x = \frac{-12+6}{6} \quad , \quad x = \frac{-12-6}{6}$$

$$\Rightarrow x = -\frac{6}{6} \quad , \quad x = -\frac{18}{6}$$

$$\Rightarrow x = -1 \quad , \quad x = -3$$

The critical values ar $x = -1$, $x = -3$

Put $x = -1$ in (iii)

$$f''(-1) = 6(-1) + 12 = 6 > 0$$

Then $f(x)$ has relative minimum at $x = -1$

Put $x = -1$ in (i)

$$f(-1) = (-1)^3 + 6(-1)^2 + 9(-1) + 2 = -1 + 6 - 9 + 2 = -2$$

Thus the relative miinimum of $f(-1) = -2$ at $x = -1$.

Put $x = -3$ in (iii)

$$f''(-3) = 6(-3) + 12 = -18 + 12 = -6 < 0$$

Then $f(x)$ has relative maximum at $x = -3$.

Put $x = -3$ in (i)

$$f(-3) = (-3)^3 + 6(-3)^2 + 9(-3) + 2 = -27 + 54 - 27 + 2 = 2$$

Then $f(x)$ has relative maximum of $f(-3) = 2$ at $x = -3$.

Q.6: (a) Suppose $f(x)$ is a differentiable function with derivative $f'(x) = (x-1)^2(x-2)(x-4)(x+5)^4$

Fiind all critical values of $f(x)$ and determine whether each corresponds to relative maximum, a relative minimu, or neither.

(b) Suppose $f(x)$ is a differentiable function with Derivative $f'(x) = \dfrac{(2x-1)(x+3)}{(x-1)^2}$.

Fiind all critical values of $f(x)$ and determine whether each corresponds to relative maximum, a relative minimu, or neither.

Solution: (a) $f'(x) = (x-1)^2(x-2)(x-4)(x+5)^4$ .....(i)

For the critical values put $f'(x) = 0$ in(i)

$$(x-1)^2(x-2)(x-4)(x+5)^4=0$$

$\Rightarrow (x-1)^2=0\ ,\ x-2=0\ ,\ x-4=0\ ,\ (x+5)^4=0$

$\Rightarrow x-1=0\ ,\ x=2\ ,\ x=4\ ,\ x+5=0$

$\Rightarrow x=1\ ,\ 2\ ,\ 4\ ,\ -5$

Thus all the critical values are $x=1,\ 2,\ 4$ *and* -5 .

Now to find the 2nd derivative of $f(x)$.

Then taking log of eq(i) of both sides

$$\ln f'(x)=\ln\left[(x-1)^2(x-2)(x-4)(x+5)^4\right]$$

$$=\ln(x-1)^2+\ln(x-2)+\ln(x-4)+\ln(x+5)^4$$

$$\Rightarrow \ln f'(x)=2\ln(x-1)+\ln(x-2)+\ln(x-4)+4\ln(x+5)$$

Now differentiate w. r. t. to x ,

$$\frac{d}{dx}\ln f'(x)=\frac{d}{dx}\left[2\ln(x-1)+\ln(x-2)+\ln(x-4)+4\ln(x+5)\right]$$

$$\Rightarrow \frac{1}{f'(x)}f''(x)=\frac{2}{x-1}+\frac{1}{x-2}+\frac{1}{x-4}+\frac{4}{x+5}$$

$$\Rightarrow f''(x)=f'(x)\left[\frac{2}{x-1}+\frac{1}{x-2}+\frac{1}{x-4}+\frac{4}{x+5}\right]\quad \text{....(ii)}$$

Put (i) in (ii)

$$f''(x)=\frac{(x-1)^2(x-2)(x-4)(x+5)^4}{(x-1)(x-2)(x-4)(x+5)}\begin{bmatrix}2(x-2)(x-4)(x+5)\\+(x-1)(x-4)(x+5)\\+(x-1)(x-2)(x+5)\\+4(x-1)(x-2)(x-4)\end{bmatrix}$$

$$\Rightarrow f''(x)=(x-1)(x+5)^3\begin{bmatrix}2(x-2)(x-4)(x+5)\\+(x-1)(x-4)(x+5)\\+(x-1)(x-2)(x+5)\\+4(x-1)(x-2)(x-4)\end{bmatrix}\quad .....(iii)$$

Put $x=1$ **in (iii) we get** $f''(1)=0$

Also put $x=-5$ **in (iii) we get** $f''(-5)=0$

Thus the critical values 1 and -5 does not give relative maximum nor relative minimum.

