

Unit – 3:**Differentiation:**

Independent variable: The domain or input values is called independent variable .

Dependent variable: The range or output values is called dependent variable .

For example if $y = x^2 + 2x + 3$

Here x is independent variable and y is dependent variable.

Rate of change : Let $y = f(x)$... (i)

Then $y + \Delta y = f(x + \Delta x)$ (ii)

Subtract (i) from (ii)

$$y + \Delta y - y = f(x + \Delta x) - f(x)$$

$$\Rightarrow \Delta y = f(x + \Delta x) - f(x) \text{ (iii)}$$

Both sides of eq(iii) divided by Δx

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ (iv), where } \Delta x = x + \Delta x - x$$

The eq(iii) shows the average rate of change of function $y = f(x)$ per unit change in x .

The average rate of change is also called the slope of the Secant line .

Example. 1: Determine the average rate change of y

per unit change in x for $y = x^2 - 6x + 5$, as x increases from $x = 1$ to $x = 3$.

Solution: Since the average rate of change in y per unit in x is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ (i)}$$

Since $y = f(x) = x^2 - 5x + 5$ (ii)

$$\begin{aligned} \text{Then } f(x + \Delta x) &= (x + \Delta x)^2 - 6(x + \Delta x) + 5 \\ &= x^2 + (\Delta x)^2 + 2x\Delta x - 6x - 6\Delta x + 5 \text{ (iii)} \end{aligned}$$

Put (ii) and (iii) in (i).

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{x^2 + (\Delta x)^2 + 2x\Delta x - 6x - 6\Delta x + 5 - (x^2 - 6x + 5)}{\Delta x} \\ &= \frac{x^2 + (\Delta x)^2 + 2x\Delta x - 6x - 6\Delta x + 5 - x^2 + 6x - 5}{\Delta x} \\ &= \frac{(\Delta x)^2 + 2x\Delta x - 6\Delta x}{\Delta x} \\ &= \frac{\Delta x(\Delta x + 2x - 6)}{\Delta x}\end{aligned}$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \Delta x + 2x - 6 \dots\dots(iv)$$

As x changes from 1 to 3, Then

$$\Delta x = 3 - 1 = 2$$

Put $x = 1$ and $\Delta x = 2$ in (iv)

$$\frac{\Delta y}{\Delta x} = 2 + 2(1) - 6 = 2 + 2 - 6 = -2 \quad \text{Ans.}$$

Definition: Derivative: The instantaneous rate of change of a function $f(x)$ with respect to independent variable x at particular point P is called derivative of $f(x)$ at that point P .

Let $y = f(x) \dots(i)$

Then $y + \Delta y = f(x + \Delta x) \dots\dots(ii)$

Subtract (i) from (ii)

$$y + \Delta y - y = f(x + \Delta x) - f(x)$$

$$\Rightarrow \Delta y = f(x + \Delta x) - f(x) \dots\dots(iii)$$

Both sides of eq(iii) divided by Δx

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{. Take } \lim_{\Delta x \rightarrow 0} \text{ on both sides}$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(iv)$$

If above limit exist then it is called derivative of $f(x)$.

Also $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ or $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = y'$ or $f'(x)$

Then eq(iv) can be written as

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\text{or } f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(v)$$

Eq(iv) or (v) is called first principle rule of derivative of $f(x)$ with respect to x .

$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ is called slope of the tangent line at particular

Point P.

The process that we find the derivative of a function is called differentiation.

Note: The slope of the secant line is called the approximate rate of change.

The slope of the tangent line is called the exact rate of change.

Example .3: Determine the derivative of a function

$f(x) = x^2 - 6x + 5$ by first principle rule at a point $P(4, -3)$

Solution: As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(i)$$

Since $f(x) = x^2 - 6x + 5 \dots\dots(ii)$

Then $f(x + \Delta x) = (x + \Delta x)^2 - 6(x + \Delta x) + 5$

$$= x^2 + (\Delta x)^2 + 2x\Delta x - 6x - 6\Delta x + 5 \dots\dots(iii)$$

Put (ii) and (iii) in (i)

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + (\Delta x)^2 + 2x\Delta x - 6x - 6\Delta x + 5 - (x^2 - 6x + 5)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + (\Delta x)^2 + 2x\Delta x - 6x - 6\Delta x + 5 - x^2 + 6x - 5}{\Delta x} \end{aligned}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^2 + 2x\Delta x - 6\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(\Delta x + 2x - 6)}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} (\Delta x + 2x - 6)$$

$$\Rightarrow f'(x) = 0 + 2x - 6$$

$$\Rightarrow f'(x) = 2x - 6 \dots\dots(iv)$$

Eq(iv) is the derivative of $f(x)$ or slope of the tangent line at any point $P(x, y)$ on the curve $f(x) = x^2 - 6x + 5$

Now derivative of $f(x)$ at point $P(4, -3)$

Then put $x = 4$ in eq(iv)

$$f'(4) = 2(4) - 6 = 8 - 6$$

$$\Rightarrow f'(4) = 2 \text{ at point } P(4, -3).$$

Equation of tangent line:

Equation of the tangent line on the $y = f(x)$ at point

$$P(x_1, y_1) \text{ is } y - y_1 = f'(x)(x - x_1).$$

Example .4: The function is $f(x) = x^2$.

(a) Find the derivative of a function at point $P(3, 9)$.

(b) Find the tangent line on a given curve $y = x^2$ at point $P(3, 9)$.

(c) View the slope of the tangent line on a curve $y = x^2$ at point $P(3, 9)$ graphically.

Solution: (a) As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(i)$$

$$\text{Since } f(x) = x^2 \dots\dots(ii)$$

$$f(x + \Delta x) = (x + \Delta x)^2$$

$$= x^2 + (\Delta x)^2 + 2x\Delta x \dots\dots(iii)$$

Put (ii) and (iii) in (i)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{x^2 + (\Delta x)^2 + 2x\Delta x - x^2}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^2 + 2x\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(\Delta x + 2x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} (\Delta x + 2x)$$

$$\Rightarrow f'(x) = 0 + 2x \quad \text{applying limits}$$

$$\Rightarrow f'(x) = 2x \quad \dots\dots(A)$$

At point $P(3, 9)$: Put $x=3$ in (A)

$$f'(3) = 2(3) = 6 \quad \dots(B)$$

(b) Slope of the tangent line at point $P(3, 9)$ on a curve

$y = x^2$ is the derivative of $f(x)$ at point $P(3, 9)$.

Thus from eq(B) the slope of the tangent line is

$$f'(3) = 6 \quad \dots(iv)$$

As equation of the tangent line on the $y = x^2$ at point

$P(3, 9)$ is $y - y_1 = f'(x)(x - x_1) \dots\dots(v)$

Put $x_1 = 3, y_1 = 9$ and $f'(x) = 6$ in (v)

$$y - 9 = 6(x - 3)$$

$$\Rightarrow y - 9 = 6x - 18$$

$$\Rightarrow 0 = 6x - y - 18 + 9$$

or $6x - y - 9 = 0$ is the required equation.

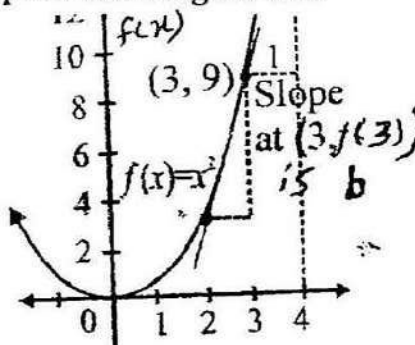
c) The graphically view of the slope of the tangent line is under.

from the graph we see

that $dy = 6$ and $\Delta x = dx = 1$

thus slope of tangent line

$$\text{at point } P(3, 9) \text{ is } = \frac{dy}{dx} = \frac{6}{1} = 6$$



3.1.6: Differentiation of $y = x^n$ by first principle rule.**Solution :** As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots (i)$$

Since $f(x) = x^n \dots\dots(ii)$

$$\begin{aligned} f(x + \Delta x) &= (x + \Delta x)^n \text{ As by binomial expansion} \\ &= x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \dots\dots(iii) \end{aligned}$$

Put (ii) and (iii) in (i)

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \dots\dots - x^n}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \dots\dots}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x \left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \dots \right)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \dots \right) \end{aligned}$$

Applying limits

$$\Rightarrow f'(x) = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}(0) + 0 + \dots + 0$$

$$\Rightarrow f'(x) = nx^{n-1} \text{ Ans.}$$

3.1.7: Differentiation of $y = (ax + b)^n$ by first principle rule.Where $n = \frac{p}{q}$, p and q are integers and $q \neq 0$ **Solution :** As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots (i)$$

Case.1: if n is positive integer .

Since $f(x) = (ax + b)^n$ (ii)

$$f(x + \Delta x) = (a(x + \Delta x) + b)^n$$

$= ((ax + b) + a\Delta x)^n$ As by binomial expansion

$$= (ax + b)^n + n(ax + b)^{n-1}(a\Delta x) + \frac{n(n-1)}{2}(ax + b)^{n-2}(a\Delta x)^2 + \dots \text{.....(iii)}$$

Put (ii) and (iii) in (i)

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left((ax + b)^n + n(ax + b)^{n-1}(a\Delta x) + \frac{n(n-1)}{2}(ax + b)^{n-2}(a\Delta x)^2 + \dots - (ax + b)^n \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{n(ax + b)^{n-1}(a\Delta x) + \frac{n(n-1)}{2}(ax + b)^{n-2}(a\Delta x)^2 + \dots}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x \left(na(ax + b)^{n-1} + \frac{n(n-1)}{2}(ax + b)^{n-2}a^2\Delta x + \dots \right)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left(na(ax + b)^{n-1} + \frac{n(n-1)}{2}(ax + b)^{n-2}a^2\Delta x + \dots \right) \end{aligned}$$

Applying limits

$$\Rightarrow f'(x) = na(ax + b)^{n-1} + \frac{n(n-1)}{2}(ax + b)^{n-2}a^2(0) + 0 + \dots$$

$$\Rightarrow f'(x) = na(ax + b)^{n-1} \quad \text{Ans.}$$

This is the generalized power rule of differentiation .