Then the function is neither at $x=1$, -5

put $x=2$ in (iii)

$$f''(2)=(2-1)(2+5)^3[0+(2-1)(2-4)(2+5)+0+0]$$
$$=7^3(-14)<0$$

Thus at $x=2$, $f(x)$ has relative maximu.

Put $x=4$ in (iii)

$$f''(4)=(4-1)(4+5)^3[0+0+(4-1)(4-2)(4+5)+0]$$
$$=(3)(9)^3(54)>0$$

Thus at $x=4$, $f(x)$ has relative minimum.

(b) $f'(x)=\dfrac{(2x-1)(x+3)}{(x-1)^2}$ .....(i)

For critical value put $f'(x)=0$ in (i)

$$\frac{(2x-1)(x+3)}{(x-1)^2}=0 \quad => (2x-1)(x+3)=0$$

$=> 2x-1=0 \quad , x+3=0$

$=> 2x=1 \quad , x=-3$

$=> x=\dfrac{1}{2}$

Also $f'(x)$ does not exist at $x=1$

Then the critical values are $x=1$, $\dfrac{1}{2}$, -3

Now from (i)

$$f'(x)=\frac{2x^2+6x-x-3}{(x-1)^2}$$

$$=> f'(x)=\frac{2x^2+5x-3}{(x-1)^2}$$

$$\frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{2x^2+5x-3}{(x-1)^2}\right)$$

$$\Rightarrow \quad f''(x) = \frac{(x-1)^2 \frac{d}{dx}(2x^2+5x-3) - (2x^2+5x-3)\frac{d}{dx}(x-1)^2}{\left[(x-1)^2\right]^2}$$

$$\Rightarrow \quad f''(x) = \frac{1}{(x-1)^4}\left[(x-1)^2(4x+4) - (2x^2+5x-3)(2)(x-1)\right]$$

$$\Rightarrow \quad f''(x) = \frac{x-1}{(x-1)^4}\left[(x-1)(4x+4) - 2(2x^2+5x-3)\right]$$

$$= \frac{1}{(x-1)^3}\left(4x^2+4x-4x-4-4x^2-10x+6\right)$$

$$\Rightarrow \quad f''(x) = \frac{2-10x}{(x-1)^3} \quad ......(ii)$$

As from (ii) $f''(x)$ does not exist at $x=1$

Then $f(x)$ is neither at $x=1$. Put $x=\frac{1}{2}$ in (ii)

$$f''\left(\frac{1}{2}\right) = \frac{2-10\left(\frac{1}{2}\right)}{\left(\frac{1}{2}-1\right)^3}$$

$$\Rightarrow \quad f''\left(\frac{1}{2}\right) = \frac{2-5}{\left(-\frac{1}{2}\right)^3} = -\frac{5}{-\frac{1}{8}} = 5\times 8 = 40 > 0$$

Thus the function has relative minimum at $x=\frac{1}{2}$

Put $x=-3$ in (ii)

$$f''(-3) = \frac{2-10(-3)}{(-3-1)^3} = \frac{2+30}{-64} = -\frac{1}{2} < 0$$

Thuis the function has relative maximum at $x=-3$.

Q.7: A company has found through experience that increasing its advertising also increases its sales up to a point. The company believes that the mathematical model connecting profit in hundreds of dollars $P(x)$ and expenditures on advertising in thousands of dollars x is:

$P(x) = 80 + 108x - x^3$, $0 \le x \le 10$

(a) Find the expenditure on advertising that leads to maximum profit.

(b) Find the maximum profit.