Case: 2: If n is negative integer or fraction . Then

$$f(x + \Delta x) = (a(x + \Delta x) + b)^n = ((ax + b) + a\Delta x)$$

$$= (ax + b)^n \left(1 + \frac{a\Delta x}{(ax + b)} \right)^n \text{ As by binomial expansion}$$

$$= (ax + b)^n \left(1 + n \left(\frac{a\Delta x}{ax + b} \right) + \frac{n(n-1)}{2} \left(\frac{a\Delta x}{ax + b} \right)^2 + \dots \right) \text{.....(iv)}$$

put (ii) and (iv) in (i)

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left((ax+b)^n \left(1 + n \left(\frac{a\Delta x}{ax+b} \right) + \frac{n(n-1)}{2} \left(\frac{a\Delta x}{ax+b} \right)^2 + \dots \right) - (ax+b)^n \right) \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(ax+b)^n \left(1 + n \left(\frac{a\Delta x}{ax+b} \right) + \frac{n(n-1)}{2} \left(\frac{a\Delta x}{ax+b} \right)^2 + \dots - 1 \right)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(ax+b)^n \left(n \left(\frac{a\Delta x}{ax+b} \right) + \frac{n(n-1)}{2} \left(\frac{a\Delta x}{ax+b} \right)^2 + \dots \right)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x \left(na(ax+b)^{n-1} + \frac{n(n-1)}{2} (ax+b)^{n-2} a^2 \Delta x + \dots \right)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \left(na(ax+b)^{n-1} + \frac{n(n-1)}{2} (ax+b)^{n-2} a^2 \Delta x + \dots \right)
 \end{aligned}$$

Applying limits

$$\Rightarrow f'(x) = na(ax+b)^{n-1} + \frac{n(n-1)}{2} (ax+b)^{n-2} a^2 (0) + 0 + \dots$$

$$\Rightarrow f'(x) = na(ax+b)^{n-1} \quad \text{Ans.}$$

This is the generalized power rule of differentiation .

Exercise 3.1:

Q.1: Find the average rate of change of the following functions over the indicated intervals .

(a): $y = x^2 + 4$ from $x = 2$ to $x = 3$

Solution: Since the average rate of change in y per unit in x is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \dots\dots(i)$$

As $y = f(x) = x^2 + 4$ (ii)

Then $f(x + \Delta x) = (x + \Delta x)^2 + 4$
 $= x^2 + (\Delta x)^2 + 2x.\Delta x + 4$ (iii)

Put (ii) and (iii) in (i)

$$\frac{\Delta y}{\Delta x} = \frac{x^2 + (\Delta x)^2 + 2x\Delta x + 4 - (x^2 + 4)}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{x^2 + (\Delta x)^2 + 2x\Delta x + 4 - x^2 - 4}{\Delta x}$$

$$= \frac{(\Delta x)^2 + 2x\Delta x}{\Delta x}$$

$$= \frac{\Delta x(\Delta x + 2x)}{\Delta x}$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \Delta x + 2x$$
(iv)

Since x changes from 2 to 3, then $\Delta x = 3 - 2 = 1$

Put $x = 2$ and $\Delta x = 1$ in (iv)

$$\frac{\Delta y}{\Delta x} = 1 + 2(2) = 1 + 4 = 5 \text{ Ans.}$$

(b) $y = x^2 + \frac{1}{3}x$ from $x = -3$ to $x = 3$

Solution: Since the average rate of change in y per unit in x is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
(i)

As $y = f(x) = x^2 + \frac{1}{3}x$ (ii)

Then $f(x + \Delta x) = (x + \Delta x)^2 + \frac{1}{3}(x + \Delta x)$
 $= x^2 + (\Delta x)^2 + 2x.\Delta x + \frac{1}{3}x + \frac{1}{3}\Delta x$ (iii)

Put (ii) and (iii) in (i)

$$\frac{\Delta y}{\Delta x} = \frac{x^2 + (\Delta x)^2 + 2x.\Delta x + \frac{1}{3}x + \frac{1}{3}\Delta x - \left(x^2 + \frac{1}{3}x\right)}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{x^2 + (\Delta x)^2 + 2x.\Delta x + \frac{1}{3}x + \frac{1}{3}\Delta x - x^2 - \frac{1}{3}x}{\Delta x}$$

$$= \frac{(\Delta x)^2 + 2x.\Delta x + \frac{1}{3}\Delta x}{\Delta x}$$

$$= \frac{\Delta x \left(\Delta x + 2x + \frac{1}{3} \right)}{\Delta x}$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \Delta x + 2x + \frac{1}{3} \dots\dots(iv)$$

Since x changes from -3 to 3 , then

$$\Delta x = 3 - (-3) = 3 + 3 = 6$$

Put $x = -3$ and $\Delta x = 6$ in (iv)

$$\frac{\Delta y}{\Delta x} = 6 + 2(-3) + \frac{1}{3} = 6 - 6 + \frac{1}{3} = \frac{1}{3} \text{ Ans.}$$

(c) $s = 2t^3 - 5t + 7$ from $t = 1$ to $t = 3$

Solution: Since the average rate of change in s per unit in t is given by

$$\frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t} \dots\dots(i)$$

$$\text{As } s = f(t) = 2t^3 - 5t + 7 \dots\dots(ii)$$

$$\text{Then } f(t + \Delta t) = 2(t + \Delta t)^3 - 5(t + \Delta t) + 7$$

$$= 2(t^3 + (\Delta t)^3 + 3t^2\Delta t + 3t(\Delta t)^2) - 5(t + \Delta t) + 7$$

$$= 2t^3 + 2(\Delta t)^3 + 6t^2\Delta t + 6t(\Delta t)^2 - 5t - 5\Delta t + 7 \dots\dots(iii)$$

Put (ii) and (iii) in (i)

$$\begin{aligned}
 \frac{\Delta s}{\Delta t} &= \frac{2t^3 + 2(\Delta t)^3 + 6t^2\Delta t + 6t(\Delta t)^2 - 5t - 5\Delta t + 7 - (2t^3 - 5t + 7)}{\Delta t} \\
 &= \frac{2t^3 + 2(\Delta t)^3 + 6t^2\Delta t + 6t(\Delta t)^2 - 5t - 5\Delta t + 7 - 2t^3 + 5t - 7}{\Delta t} \\
 &= \frac{2(\Delta t)^3 + 6t^2\Delta t + 6t(\Delta t)^2 - 5\Delta t}{\Delta t} \\
 \Rightarrow \frac{\Delta s}{\Delta t} &= \frac{\Delta t(2(\Delta t)^2 + 6t^2 + 6t\Delta t - 5)}{\Delta t} \\
 &= 2(\Delta t)^2 + 6t^2 + 6t\Delta t - 5 \quad \dots\dots(iv)
 \end{aligned}$$

Since t changes from 1 to 3, then

$$\Delta t = 3 - 1 = 2$$

Put $t = 1$ and $\Delta t = 2$ in (iv)

$$\begin{aligned}
 \frac{\Delta s}{\Delta t} &= 2(2)^2 + 6(1)^2 + 6(1)(2) - 5 \\
 &= 8 + 6 + 12 - 5 = 21 \quad \text{Ans.}
 \end{aligned}$$

(d) $h = \sqrt{2t} + 4$ from $t = 8$ to $t = 8.5$

Solution: Since the average rate of change in h per unit in t is given by

$$\frac{\Delta h}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t} \quad \dots\dots(i)$$

$$\text{As } h = f(t) = \sqrt{2t} + 4 \quad \dots\dots(ii)$$

$$\text{Then } f(t + \Delta t) = \sqrt{2(t + \Delta t)} + 4 \quad \dots\dots(iii)$$

Put (ii) and (iii) in (i)

$$\begin{aligned}
 \frac{\Delta h}{\Delta t} &= \frac{\sqrt{2(t + \Delta t)} + 4 - (\sqrt{2t} + 4)}{\Delta t} \\
 &= \frac{\sqrt{2t + 2\Delta t} + 4 - \sqrt{2t} - 4}{\Delta t} \\
 &= \frac{\sqrt{2t + 2\Delta t} - \sqrt{2t}}{\Delta t} \quad \dots\dots(v)
 \end{aligned}$$

Since t changes from 8 to 8.5, then

$$\Delta t = 8.5 - 8 = 0.5$$

Put $t = 8$ and $\Delta t = 0.5$ in (v)

$$\frac{\Delta h}{\Delta t} = \frac{\sqrt{2(8) + 2(0.5)} - \sqrt{2(8)}}{0.5}$$

$$\Rightarrow \frac{\Delta h}{\Delta t} = \frac{\sqrt{17} - \sqrt{16}}{0.5} = 0.25 \text{ Ans.}$$

Q.2: Use definition for the rate of change to find out the average rate of change over the specified interval for the following functions :

(a) $s = 2t - 3$ from $t = 2$ to $t = 5$

Solution: Since the average rate of change in s per unit in t is given by

$$\frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t} \dots\dots(i)$$

$$\text{As } s = f(t) = 2t - 3 \dots\dots(ii)$$

$$\begin{aligned} \text{Then } f(t + \Delta t) &= 2(t + \Delta t) - 3 \\ &= 2t + 2\Delta t - 3 \dots\dots(iii) \end{aligned}$$

Put (iii) and (ii) in (i) .

$$\begin{aligned} \frac{\Delta s}{\Delta t} &= \frac{2t + 2\Delta t - 3 - (2t - 3)}{\Delta t} \\ &= \frac{2t + 2\Delta t - 3 - 2t + 3}{\Delta t} \end{aligned}$$

$$\Rightarrow \frac{\Delta s}{\Delta t} = \frac{2\Delta t}{\Delta t} = 2 \text{ Ans.}$$

As from $t = 2$ to $t = 5$

$$\text{Then } \Delta t = 5 - 2 = 3$$

(b) $y = x^2 - 6x + 8$ from $x = 3$ to $x = 3.1$

Solution: Since the average rate of change in y per unit in x is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(i)$$

$$\text{As } y = f(x) = x^2 - 6x + 8 \dots\dots(ii)$$

$$\begin{aligned}\text{Then } f(x + \Delta x) &= (x + \Delta x)^2 - 6(x + \Delta x) + 8 \\ &= x^2 + (\Delta x)^2 + 2x\Delta x - 6x - 6\Delta x + 8 \dots\dots(iii)\end{aligned}$$

Put (ii) and (iii) in (i)

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{x^2 + (\Delta x)^2 + 2x\Delta x - 6x - 6\Delta x + 8 - (x^2 - 6x + 8)}{\Delta x} \\ \frac{\Delta y}{\Delta x} &= \frac{x^2 + (\Delta x)^2 + 2x\Delta x - 6x - 6\Delta x + 8 - x^2 + 6x - 8}{\Delta x} \\ &= \frac{(\Delta x)^2 + 2x\Delta x - 6\Delta x}{\Delta x} \\ &= \frac{\Delta x(\Delta x + 2x - 6)}{\Delta x} \\ \Rightarrow \frac{\Delta y}{\Delta x} &= 2x + \Delta x - 6 \dots\dots(iv)\end{aligned}$$

Since x changes from 3 to 3.1, then

$$\Delta x = 3.1 - 3 = 0.1$$

Put $x = 3$ and $\Delta x = 0.1$ in (iv)

$$\frac{\Delta y}{\Delta x} = 2(3) + 0.1 - 6 = 0.1 + 6 - 6 = 0.1 \quad \text{Ans.}$$

(c): $A = \pi r^2$ from $r = 2$ to $r = 2.1$

Solution: Since the average rate of change in A per unit in r is given by

$$\frac{\Delta A}{\Delta r} = \frac{f(r + \Delta r) - f(r)}{\Delta r} \dots\dots(i)$$

$$\text{As } A = f(r) = \pi r^2 \dots\dots(ii)$$

$$\begin{aligned}f(r + \Delta r) &= \pi(r + \Delta r)^2 = \pi(r^2 + (\Delta r)^2 + 2r\Delta r) \\ &= \pi r^2 + \pi(\Delta r)^2 + 2\pi r\Delta r \dots\dots(iii)\end{aligned}$$

Put (ii) and (iii) in (i)

$$\begin{aligned}\frac{\Delta A}{\Delta r} &= \frac{\pi r^2 + \pi(\Delta r)^2 + 2r\pi\Delta r - \pi r^2}{\Delta r} \\ &= \frac{\pi(\Delta r)^2 + 2r\pi\Delta r}{\Delta r} = \frac{\Delta r(\pi\Delta r + 2r\pi)}{\Delta r}\end{aligned}$$

$$\Rightarrow \frac{\Delta A}{\Delta r} = \pi\Delta r + 2r\pi \dots\dots\dots(iv)$$

As r changes from 2 to 2.1 , then

$$\Delta r = 2.1 - 2 = 0.1$$

Put $r = 2$ and $\Delta r = 0.1$ in (iv)

$$\frac{\Delta A}{\Delta r} = \pi(0.1) + 2(2)\pi = 0.1\pi + 4\pi = 4.1\pi$$

(d) $h = \sqrt{t} - 9$ from $t = 9$ to $t = 16$

Solution: Since the average rate of change in h per unit in t is given by

$$\frac{\Delta h}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t} \dots\dots(i)$$

$$\text{As } h = f(t) = \sqrt{t} - 9 \dots\dots(ii)$$

$$\text{Then } f(t + \Delta t) = \sqrt{t + \Delta t} - 9 \dots\dots(iii)$$

Put (ii) and (iii) in (i)

$$\begin{aligned}\frac{\Delta h}{\Delta t} &= \frac{\sqrt{t + \Delta t} - 9 - (\sqrt{t} - 9)}{\Delta t} \\ &= \frac{\sqrt{t + \Delta t} - 9 - \sqrt{t} + 9}{\Delta t}\end{aligned}$$

$$\Rightarrow \frac{\Delta h}{\Delta t} = \frac{\sqrt{t + \Delta t} - \sqrt{t}}{\Delta t} \dots\dots(iv)$$

Eq (iv) multiplied and divided by $\sqrt{t + \Delta t} + \sqrt{t}$

$$\frac{\Delta h}{\Delta t} = \frac{\sqrt{t + \Delta t} - \sqrt{t}}{\Delta t} \times \frac{\sqrt{t + \Delta t} + \sqrt{t}}{\sqrt{t + \Delta t} + \sqrt{t}}$$

$$\begin{aligned}
 &= \frac{(\sqrt{t+\Delta t})^2 - (\sqrt{t})^2}{\Delta t(\sqrt{t+\Delta t} + \sqrt{t})} \\
 &= \frac{t + \Delta t - t}{\Delta t(\sqrt{t+\Delta t} + \sqrt{t})} = \frac{\Delta t}{\Delta t(\sqrt{t+\Delta t} + \sqrt{t})} \\
 \Rightarrow \frac{\Delta h}{\Delta t} &= \frac{1}{\sqrt{t+\Delta t} + \sqrt{t}} \dots\dots(v)
 \end{aligned}$$

As t changes from 9 to 16, then

$$\Delta t = 16 - 9 = 7$$

Put $t = 9$ and $\Delta t = 7$ in (v)

$$\begin{aligned}
 \frac{\Delta h}{\Delta t} &= \frac{1}{\sqrt{9+7} + \sqrt{9}} = \frac{1}{\sqrt{16} + 3} \\
 &= \frac{1}{4+3} = \frac{1}{7}
 \end{aligned}$$

Q.3: A ball is thrown straight up. Its height after t seconds is given by the formula $h = -16t^2 + 80t$. Use definition for the rate of change to determine the average velocity $\frac{\Delta h}{\Delta t}$ for the specified intervals.

(a) From $t = 2$ to $t = 2.1$ (b) From $t = 2$ to $t = 2.01$

Solution: Since the average velocity of change is given by

$$\frac{\Delta h}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} \dots\dots(i)$$

$$\text{As } h = f(t) = -16t^2 + 80t \dots\dots(ii)$$

$$\begin{aligned}
 \Rightarrow f(t+\Delta t) &= -16(t+\Delta t)^2 + 80(t+\Delta t) \\
 &= -16(t^2 + (\Delta t)^2 + 2t\Delta t) + 80t + 80\Delta t \\
 &= -16t^2 - 16(\Delta t)^2 - 32t\Delta t + 80t + 80\Delta t \dots\dots(iii)
 \end{aligned}$$

Put (ii) and (iii) in (i)

$$\frac{\Delta h}{\Delta t} = \frac{-16t^2 - 16(\Delta t)^2 - 32t\Delta t + 80t + 80\Delta t - (-16t^2 + 80t)}{\Delta t}$$

$$= \frac{-16t^2 - 16(\Delta t)^2 - 32t\Delta t + 80t + 80\Delta t - 16t^2 - 80t}{\Delta t}$$

$$= \frac{-16(\Delta t)^2 - 32t\Delta t + 80\Delta t}{\Delta t}$$

$$= \frac{\Delta t(-16\Delta t - 32t + 80)}{\Delta t}$$

$$\Rightarrow \frac{\Delta h}{\Delta t} = -16\Delta t - 32t + 80 \quad \dots (iv)$$

(a) From $t = 2$ to $t = 2.1$

$$\text{Then } \Delta t = 2.1 - 2 = 0.1$$

Put $t = 2$ and $\Delta t = 0.1$ in (iv)

$$\Rightarrow \frac{\Delta h}{\Delta t} = -16(0.1) - 32(2) + 80$$

$$= -1.6 - 64 + 80 = 14.4 \quad \text{Ans.}$$

(b) From $t = 2$ to $t = 2.01$

$$\text{Then } \Delta t = 2.01 - 2 = 0.01$$

Put $t = 2$ and $\Delta t = 0.01$ in (iv)

$$\Rightarrow \frac{\Delta h}{\Delta t} = -16(0.01) - 32(2) + 80$$

$$= -0.16 - 64 + 80 = 15.84 \quad \text{Ans.}$$

Q.4: The rate of change of price is called inflation. The price

P in rupees after t years is $P(t) = 3t^2 + t + 1$. Use

definition for the rate of change to determine the

average rate of change of inflation from $t = 3$ to $t = 5$

years. What the rate of change means ?. Explain.

Solution: The average rate of change of inflation is given by

$$\frac{\Delta P}{\Delta t} = \frac{p(t + \Delta t) - p(t)}{\Delta t} \quad \dots (i)$$

$$\text{As } p(t) = 3t^2 + t + 1 \quad \dots (ii)$$

$$p(t + \Delta t) = 3(t + \Delta t)^2 + t + \Delta t + 1$$

$$\begin{aligned}
 &= 3(t^2 + (\Delta t)^2 + 2t\Delta t) + t + \Delta t + 1 \\
 &= 3t^2 + 3(\Delta t)^2 + 6t\Delta t + t + \Delta t + 1 \dots\dots (iii)
 \end{aligned}$$

Put (ii) and (iii) in (i)

$$\begin{aligned}
 \frac{\Delta P}{\Delta t} &= \frac{3t^2 + 3(\Delta t)^2 + 6t\Delta t + t + \Delta t + 1 - (3t^2 + t + 1)}{\Delta t} \\
 &= \frac{3t^2 + 3(\Delta t)^2 + 6t\Delta t + t + \Delta t + 1 - 3t^2 - t - 1}{\Delta t} \\
 &= \frac{3(\Delta t)^2 + 6t\Delta t + \Delta t}{\Delta t} \\
 &= \frac{\Delta t(3\Delta t + 6t + 1)}{\Delta t} \\
 \Rightarrow \frac{\Delta P}{\Delta t} &= 3\Delta t + 6t + 1 \dots\dots (iv)
 \end{aligned}$$

Since t changes from $t = 3$ to $t = 5$, then

$$\Delta t = 5 - 3 = 2$$

Put $t = 3$ and $\Delta t = 2$ in (iv)

$$\begin{aligned}
 \frac{\Delta P}{\Delta t} &= 3(2) + 6(3) + 1 = 6 + 18 + 1 \\
 &= 25
 \end{aligned}$$

$$\text{Thus } \Delta t = 2 \text{ and } \frac{\Delta P}{\Delta t} = 25$$

Thus the average rate of inflation per year is Rs 25 rupees per year.

Q.5: A farmer plants x acres of sugar beets. The profit generated is $f(x) = 1800x - 9x^2$. Determine the average rate of change of the profit, when the planted area is in between $x = 20$ acres and $x = 50$ acres. what the rate of change means? Explain.