Solution: Given $P(x) = 80 + 108x - x^3$ ....(i)

The first derivative of (i) is

$$P'(x) = \frac{d}{dx}\left(80 + 108x - x^3\right) \Rightarrow P'(x) = 108 - 3x^2 \quad ......(ii)$$

The 2nd derivative is

$$P''(x) = \frac{d}{dx}\left(108 - 3x^2\right) = -6x \quad .....(iii)$$

Put $P'(x) = 0$ in (ii)

$108 - 3x^2 = 0$

$\Rightarrow 108 = 3x^2$ $\quad or \quad x^2 = 36$

$\Rightarrow x = \pm 6$

But given that $0 \le x \le 10$ then $x = 6$

put $x = 6$ in (iii)

$P''(6) = -6(6) = -36 < 0$

Thus the maximum profit is get at $x = 6$

(a) The expenditure on advertising that leads to maximum Profit at $x = 6$.

(b) Maximum profit: put $x = 6$ in (i)

$P(6) = 80 + (108)(6) - (6)^3 = 80 + 648 - 216$

$\Rightarrow P(6) = 512$ *hundreds dollars.*

Q.8: The total profit $P(x)$ (in thousands of dollars) from the sale of x hundred thousands of automobile tyres is approximated by $P(x) = -x^3 + 9x^2 + 120x - 400$, $3 \le x \le 15$

Find the number of hundred thousands of tyres that must be sold to maximize profit. Find the maximum profit.

Solutiuon: $P(x) = -x^3 + 9x^2 + 120x - 400$ **...(i)**

The first derivative of (i) is

$$\frac{d}{dx}P(x) = \frac{d}{dx}\left(-x^3 + 9x^2 + 120x - 400\right)$$

=> $P'(x) = -3x^2 + 18x + 120$ ....(ii)

The second derivative is

$$\frac{d}{dx}P'(x) = \frac{d}{dx}\left(-3x^2 + 18x + 120\right) \Rightarrow P''(x) = -6x + 18 \quad ....(iii)$$

Put $P'(x) = 0$ **in (ii)**

$$-3x^2 + 18x + 120 = 0$$

=> $-3\left(x^2 - 6x - 40\right) = 0$ => $x^2 - 6x - 40 = 0$

=> $x^2 + 4x - 10x - 40 = 0$ => $x(x+4) - 10(x+4) = 0$

=> $(x+4)(x-10) = 0$

=> $x + 4 = 0$, $x - 10 = 0$

=> $x = -4$, $x = 10$

Since $3 \le x \le 15$ **then we take** $x = 10$

Put $x = 10$ **in (iii)**

$P(10) = -6(10) + 18 = -60 + 18 = -42 < 0$

(a) Thus the the number of tyres that must be sold to maximize the profit $x = 10$ *hundred thousands tyres*

or $x = (10)(100)(1000) = 1000{,}000$ **tyres.**

(b) Maximum profit: put $x = 10$ **in (i)**

$P(10) = -(10)^3 + 9(10)^2 + 120(10) - 400$

=> $P(10) = -1000 + 900 + 1200 - 400$

$= 700$ *thounsands dollars*

Or $P(10) = (700)(1000) = 700000$ *dollars.*

Q.9: The percent of concentration of a drug in the

bloodstream x hours after the drug is administered

is given by $K(x)=\dfrac{4x}{3x^2+27}$

(a) On what time intervals is the concentration of the drug increases ?.

(b) On what time intervals it decreases ?

(c) Find the time at which the concentration is a maximum .

(d) Find the maximum concentration .

Solution: $K(x)=\dfrac{4x}{3x^2+27}$ ....(i)

Differentiate (i) w . r.t . to x .