Solution:

As by definition of average change of $f(x)$ is

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(a)$$

Given $f(x) = 1800x - 9x^2 \dots\dots(i)$

$$\begin{aligned} \Rightarrow f(x + \Delta x) &= 1800(x + \Delta x) - 9(x + \Delta x)^2 \\ &= 1800x + 1800\Delta x - 9(x^2 + 2x\Delta x + (\Delta x)^2) \\ &= 1800x + 1800\Delta x - 9x^2 - 18x\Delta x - 9(\Delta x)^2 \dots(ii) \end{aligned}$$

Put (i) and (ii) in (a)

$$\begin{aligned} \frac{\Delta f(x)}{\Delta x} &= \frac{1800x + 1800\Delta x - 9x^2 - 18x\Delta x - 9(\Delta x)^2 - (1800x - 9x^2)}{\Delta x} \\ &= \frac{1800x + 1800\Delta x - 9x^2 - 18x\Delta x - 9(\Delta x)^2 - 1800x + 9x^2}{\Delta x} \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{\Delta f(x)}{\Delta x} &= \frac{1800\Delta x - 18x\Delta x - 9(\Delta x)^2}{\Delta x} \\ &= \frac{\Delta x(1800 - 18x - 9\Delta x)}{\Delta x} \end{aligned}$$

$$\Rightarrow \frac{\Delta f(x)}{\Delta x} = 1800 - 18x - 9\Delta x \dots\dots(b)$$

As given $x = 20$ to increase $x = 50$

Now $\Rightarrow \Delta x = 50 - 20 = 30$

Put $x = 50$ and $\Delta x = 30$ in (b)

$$\begin{aligned} \Rightarrow \frac{\Delta f(x)}{\Delta x} &= 1800 - 18(50) - 9(30) \\ &= 1800 - 900 - 270 = 630 \text{ Ans.} \end{aligned}$$

The profit is increasing at the rate of Rs 630 per acre

When number of acrs increases from 20 to 30 .

Q.6: Use first principle rule to determine the derivative of the following functions.

(a) $f(x) = 3x$

Solution: As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots (i)$$

As $f(x) = 3x \dots\dots (ii)$

$$f(x + \Delta x) = 3(x + \Delta x) = 3x + 3\Delta x \dots\dots (iii)$$

Put (ii) and (iii) in (i)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{3x + 3\Delta x - 3x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{3\Delta x}{\Delta x}$$

$$\Rightarrow f'(x) = 3 \text{ Ans.}$$

(b) $f(x) = (5x + 6)^{\frac{1}{2}}$

Solution: As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots (i)$$

As $f(x) = (5x + 6)^{\frac{1}{2}} \dots\dots (ii)$

$$f(x + \Delta x) = [5(x + \Delta x) + 6]^{\frac{1}{2}}$$

$$= (5x + 6 + 5\Delta x)^{\frac{1}{2}} = \left[(5x + 6) \left(1 + \frac{5\Delta x}{5x + 6} \right) \right]^{\frac{1}{2}}$$

$$= (5x + 6)^{\frac{1}{2}} \left(1 + \frac{5\Delta x}{5x + 6} \right)^{\frac{1}{2}} \text{ expand by binomial expansion}$$

$$= (5x + 6)^{\frac{1}{2}} \left(1 + \frac{1}{2} \left(\frac{5\Delta x}{5x + 6} \right) + \frac{\frac{1}{2} \left(\frac{1}{2} - 1 \right)}{2!} \left(\frac{5\Delta x}{5x + 6} \right)^2 + \dots\dots \right)$$

$$= (5x + 6)^{\frac{1}{2}} + \frac{1}{2} (5x + 6)^{\frac{1}{2}} \left(\frac{5\Delta x}{5x + 6} \right) + \frac{1}{2!} \cdot \frac{1}{2} \left(\frac{1 - 2}{2} \right) (5x + 6)^{\frac{1}{2}} \left(\frac{5\Delta x}{5x + 6} \right)^2$$

$$= (5x + 6)^{\frac{1}{2}} + \frac{1}{2} (5x + 6)^{\frac{1}{2} - 1} (5\Delta x) - \frac{1}{8} (5x + 6)^{\frac{1}{2} - 2} (5\Delta x)^2$$

$$= (5x+6)^{\frac{1}{2}} + \frac{1}{2}(5x+6)^{-\frac{1}{2}}(5\Delta x) - \frac{1}{8}(5x+6)^{-\frac{3}{2}}(5\Delta x)^2 \dots\dots(iii)$$

Put (ii) and (iii) in (i)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[(5x+6)^{\frac{1}{2}} + \frac{1}{2}(5x+6)^{-\frac{1}{2}}(5\Delta x) - \frac{1}{8}(5x+6)^{-\frac{3}{2}}(5\Delta x)^2 \right] - (5x+6)^{\frac{1}{2}}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{1}{2}(5x+6)^{-\frac{1}{2}}(5\Delta x) - \frac{1}{8}(5x+6)^{-\frac{3}{2}}(5\Delta x)^2 \right]$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} \left[\frac{1}{2}(5x+6)^{-\frac{1}{2}}(5) - \frac{1}{8}(5x+6)^{-\frac{3}{2}}(25\Delta x) \right]$$

$$= \lim_{\Delta x \rightarrow 0} \left[\frac{1}{2}(5x+6)^{-\frac{1}{2}}(5) - \frac{1}{8}(5x+6)^{-\frac{3}{2}}(25\Delta x) \right]$$

$$= \frac{1}{2}(5x+6)^{-\frac{1}{2}}(5) - 0 \text{ Applying limits}$$

$$= \frac{5}{2(5x+6)^{\frac{1}{2}}} = \frac{5}{2\sqrt{5x+6}} \text{ Ans.}$$

(c) $f(x) = x^2 + 1$

Solution : As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \dots\dots(i)$$

As $f(x) = x^2 + 1 \dots\dots(ii)$

Then $f(x+\Delta x) = (x+\Delta x)^2 + 1$

$$= x^2 + (\Delta x)^2 + 2x\Delta x + 1 \dots\dots(iii)$$

Put (ii) and (iii) in (i)

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + (\Delta x)^2 + 2x\Delta x + 1 - (x^2 + 1)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + (\Delta x)^2 + 2x\Delta x + 1 - x^2 - 1}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^2 + 2x\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(\Delta x + 2x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (\Delta x + 2x) = 0 + 2x
 \end{aligned}$$

Thus $f'(x) = 2x$ Ans.

(d) $f(x) = 12 - x^2$

Solution : As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots (i)$$

As $f(x) = 12 - x^2 \dots\dots(ii)$

$$\begin{aligned}
 \text{Then } f(x + \Delta x) &= 12 - (x + \Delta x)^2 \\
 &= 12 - (x^2 + (\Delta x)^2 + 2x\Delta x) \\
 &= 12 - x^2 - (\Delta x)^2 - 2x\Delta x \dots\dots(iii)
 \end{aligned}$$

Put (ii) and (iii) in (i)

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{12 - x^2 - (\Delta x)^2 - 2x\Delta x - (12 - x^2)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{12 - x^2 - (\Delta x)^2 - 2x\Delta x - 12 + x^2}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-(\Delta x)^2 - 2x\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(-\Delta x - 2x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (-\Delta x - 2x) = 0 - 2x
 \end{aligned}$$

Thus $f'(x) = -2x$ Ans.

(e) $f(x) = 16x^2 - 7x$

Solution : As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots (i)$$

As $f(x) = 16x^2 - 7x \dots\dots (ii)$

$$\begin{aligned} \text{Then } f(x + \Delta x) &= 16(x + \Delta x)^2 - 7(x + \Delta x) \\ &= 16(x^2 + (\Delta x)^2 + 2x\Delta x) - 7x - 7\Delta x \\ &= 16x^2 + 16(\Delta x)^2 + 32x\Delta x - 7x - 7\Delta x \dots\dots (iii) \end{aligned}$$

Put (ii) and (iii) in (i)

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{16x^2 + 16(\Delta x)^2 + 32x\Delta x - 7x - 7\Delta x - (16x^2 - 7x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{16x^2 + 16(\Delta x)^2 + 32x\Delta x - 7x - 7\Delta x - 16x^2 + 7x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{16(\Delta x)^2 + 32x\Delta x - 7\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(16\Delta x + 32x - 7)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (16\Delta x + 32x - 7) = 0 + 32x - 7 \end{aligned}$$

$\Rightarrow f'(x) = 32x - 7$ Ans.

(f) $f(x) = \frac{7}{x}$

Solution : As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots (i)$$

As $f(x) = \frac{7}{x} \dots\dots (ii)$

Then $f(x + \Delta x) = \frac{7}{x + \Delta x} \dots\dots (iii)$

Put (ii) and (iii) in (i)

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\frac{7}{x+\Delta x} - \frac{7}{x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(\frac{7}{x+\Delta x} - \frac{7}{x} \right) \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(\frac{7x - 7x - 7\Delta x}{x(x+\Delta x)} \right) \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(\frac{-7\Delta x}{x(x+\Delta x)} \right)
 \end{aligned}$$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{-7}{x(x+\Delta x)} \\
 &= \frac{-7}{x(x+0)} = -\frac{7}{x^2} \text{ Ans.}
 \end{aligned}$$

Q.7: Use the graph of a function $f(x) = x^2 - 7x + 6$ to do the following :

- Find the derivative of the function at point P(5, -4) .
- Find the slope of tangent line on the curve $y = x^2 - 7x + 6$ at a point (5 , -4)
- View the slope of the tangent line on the curve at P(6, 0).

Solution: As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \dots\dots(i)$$

$$\text{Given } f(x) = x^2 - 7x + 6 \dots\dots(ii)$$

$$\begin{aligned}
 f(x+\Delta x) &= (x+\Delta x)^2 - 7(x+\Delta x) + 6 \\
 &= x^2 + 2x\Delta x + (\Delta x)^2 - 7x - 7\Delta x + 6 \dots(iii)
 \end{aligned}$$

Put (ii) and (iii) in (i)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 7x - 7\Delta x + 6 - (x^2 - 7x + 6)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 7x - 7\Delta x + 6 - x^2 + 7x - 6}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2 - 7\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x - 7)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 7)$$

$$\Rightarrow f'(x) = 2x + 0 - 7$$

$$\Rightarrow f'(x) = 2x - 7 \dots\dots(iv)$$

Eq(iv) is the derivative of $f(x)$ at any point $P(x, y)$

(a) The derivative of $f(x)$ at point $(5, -4)$.

Put $x = 5$ in (iv)

$$\text{Slope} = 2(5) - 7 = 10 - 7 = 3$$

(b) The slope of the tangent line at $(5, -4)$.

Put $x = 5$ in (iv)

$$\text{Slope} = 2(5) - 7 = 10 - 7 = 3$$

(c) The slope of the tangent line at $(6, 0)$.

Put $x = 6$ in (iv)

$$\text{Slope} = 2(6) - 7 = 12 - 7 = 5$$

Q.8: Use definition of derivative to determine the slope of the tangent line to the curve at a given point and then find out the tangent line equation on the curve at the same point, for the following curves.

(a) $f(x) = -x^2 + 7x$, $x = 3$

Solution: Slope of the tangent line is given by

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(i)$$

As $f(x) = -x^2 + 7x \dots\dots(ii)$

$$\begin{aligned}
 \text{Then } f(x + \Delta x) &= -(x + \Delta x)^2 + 7(x + \Delta x) \\
 &= -(x^2 + (\Delta x)^2 + 2x\Delta x) + 7x + 7\Delta x \\
 &= -x^2 - (\Delta x)^2 - 2x\Delta x + 7x + 7\Delta x \dots\dots(iii)
 \end{aligned}$$

Put (ii) and (iii) in (i)

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{-x^2 - (\Delta x)^2 - 2x\Delta x + 7x + 7\Delta x - (-x^2 + 7x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-x^2 - (\Delta x)^2 - 2x\Delta x + 7x + 7\Delta x + x^2 - 7x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-(\Delta x)^2 - 2x\Delta x + 7\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(-\Delta x - 2x + 7)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (-\Delta x - 2x + 7) = 0 - 2x + 7 \\
 \Rightarrow f'(x) &= -2x + 7 \dots(iv)
 \end{aligned}$$

Eq(iv) shows the slope of the tangent line at any point to the given curve.

Now the slope of the tangent line at $x = 3$

Then Put $x = 3$ in (iv)

$$f'(3) = -2(3) + 7 = -6 + 7 = 1$$

Now we find the point $(3, f(3))$ on the given curve.

Put $x = 3$ in (ii)

$$f(3) = -(3)^2 + 7(3) = -9 + 21 = 12$$

Now equation of tangent line at the point $(3, 12)$ is

$$y - y_1 = f'(3)(x - x_1) \dots\dots(v)$$

Put $x_1 = 3$, $y_1 = 12$ and $f'(3) = 1$ in (v)

$$y - 12 = 1(x - 3)$$

This is the required tangent line equation.

(b) $f(x) = 6x^2 - 11x - 10$, $x = 1$

Solution: Slope of the tangent line is given by

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots (i)$$

As $f(x) = 6x^2 - 11x - 10 \dots\dots (ii)$

$$\begin{aligned} \text{Then } f(x + \Delta x) &= 6(x + \Delta x)^2 - 11(x + \Delta x) - 10 \\ &= 6(x^2 + (\Delta x)^2 + 2x\Delta x) - 11x - 11\Delta x - 10 \\ &= 6x^2 + 6(\Delta x)^2 + 12x\Delta x - 11x - 11\Delta x - 10 \dots\dots (iii) \end{aligned}$$

Put (ii) and (iii) in (i)

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(\begin{array}{l} 6x^2 + 6(\Delta x)^2 + 12x\Delta x - 11x - 11\Delta x - 10 \\ - (6x^2 - 11x - 10) \end{array} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(\begin{array}{l} 6x^2 + 6(\Delta x)^2 + 12x\Delta x - 11x - 11\Delta x - 10 \\ - 6x^2 + 11x + 10 \end{array} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{6(\Delta x)^2 + 12x\Delta x - 11\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(6\Delta x + 12x - 11)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (6\Delta x + 12x - 11) = 0 + 12x - 11 \end{aligned}$$

$$\Rightarrow f'(x) = 12x - 11 \dots\dots (iv)$$

Eq(iv) shows the slope of the tangent line at any point to the given curve .

Now the slope of the tangent line at $x = 1$

Then Put $x = 1$ in (iv)

$$f'(1) = 12(1) - 11 = 12 - 11 = 1$$

Now we find the point $(1, f(1))$ on the given curve .

Put $x = 1$ in (ii)

$$f(1) = 6(1)^2 - 11(1) - 10 = 6 - 11 - 10 = -15$$

Now equation of tangent line at the point $(1, -15)$ is

$$y - y_1 = f'(1)(x - x_1) \dots\dots (v)$$

Put $x_1 = 1, y_1 = -15$ and $f'(1) = 1$ in (v)

$$y - (-15) = 1(x - 1)$$

$$\Rightarrow y + 15 = x - 1$$

This is the required tangent line equation .

(c) $f(x) = 3x^2 - 6x - 10$, $x = 0$

Solution: Slope of the tangent line is given by

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots (i)$$

As $f(x) = 3x^2 - 6x - 10 \dots\dots (ii)$

$$\begin{aligned} \text{Then } f(x + \Delta x) &= 3(x + \Delta x)^2 - 6(x + \Delta x) - 10 \\ &= 3(x^2 + (\Delta x)^2 + 2x\Delta x) - 6x - 6\Delta x - 10 \\ &= 3x^2 + 3(\Delta x)^2 + 6x\Delta x - 6x - 6\Delta x - 10 \dots\dots (iii) \end{aligned}$$

Put (ii) and (iii) in (i)

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(\begin{aligned} &3x^2 + 3(\Delta x)^2 + 6x\Delta x - 6x - 6\Delta x - 10 \\ &- (3x^2 - 6x - 10) \end{aligned} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(\begin{aligned} &3x^2 + 3(\Delta x)^2 + 6x\Delta x - 6x - 6\Delta x - 10 \\ &- 3x^2 + 6x + 10 \end{aligned} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{3(\Delta x)^2 + 6x\Delta x - 6\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(3\Delta x + 6x - 6)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (3\Delta x + 6x - 6) = 0 + 6x - 6 \end{aligned}$$

$$\Rightarrow f'(x) = 6x - 6 \dots\dots (iv)$$

Eq(iv) shows the slope of the tangent line at any point to the given curve .

Now the slope of the tangent line at $x = 0$

Then Put $x = 0$ in (iv)

$$f'(0) = 6(0) - 6 = -6$$

Now we find the point $(0, f(0))$ on the given curve .

Put $x = 0$ in (ii)

$$f(0) = 3(0)^2 - 6(0) - 10 = 0 - 0 - 10 = -10$$

Now equation of tangent line at the point $(0, -10)$ is

$$y - y_1 = f'(0)(x - x_1) \dots\dots(v)$$

Put $x_1 = 0$, $y_1 = -10$ and $f'(0) = -6$ in (v)

$$y - (-10) = -6(x - 0)$$

$$\Rightarrow y + 10 = -6(x - 0)$$

This is the required tangent line equation .

(d) $f(x) = 2x^2 + 3x - 4$, $x = 1$

Solution: Slope of the tangent line is given by

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(i)$$

As $f(x) = 2x^2 + 3x - 4 \dots\dots(ii)$

$$\begin{aligned} \text{Then } f(x + \Delta x) &= 2(x + \Delta x)^2 + 3(x + \Delta x) - 4 \\ &= 2(x^2 + (\Delta x)^2 + 2x\Delta x) + 3x + 3\Delta x - 4 \\ &= 2x^2 + 2(\Delta x)^2 + 4x\Delta x + 3x + 3\Delta x - 4 \dots\dots(iii) \end{aligned}$$

Put (ii) and (iii) in (i)

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(2x^2 + 2(\Delta x)^2 + 4x\Delta x + 3x + 3\Delta x - 4 - (2x^2 + 3x - 4) \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(2x^2 + 2(\Delta x)^2 + 4x\Delta x + 3x + 3\Delta x - 4 - 2x^2 - 3x + 4 \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{2(\Delta x)^2 + 4x\Delta x + 3\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2\Delta x + 4x + 3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2\Delta x + 4x + 3) = 0 + 4x + 3 \\ \Rightarrow f'(x) &= 4x + 3 \dots\dots(iv) \end{aligned}$$

Eq(iv) shows the slope of the tangent line at any point to the given curve .

Now the slope of the tangent line at $x = 1$

Then Put $x=1$ in (iv)

$$f'(1) = 4(1) + 3 = 4 + 3 = 7$$

Now we find the point $(1, f(1))$ on the given curve .

Put $x=1$ in (ii)

$$f(1) = 2(1)^2 + 3(1) - 4 = 2 + 3 - 4 = 1$$

Now equation of tangent line at the point $(1, 1)$ is

$$y - y_1 = f'(1)(x - x_1) \dots\dots(v)$$

Put $x_1 = 1$, $y_1 = 1$ and $f'(1) = 7$ in (v)

$$y - 1 = 7(x - 1)$$

This is the required tangent line equation .

Theorems of Differentiation :

(i) Constant rule $\frac{d}{dx}(c) = 0$, where c is constant .