$$K'(x)=\frac{d}{dx}\left(\frac{4x}{3x^2+27}\right)$$

$$\Rightarrow\ K'(x)=\frac{(3x^2+27)\frac{d}{dx}(4x)-(4x)\frac{d}{dx}(3x^2+27)}{(3x^2+27)^2}$$

$$\Rightarrow\ K'(x)=\frac{4(3x^2+27)-(4x)(6x)}{(3x^2+27)^2}=\frac{12x^2+108-24x^2}{(3x^2+27)^2}$$

$$\Rightarrow\ K'(x)=\frac{108-12x^2}{(3x^2+27)^2}\ ....(ii)$$

Put $K'(x)=0$ in (ii) $\Rightarrow$ $\dfrac{108-12x^2}{(3x^2+27)^2}=0$

$\Rightarrow\ 108-12x^2=0$

$\Rightarrow\ 12x^2=108$ $\Rightarrow$ $x^2=9$

$\Rightarrow\ x=\pm 3$

Since x represents hour so x is non –negative

Then we take $x=3$

To find the intervals where the function is maximum or minimum then we take a point left to 3 and a point right to 3. In this case we use first derivative test .

let put $x=2$ in (ii)

$$K'(2)=\frac{108-12(2)^2}{\left(3(2)^2+27\right)^3}=\frac{108-48}{(12+27)^3}=\frac{60}{(39)^3}>0$$

Put $x=4$ **in (ii)**

$$K'(4)=\frac{108-12(4)^2}{\left(3(4)^2+27\right)^2}=\frac{108-192}{(48+27)^2}=-\frac{84}{(75)^2}<0$$

(a) Thus the concentration of the drug increase in the interval for $x<3$ **. Since** x **is non-negative , then the concentration of the drug increase in the inteval** $(0\ ,3)$**.**

(b) The concentration of the drug decrease in the inteval $(3\ ,\ \infty)$**.**

(c) The time at which the concentration is a maximum $x=3$

(d) Put $x=3$ **in (i)**

$$K(3)=\frac{4(3)}{3(3)^2+27}=\frac{12}{27+27}=\frac{12}{54}=0.22\ \%$$

$$OR\quad K(3)=\frac{0.22}{100}=0.0022$$

<u>Q.10</u>: A diesel generator burns fuel at the rate of

$$G(x)=\frac{1}{48}\left(\frac{300}{x}+2x\right)\ ,$$ **gallons per hour when producing** x **thousands kilowatt hours of electricity .**

Suppose that fuel costs 2.25 dollars a gallon and find the value of x **that leads to minimum total cost if the generator is operated for 32 hours . Find the minimum cost .**

<u>Solution</u>: Given that fuels in gallons per hour given is

$$G(x)=\frac{1}{48}\left(\frac{300}{x}+2x\right)\quad \text{where } x>0$$

Since total hours = 32 hours

Per gallons cost = 2.25 dollars .

Thus the total cost in dollars

$$C(x) = (\text{number of gallons per hour}) \times (\text{the number of hours}) \times (\text{the cost per gallons}).$$

Thus $C(x) = \frac{1}{48}\left(\frac{300}{x} + 2x\right)(32)(2.25) = \frac{72}{48}\left(\frac{300}{x} + 2x\right)$

$\Rightarrow \quad C(x) = \frac{3}{2}\left(\frac{300}{x} + 2x\right) \quad \text{.....(i)}$

The first derivative of (i) is given by

$$\frac{d}{dx}C(x) = \frac{3}{2}\frac{d}{dx}\left(\frac{300}{x} + 2x\right)$$

$\Rightarrow \quad C'(x) = \frac{3}{2}\left(-\frac{300}{x^2} + 2\right)$ **or** $\Rightarrow \quad C'(x) = -\frac{450}{x^2} + 3$ **...(ii)**

The 2nd derivative of $C(x)$ **is given by**

$$\frac{d}{dx}C'(x) = \frac{d}{dx}\left(-\frac{450}{x^2} + 3\right) \Rightarrow C''(x) = (-450)\frac{d}{dx}\left(x^{-2}\right) + 0$$

$\Rightarrow \quad C''(x) = (-450)(-2)x^{-3} = \frac{900}{x^3} \quad \text{.....}(iii)$

For critical value put $f'(x) = 0$ **in (ii)**

$$-\frac{450}{x^2} + 3 = 0$$

$\Rightarrow \quad -450 + 3x^2 = 0 \qquad \Rightarrow \quad 3x^2 = 450$

$\Rightarrow \quad x^2 = 150 \qquad \Rightarrow \quad x = \pm\sqrt{150}$

Since x **shows hours so x is positive**

Then we take the critical value of $x = \sqrt{150}$ **put in (iii)**

$$C''(28.3) = \frac{900}{\left(\sqrt{150}\right)^3} > 0$$

Thus the cost will be minimum when $x = \sqrt{150}$ **.**

Then the minimum cost will be obtained by putting

$x = \sqrt{150} = 12.2$ **in (i)**

$$C(12.2)=\frac{3}{2}\left(\frac{300}{12.2}+2(12.2)\right)=73.48 \quad dollars.$$

Review Exercise. 4:

Q.1: Chose the correct option .