Proof: Let $f(x) = c \dots\dots(i)$

$$f(x + \Delta x) = c \dots\dots(ii)$$

As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(iii)$$

Put (i) , (ii) in (iii)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{c - c}{\Delta x} = 0$$

(ii) Constant multiple rule: $\frac{d}{dx}[cf(x)] = c \frac{d}{dx} f(x)$

Proof: Let $y = cf(x) \dots\dots(i)$,

$$y + \Delta y = cf(x + \Delta x) \dots\dots(ii)$$

$$(ii) - (i) \Rightarrow \Delta y = cf(x + \Delta x) - cf(x)$$

$$\Rightarrow \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{cf(x + \Delta x) - cf(x)}{\Delta x}$$

$$\Rightarrow \frac{dy}{dx} = c \cdot \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\Rightarrow \frac{dy}{dx} = c \cdot \frac{d}{dx} [f(x)]$$

$$\Rightarrow \frac{d}{dx} [cf(x)] = c \frac{d}{dx} (f(x))$$

(iii) Sum Rule: $\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$

Proof: Let $y = f(x) + g(x)$ ----- (i)

$$y + \Delta y = f(x + \Delta x) + g(x + \Delta x) \text{ ----- (ii)}$$

Now (ii) - (i)

$$\Rightarrow \Delta y = f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)$$

Now

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x) + g(x + \Delta x) - g(x)}{\Delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} f(x) + \frac{d}{dx} g(x) \quad \text{Proved}$$

(iv) Difference rule: $\frac{d}{dx} [f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$

(v) Product Rules:

$$\frac{d}{dx} [f(x) \cdot g(x)] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

Proof: Let $y = f(x) \cdot g(x)$ ----- (i)

$$y + \Delta y = f(x + \Delta x) \cdot g(x + \Delta x) \text{ ----- (ii)}$$

Now (ii) - (i)

$$\Rightarrow \Delta y = f(x + \Delta x)g(x + \Delta x) - f(x) \cdot g(x)$$

$$\Rightarrow \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x}$$

$$\begin{aligned}
\Rightarrow \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) \right. \\
&\quad \left. + f(x+\Delta x)g(x) - f(x)g(x) \right) \\
\Rightarrow \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x)}{\Delta x} \{g(x+\Delta x) - g(x)\} \\
&\quad + \lim_{\Delta x \rightarrow 0} g(x) \left\{ \frac{f(x+\Delta x) - f(x)}{\Delta x} \right\} \\
\Rightarrow \frac{dy}{dx} &= f(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&\quad + g(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
\Rightarrow \frac{dy}{dx} &= f(x) \cdot \frac{d}{dx}(g(x)) + g(x) \cdot \frac{d}{dx}(f(x)) \\
\Rightarrow \frac{d}{dx}(f \cdot g) &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \quad \text{Proved}
\end{aligned}$$

(vi) Quotient Rule:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2}$$

Proof: Let $y = \frac{f(x)}{g(x)}$ ----- (i)

$$y + \Delta y = \frac{f(x+\Delta x)}{g(x+\Delta x)} \text{ ----- (ii)}$$

Now (ii) - (i)

$$\Rightarrow \Delta y = \frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}$$

$$\Rightarrow \Delta y = \frac{f(x+\Delta x)g(x) - g(x+\Delta x)f(x)}{g(x+\Delta x)g(x)}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x \cdot g(x+\Delta x)g(x)} \left(f(x+\Delta x)g(x) - f(x)g(x) \right. \\
\left. - g(x+\Delta x)f(x) + f(x)g(x) \right)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{g(x+\Delta x)g(x)} \left\{ g(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \right. \\ &\quad \left. - f(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \right\} \\ \frac{dy}{dx} &= \frac{1}{g(x)g(x)} \left\{ g(x) \cdot \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x) \right\} \\ \frac{dy}{dx} &= \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2} \\ \Rightarrow \frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) &= \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2} \text{ Proved}\end{aligned}$$

Example 5: Differentiate the following constant functions .

- (a) $f(x) = 13$ (b) $f(x) = 3$ (c) $f(x) = 7\pi$
(d) $f(x) = \sqrt{149}$

Solution: (a) $f(x) = 13$, differentiate w . r . t . to x

$$\frac{d}{dx} (f(x)) = \frac{d}{dx} (13) = 0$$

(b) $f(x) = 3$, differentiate w . r . t . to x

$$\frac{d}{dx} (f(x)) = \frac{d}{dx} (3) = 0$$

(c) $f(x) = 7\pi$, differentiate w . r . t . to x

$$\frac{d}{dx} (f(x)) = \frac{d}{dx} (7\pi) = 0$$

(d) $f(x) = \sqrt{149}$, differentiate w . r . t . to x

$$\frac{d}{dx} (f(x)) = \frac{d}{dx} (\sqrt{149}) = 0$$

Example 6: Differentiate the following functions .

(a) $f(x) = 4x^3$ (b) $f(x) = 0.555x^6$

Solution: (a) $f(x) = 4x^3$, differentiate w . r . t . to x

$$\begin{aligned}\frac{d}{dx}(f(x)) &= \frac{d}{dx}(4x^3) \\ &= 4(3)x^{3-1} = 12x^2\end{aligned}$$

(b) $f(x) = 0.555x^6$, differentiate w . r . t . to x

$$\begin{aligned}\frac{d}{dx}(f(x)) &= \frac{d}{dx}(0.555x^6) \\ &= (0.555)(6)x^{6-1} = 3.33x^5\end{aligned}$$

Example 8: Differentiate the following functions .

(a) $y = 4x^3 - 2x^2 + 5x$ (b) $y = (x^2 - 2x)(x^3 - 3)$

(c) $y = \frac{x^2 + 13x + 9}{x^2 + 11x + 3}$

Solution: (a) $y = 4x^3 - 2x^2 + 5x$, differentiate w . r . t . to x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(4x^3 - 2x^2 + 5x) \quad \text{use sum and difference rule} \\ &= \frac{d}{dx}(4x^3) - \frac{d}{dx}(2x^2) + \frac{d}{dx}(5x) \\ &= 4(3)x^2 - 2(2)x + 5 \\ &= 12x^2 - 4x + 5\end{aligned}$$

(b) $y = (x^2 - 2x)(x^3 - 3)$, differentiate w . r . t . to x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}((x^2 - 2x)(x^3 - 3)) \quad \text{use product rule} \\ &= (x^2 - 2x)\frac{d}{dx}(x^3 - 3) + (x^3 - 3)\frac{d}{dx}(x^2 - 2x) \\ &= (x^2 - 2x)(3x^2) + (x^3 - 3)(2x - 2) \\ &= 3x^4 - 6x^3 + 2x^4 - 2x^3 - 6x + 6 \\ &= 5x^4 - 8x^3 - 6x + 6 \quad \text{Ans.}\end{aligned}$$

(c) $y = \frac{x^2 + 13x + 9}{x^2 + 11x + 3}$, differentiate w.r.t. to x

$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2 + 13x + 9}{x^2 + 11x + 3} \right)$ use quotient rule

$$= \frac{(x^2 + 11x + 3) \frac{d}{dx} (x^2 + 13x + 9) - (x^2 + 13x + 9) \frac{d}{dx} (x^2 + 11x + 3)}{(x^2 + 11x + 3)^2}$$

$$= \frac{(x^2 + 11x + 3)(2x + 13) - (x^2 + 13x + 9)(2x + 11)}{(x^2 + 11x + 3)^2}$$

$$= \frac{1}{(x^2 + 11x + 3)^2} \left(\begin{matrix} 2x^3 + 13x^2 + 22x^2 + 143x + 6x + 39 \\ -2x^3 - 11x^2 - 26x^2 - 143x - 18x - 99 \end{matrix} \right)$$

$$= \frac{1}{(x^2 + 11x + 3)^2} (-2x^2 - 12x - 60)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2(x^2 + 6x + 30)}{(x^2 + 11x + 3)^2} \quad \text{Ans.}$$

Application of Differentiation :

Marginal Cost or Marginal Analysis: The rate of change of cost is called **Marginal cost**. It is obtained by taking the first derivative of the cost. Similarly the Marginal profit and Marginal revenue is the first derivative of profit and revenue.

Example 9: The cost in (million) dollars to produce

x units of wheat is given by $C(x) = 5000 + 20x + 10\sqrt{x}$.

Find the marginal cost, when

(a) $x = 9$ units (b) $x = 16$ units (c) $x = 25$ units

(d) As more wheat is produce, what happens to the Marginal cost.

Solution: The cost function is given

$$C(x) = 5000 + 20x + 10\sqrt{x} \quad , \text{differentiation w.r.t. to } x$$

$$\begin{aligned}\frac{d}{dx}(C(x)) &= \frac{d}{dx}(5000 + 20x + 10\sqrt{x}) \\ \Rightarrow C'(x) &= \frac{d}{dx}(5000) + 20\frac{d}{dx}(x) + 10\frac{d}{dx}(x^{\frac{1}{2}}) \\ &= 0 + 20(1) + (10)\left(\frac{1}{2}\right)x^{\frac{1}{2}-1} \\ &= 20 + 5x^{-\frac{1}{2}} \\ \Rightarrow C'(x) &= 20 + \frac{5}{\sqrt{x}} \dots\dots(i)\end{aligned}$$

(a) **Marginal cost when $x = 9$ units :**

Put $x = 9$ in (i)

$$\begin{aligned}C'(9) &= 20 + \frac{5}{\sqrt{9}} = 20 + \frac{5}{3} \\ &= \frac{60 + 5}{3} = \frac{65}{3} = 21.67 \text{ dollars.}\end{aligned}$$

(b) **Marginal cost when $x = 16$ units :**

Put $x = 16$ in (i)

$$\begin{aligned}C'(16) &= 20 + \frac{5}{\sqrt{16}} = 20 + \frac{5}{4} \\ &= \frac{80 + 5}{4} = \frac{85}{4} = 21.25 \text{ dollars.}\end{aligned}$$

(c) **Marginal cost when $x = 25$ units :**

Put $x = 25$ in (i)

$$\begin{aligned}C'(25) &= 20 + \frac{5}{\sqrt{25}} = 20 + \frac{5}{5} \\ &= 20 + 1 = 21 \text{ dollars.}\end{aligned}$$

(d) **When more wheat is produce then Marginal cost**

is decreases and approaches to 21 dollars .

Exercise 3.2:**Q.1:** Differentiate $f(x) + g(x)$ and $f(x) - g(x)$ if

(a) $f(x) = 3x + 7$ and $g(x) = 6x^2 + 2x - 3$

Solution: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

$$= \frac{d}{dx}(3x + 7) + \frac{d}{dx}(6x^2 + 2x - 3)$$

$$= 3 + 0 + 12x + 2 - 0$$

$$= 12x + 5 \text{ Ans.}$$

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$$

$$= \frac{d}{dx}(3x + 7) - \frac{d}{dx}(6x^2 + 2x - 3)$$

$$= 3 + 0 - (12x + 2 - 0)$$

$$= 3 - 12x - 2$$

$$= 1 - 12x \text{ Ans.}$$

(b) $f(x) = 17x^2 - 15$ and $g(x) = \frac{1}{3}x^2 + \frac{1}{2}x - 5$

Solution: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

$$= \frac{d}{dx}(17x^2 - 15) + \frac{d}{dx}\left(\frac{1}{3}x^2 + \frac{1}{2}x - 5\right)$$

$$= (17)(2x) - 0 + \frac{1}{3}(2x) + \frac{1}{2} - 0$$

$$= 34x + \frac{2}{3}x + \frac{1}{2} = \frac{102x + 2x}{3} + \frac{1}{2}$$

$$= \frac{104}{3}x + \frac{1}{2} \text{ Ans.}$$

Now

$$\begin{aligned}
 \frac{d}{dx}(f(x) - g(x)) &= \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x)) \\
 &= \frac{d}{dx}(17x^2 - 15) - \frac{d}{dx}\left(\frac{1}{3}x^2 + \frac{1}{2}x - 5\right) \\
 &= (17)(2x) - 0 - \left(\frac{1}{3}(2x) + \frac{1}{2} - 0\right) \\
 &= 34x - \frac{2}{3}x - \frac{1}{2} = \frac{102x - 2x}{3} - \frac{1}{2} \\
 &= \frac{100}{3}x - \frac{1}{2} \quad \text{Ans.}
 \end{aligned}$$

(c) $f(x) = x^3 - \frac{3}{2}$ and $g(x) = 3x^3 - 4x^2 + 2$

Solution: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

$$\begin{aligned}
 &= \frac{d}{dx}\left(x^3 - \frac{3}{2}\right) + \frac{d}{dx}(3x^3 - 4x^2 + 2) \\
 &= 3x^2 - 0 + 3(3)x^2 - 4(2)x + 0 \\
 &= 3x^2 + 9x^2 - 8x \\
 &= 12x^2 - 8x
 \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dx}(f(x) - g(x)) &= \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x)) \\
 &= \frac{d}{dx}\left(x^3 - \frac{3}{2}\right) - \frac{d}{dx}(3x^3 - 4x^2 + 2) \\
 &= 3x^2 - 0 - \{3(3)x^2 - 4(2)x + 0\} \\
 &= 3x^2 - 9x^2 + 8x \\
 &= -6x^2 + 8x
 \end{aligned}$$

(d) $f(x) = 4x^3 - 5x$ and $g(x) = \frac{3}{5}x^2 - 2x$

Solution: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

$$\begin{aligned}
 &= \frac{d}{dx}(4x^3 - 5x) + \frac{d}{dx}\left(\frac{3}{5}x^2 - 2x\right) \\
 &= 12x^2 - 5 + \frac{3}{5}(2x) - 2 \\
 &= 12x^2 + \frac{6}{5}x - 7
 \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dx}(f(x) - g(x)) &= \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x)) \\
 &= \frac{d}{dx}(4x^3 - 5x) - \frac{d}{dx}\left(\frac{3}{5}x^2 - 2x\right) \\
 &= 12x^2 - 5 - \left(\frac{3}{5}(2x) - 2\right) \\
 &= 12x^2 - 5 - \frac{6}{5}x + 2 \\
 &= 12x^2 - \frac{6}{5}x - 3
 \end{aligned}$$

Q.2: Use the product rule to find out the derivative of the following functions.

(a) $y = (x^2 - 2)(3x + 1)$

Solution: Differentiate w . r . t . to x

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx}((x^2 - 2)(3x + 1)) \quad \text{use product rule} \\
 \Rightarrow y' &= (x^2 - 2)\frac{d}{dx}(3x + 1) + (3x + 1)\frac{d}{dx}(x^2 - 2) \\
 &= (x^2 - 2)(3) + (3x + 1)(2x) \\
 &= 3x^2 - 6 + 6x^2 + 2x \\
 \Rightarrow y' &= 9x^2 + 2x - 6 \quad \text{Ans.}
 \end{aligned}$$

(b) $y = (7x^4 + 2x)(x^2 - 4)$

Solution: Differentiate w . r . t . to x

$$\frac{dy}{dx} = \frac{d}{dx} ((7x^4 + 2x)(x^2 - 4)) \quad \text{use product rule}$$

$$\begin{aligned} \Rightarrow y' &= (7x^4 + 2x) \frac{d}{dx} (x^2 - 4) + (x^2 - 4) \frac{d}{dx} (7x^4 + 2x) \\ &= (7x^4 + 2x)(2x) + (x^2 - 4)(28x^3 + 2) \\ &= 14x^5 + 4x^2 + 28x^5 + 2x^2 - 112x^3 - 8 \\ \Rightarrow y' &= 42x^5 - 112x^3 + 6x^2 - 8 \quad \text{Ans.} \end{aligned}$$

(c) $y = (2x - 3)(\sqrt{x} - 1)$

Solution: Differentiate w . r . t . to x ,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left((2x - 3) \left(x^{\frac{1}{2}} - 1 \right) \right) \quad \text{use product rule} \\ &= (2x - 3) \frac{d}{dx} \left(x^{\frac{1}{2}} - 1 \right) + \left(x^{\frac{1}{2}} - 1 \right) \frac{d}{dx} (2x - 3) \\ &= (2x - 3) \left(\frac{1}{2} x^{\frac{1}{2} - 1} \right) + \left(x^{\frac{1}{2}} - 1 \right) (2) \\ &= (2x - 3) \left(\frac{1}{2} x^{-\frac{1}{2}} \right) + 2x^{\frac{1}{2}} - 2 \\ &= x^{1 - \frac{1}{2}} - \frac{3}{2} x^{\frac{1}{2}} + 2x^{\frac{1}{2}} - 2 = x^{\frac{1}{2}} - \frac{3}{2} x^{\frac{1}{2}} + 2x^{\frac{1}{2}} - 2 \\ &= 3x^{\frac{1}{2}} - \frac{3}{2} x^{\frac{1}{2}} - 2 \quad \text{Ans.} \end{aligned}$$

(d) $y = (-3\sqrt{x} + 6)(4\sqrt{x} - 2)$

Solution: Differentiate w . r . t . to x ,

$$\frac{dy}{dx} = \frac{d}{dx} \left((-3x^{\frac{1}{2}} + 6) \left(4x^{\frac{1}{2}} - 2 \right) \right) \quad \text{use product rule}$$

$$\begin{aligned}
&= \left(-3x^{\frac{1}{2}} + 6 \right) \frac{d}{dx} \left(4x^{\frac{1}{2}} - 2 \right) + \left(4x^{\frac{1}{2}} - 2 \right) \frac{d}{dx} \left(-3x^{\frac{1}{2}} + 6 \right) \\
&= \left(-3x^{\frac{1}{2}} + 6 \right) \left(4 \left(\frac{1}{2} \right) x^{\frac{1}{2}-1} \right) + \left(4x^{\frac{1}{2}} - 2 \right) \left(-3 \left(\frac{1}{2} \right) x^{\frac{1}{2}-1} \right) \\
&= \left(-3x^{\frac{1}{2}} + 6 \right) \left(2x^{-\frac{1}{2}} \right) + \left(4x^{\frac{1}{2}} - 2 \right) \left(-\frac{3}{2} x^{-\frac{1}{2}} \right) \\
&= -6x^{\frac{1}{2}-\frac{1}{2}} + 12x^{-\frac{1}{2}} - 6x^{\frac{1}{2}-\frac{1}{2}} + 3x^{-\frac{1}{2}} \\
&= -6x^0 + 12x^{-\frac{1}{2}} - 6x^0 + 3x^{-\frac{1}{2}} \\
&= -6 + 12x^{-\frac{1}{2}} - 6 + 3x^{-\frac{1}{2}} \\
&\Rightarrow y' = -12 + 15x^{-\frac{1}{2}} \text{ Ans.}
\end{aligned}$$

Q.3: Use the quotient rule to find out the derivative of the following functions .

(a) $y = \frac{3x-5}{x-4}$

Solution: Differentiation w . r . t . to x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{3x-5}{x-4} \right) \text{ Use quotient rule} \\
&= \frac{(x-4) \frac{d}{dx} (3x-5) - (3x-5) \frac{d}{dx} (x-4)}{(x-4)^2} \\
&= \frac{(x-4)(3) - (3x-5)(1)}{(x-4)^2} \\
&= \frac{3x-12-3x+5}{(x-4)^2} \\
&\Rightarrow y' = -\frac{7}{(x-4)^2} \text{ Ans.}
\end{aligned}$$

$$(b) \quad y = \frac{-x^2 + 6x}{4x^3 + 1}$$

Solution: Differentiation w . r . t . to x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{-x^2 + 6x}{4x^3 + 1} \right) \quad \text{Use quotient rule}$$

$$\begin{aligned} \Rightarrow y' &= \frac{(4x^3 + 1) \frac{d}{dx}(-x^2 + 6x) - (-x^2 + 6x) \frac{d}{dx}(4x^3 + 1)}{(4x^3 + 1)^2} \\ &= \frac{(4x^3 + 1)(-2x + 6) - (-x^2 + 6x)(12x^2)}{(4x^3 + 1)^2} \\ &= \frac{-8x^4 + 24x^3 - 2x + 6 + 12x^4 - 72x^3}{(4x^3 + 1)^2} \\ &= \frac{4x^4 - 48x^3 - 2x + 6}{(4x^3 + 1)^2} \quad \text{Ans.} \end{aligned}$$

$$(c) \quad y = \frac{5x + 6}{\sqrt{x}}$$

Solution: Differentiation w . r . t . to x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{5x + 6}{x^{\frac{1}{2}}} \right) \quad \text{Use quotient rule}$$

$$\begin{aligned} &= \frac{(x^{\frac{1}{2}}) \frac{d}{dx}(5x + 6) - (5x + 6) \frac{d}{dx}(x^{\frac{1}{2}})}{(x^{\frac{1}{2}})^2} \end{aligned}$$

$$= \frac{1}{x} \left((x^{\frac{1}{2}})(5) - (5x + 6) \left(\frac{1}{2} x^{\frac{1}{2}-1} \right) \right)$$

$$= \frac{1}{x} \left(5x^{\frac{1}{2}} - \frac{1}{2}(5x + 6)(x^{-\frac{1}{2}}) \right)$$

$$\begin{aligned}
 &= \frac{1}{x} \left(5x^{\frac{1}{2}} - \frac{5}{2}x^{-\frac{1}{2}} - 3x^{-\frac{1}{2}} \right) \\
 &= \frac{1}{x} \left(5x^{\frac{1}{2}} - \frac{5}{2}x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} \right) \\
 \Rightarrow y' &= \frac{1}{x} \left(\frac{10x^{\frac{1}{2}} - 5x^{\frac{1}{2}}}{2} - \frac{3}{x^{\frac{1}{2}}} \right) \\
 \Rightarrow y' &= \frac{1}{x} \left(\frac{5}{2}\sqrt{x} - \frac{3}{\sqrt{x}} \right) \quad \text{Ans.}
 \end{aligned}$$

$$(d) \quad f(p) = \frac{(2p+3)(4p-1)}{(3p+2)}$$

$$\begin{aligned}
 \text{Solution: } f(p) &= \frac{(2p+3)(4p-1)}{(3p+2)} \\
 \Rightarrow f(p) &= \frac{8p^2 - 2p + 12p - 3}{3p+2} \\
 \Rightarrow f(p) &= \frac{8p^2 + 10p - 3}{3p+2}
 \end{aligned}$$