(i) If $y=3t^2+4t-5$ **then** $f'(t)$ **is**

(a) $3t^2-4t+5$ (b) $6t+4$ (c) 12 (d) $6t-5$

(ii) If $f(t)=3t^2+4t-5$ **then** $f''(t)$ **is**

(a) $3t^2-4t+5$ (b) $6t-4$ (c) 6 (d) $6t+5$

(iii) If $y=e^{mx}$ *then* $\frac{d^n y}{dx^n}=$

(a) $me^x\log(e^x)$ (b) $m^n e^{mx}$ (c) me^{nx} (d) $me^n e^{mx}$

(iv) The fifth derivative of $f(x)=e^x$ **is**

(a) e^x (b) e^{x+5} (c) $5e^x$ (d) e^{5x}

(v) The fourth derivative of $y=\sin x$ **is**

(a) $\frac{d^4y}{dx^4}=\sin x$ (b) $\frac{d^4y}{dx^4}=\cos x$

(c) $\frac{d^4y}{dx^4}=-\sin x$ (d) $\frac{d^4y}{dx^4}=-\cos x$

(vi) If $f(x)$ **and its derivative at** $x=x_o$ **are** $f'(x_o)$, $f''(x_o)$, $f'''(x_o)$, ..., $f^n(x_o)$ **then the nth order polynomial** $\ln(x)$ **of** $f(x)$ **will be**

(a) $f(x)+(x-x_o)f'(x)+.........+\frac{(x-x_o)^n}{n!}f^n(x)$

(b) $f(x_o)+(x-x_o)f'(x_o)+.........+\frac{(x-x_o)^n}{n!}f^n(x_o)$

(c) $f(x+h)+(x+h)f'(h)+.........+\frac{(x-h)^n}{n!}f^n(h)$

(d) $f(x-h)+(x-h)f'(h)+.........+\frac{(x-h)}{n!}f^n(h)$

(vii) To calculate the first five terms of talor series for $f(x)=e^{2x}$ the MAPLE command is used

(a) $taylor\left(x,\ e^{2x}=0\right)$ (b) $taylor\left(e^{2x}\right)$

(c) $taylor\left(e^{2x},\ x=0\ ,5\right)$ (d) $taylor\left(e^{2x},\ x=5\right)$

(viii) The equation of normal at point $(x_o\ ,\ y_o)$ is

(a) $x-x_o=\frac{1}{m}(y-y_o)$ (b) $y-y_o=\frac{1}{m}(x-x_o)$

(c) $y-y_o=m(x-x_o)$ (d) $y-y_o=-\frac{1}{m}(x-x_o)$

(ix) The angle of intersection of the two curves can be calculated by using the formulas.

(a) $\theta=\sin^{-1}\frac{1+m_1m_2}{1-m_1m_2}$ (b) $\theta=\tan^{-1}\frac{1-m_1m_2}{1+m_1m_2}$

(c) $\theta=\tan^{-1}\frac{m_1-m_2}{1+m_1m_2}$ (d) $\theta=\sin^{-1}\frac{m_1+m_2}{1+m_1m_2}$

(x) If $f(x)$ is differentiable on the open interval $(a,\ b)$ the $f(x)$ is strictly increasing if :

(a) $f(x)>0$ (b) $f'(x)>0$ (c) $f'(x)>0$ (d) $f''(x)<0$

<u>Answers</u>: (i) b (ii) c (iii) d (iv) a (v) a
(vi) b (vii) c (viii) d (ix) c (x) b