Differentiation w . r . t . to p

$$\begin{aligned}
 \frac{d}{dp}(f(p)) &= \frac{d}{dp} \left(\frac{8p^2 + 10p - 3}{3p+2} \right) \quad \text{Use quotient rule} \\
 \Rightarrow f'(p) &= \frac{(3p+2) \frac{d}{dp}(8p^2 + 10p - 3) - (8p^2 + 10p - 3) \frac{d}{dp}(3p+2)}{(3p+2)^2} \\
 &= \frac{(3p+2)(16p+10) - (8p^2 + 10p - 3)(3)}{(3p+2)^2} \\
 &= \frac{48p^2 + 30p + 32p + 20 - 24p^2 - 30p + 9}{(3p+2)^2}
 \end{aligned}$$

$$\Rightarrow f'(p) = \frac{24p^2 + 32p + 29}{(3p + 2)^2} \quad \text{Ans. (h)}$$

Q.4: Find an equation of a tangent line to the graph of the function at the particular point in the following problems.

(a) $f(x) = 3x - 7$ at $(3, 2)$

Solution: $f(x) = 3x - 7$ Differentiation w . r . t. to x

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(3x - 7)$$

$$\Rightarrow f'(x) = 3 \quad \dots(i)$$

Now the slope of the tangent line at point $(3, 2)$

Put $x = 3$ in (i)

$$f'(3) = 3$$

Now equation of the tangent line at point $(3, 2)$ is

$$y - y_1 = f'(3)(x - x_1) \quad \dots(ii)$$

Put $x_1 = 3$, $y_1 = 2$ and $f'(3) = 3$ in (ii)

$$y - 2 = 3(x - 3)$$

$$\Rightarrow y = 3x - 9 + 2$$

$$\Rightarrow y = 3x - 7 \quad \text{Ans.}$$

(b) $f(x) = x^3$ at $x = -\frac{1}{2}$

Solution: $f(x) = x^3 \dots(i)$ Differentiation w . r . t. to x

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^3)$$

$$\Rightarrow f'(x) = 3x^2 \quad \dots(ii)$$

Put $x = -\frac{1}{2}$ in (ii)

$$f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$$

Now the slope of the tangent line at point $\left(-\frac{1}{2}, -\frac{1}{8}\right)$

Put $x = -\frac{1}{2}$ in (i)

$$f'\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^2 = \frac{3}{4}$$

Now equation of the tangent line at point $\left(-\frac{1}{2}, -\frac{1}{8}\right)$ is

$$y - y_1 = f'\left(-\frac{1}{2}\right)(x - x_1) \dots\dots(iii)$$

Put $x_1 = -\frac{1}{2}$, $y_1 = -\frac{1}{8}$ and $f'\left(-\frac{1}{2}\right) = \frac{3}{4}$ in (iii)

$$y - \left(-\frac{1}{8}\right) = \frac{3}{4}\left(x - \left(-\frac{1}{2}\right)\right)$$

$$\Rightarrow y + \frac{1}{8} = \frac{3}{4}\left(x + \frac{1}{2}\right) \text{ Multiplied both sides by 8}$$

$$\Rightarrow 8y + 1 = 6\left(x + \frac{1}{2}\right)$$

$$\Rightarrow 8y + 1 = 6x + 3$$

$$\Rightarrow 0 = 6x - 8y + 3 - 1$$

$$\Rightarrow 6x - 8y + 2 = 0$$

$$\Rightarrow 2(3x - 4y + 1) = 0$$

$$\Rightarrow 3x - 4y + 1 = 0 \text{ Ans.}$$

$$(c) f(x) = \frac{1}{x+3} \text{ at } x = 2$$

Solution: $f(x) = \frac{1}{x+3} \dots(i)$ Differentiation w . r . t. to x

$$\begin{aligned}\frac{d}{dx}(f(x)) &= \frac{d}{dx}\left(\frac{1}{x+3}\right) \\ &= \frac{(x+3)\frac{d}{dx}(1) - (1)\frac{d}{dx}(x+3)}{(x+3)^2} \\ \Rightarrow f'(x) &= \frac{(x+3)(0) - 1}{(x+3)^2} \\ &= -\frac{1}{(x+3)^2} \dots\dots(ii)\end{aligned}$$

Put $x = 2$ in (i)

$$f(2) = \frac{1}{2+3} = \frac{1}{5}$$

Now the slope of the tangent line at point $\left(2, \frac{1}{5}\right)$

Put $x = 2$ in (ii)

$$f'(2) = -\frac{1}{(2+3)^2} = -\frac{1}{25}$$

Now equation of the tangent line at point $\left(2, \frac{1}{5}\right)$ is

$$y - y_1 = f'(2)(x - x_1) \dots\dots(iii)$$

Put $x_1 = 2$, $y_1 = \frac{1}{5}$ and $f'(2) = -\frac{1}{25}$ in (iii)

$$y - \frac{1}{5} = -\frac{1}{25}(x - 2)$$

$$\Rightarrow 25y - 5 = -(x - 2) \text{ Multiplied both sides by 25}$$

$$\Rightarrow 25y - 5 = -x + 2$$

$$\Rightarrow x + 25y - 5 - 2 = 0$$

$$\Rightarrow x + 25y - 7 = 0 \text{ Ans.}$$

$$(d) f(x) = \frac{x}{x-2} \text{ at } (3, 3)$$

Solution: $f(x) = \frac{x}{x-2}$

Differentiation w . r . t . to x

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{x}{x-2}\right) \text{ use quotient rule}$$

$$= \frac{(x-2)\frac{d}{dx}(x) - (x)\frac{d}{dx}(x-2)}{(x-2)^2}$$

$$\Rightarrow f'(x) = \frac{(x-2)(1) - (x)(1)}{(x-2)^2}$$

$$= \frac{x-2-x}{(x-2)^2}$$

$$\Rightarrow f'(x) = -\frac{2}{(x-2)^2} \dots\dots(i)$$

Now the slope of the tangent line at point at $(3, 3)$

Put $x = 3$ in (i)

$$f'(3) = -\frac{2}{(3-2)^2} = -\frac{2}{(1)^2} = -2$$

Now equation of the tangent line at point $(3, 3)$ is

$$y - y_1 = f'(3)(x - x_1) \dots\dots(ii)$$

Put $x_1 = 3$, $y_1 = 3$ and $f'(3) = -2$ in (ii)

$$y - 3 = -2(x - 3)$$

$$\Rightarrow y - 3 = -2x + 6$$

$$\Rightarrow y = -2x + 6 + 3$$

$$\Rightarrow y = -2x + 9 \text{ Ans.}$$

Q.5: For a thin lens of constant focal length P , the object distance x and the image distance y are related by

the formula $\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$

(a) Solve the above equation for y in terms of x and p .

(b) Determine the rate of change of y with respect to x .

Solution: (a) $\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$

$$\Rightarrow \frac{1}{y} = \frac{1}{p} - \frac{1}{x}$$

$$\Rightarrow \frac{1}{y} = \frac{x-p}{xp}$$

$$\Rightarrow xp = y(x-p)$$

$$\text{or } y(x-p) = xp$$

$$\Rightarrow y = \frac{xp}{x-p} \dots\dots(i)$$

(b) Differentiation (i) w . r . t . to x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{xp}{x-p} \right) \quad \text{use quotient rule}$$

$$\Rightarrow y' = \frac{(x-p) \frac{d}{dx}(xp) - (xp) \frac{d}{dx}(x-p)}{(x-p)^2}$$

Since p is constant then $\frac{dp}{dx} = 0$

$$\text{Now } y' = \frac{(x-p)(p) - (xp)(1)}{(x-p)^2}$$

$$= \frac{xp - p^2 - xp}{(x-p)^2}$$

$$\Rightarrow y' = -\frac{p^2}{(x-p)^2} \quad \text{Ans.}$$

Chain Rule: If y is the function of u and u is the function of x . i.e.

$y = f(u)$ and $u = g(x)$ Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots(A)$$

Eq(A) is called chain rule of differentiation .

Composite Function: A function $h(x)$ is a composite function of functions $f(x)$ and $g(x)$ if
 $h(x) = f(g(x))$.

The domain of $h(x)$ is the set of all real numbers x such that x is in the domain of $g(x)$ and $g(x)$ is the domain of $f(x)$.

Derivative of composite function:

If y is the function of u and u is the function of x . i.e.

$y = f(u)$ and $u = g(x)$. Then its derivative is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots(B)$$

Generalized power rule: If $y = (f(x))^n$ then

$\frac{dy}{dx} = n(f(x))^{n-1} \cdot \frac{d}{dx}(f(x))$ is called generalized power rule .

Example 10: Differentiate the following functions
 w . r . t . x

(a) $f(x) = (4x - 3)^3$ (b) $f(x) = \sqrt{15x^2 + 1}$

Solution: (a) $f(x) = (4x - 3)^3$ Differentiate w. r. t. x .

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}((4x - 3)^3) \text{ use generalized power rule}$$

$$= 3(4x - 3) \frac{d}{dx}(4x - 3)$$

$$= 3(4x - 3)(4)$$

$$\Rightarrow f'(x) = 12(4x - 3) \text{ Ans.}$$

(b) $f(x) = \sqrt{15x^2 + 1}$, Differentiate w. r. t. x .

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(15x^2 + 1)^{\frac{1}{2}} \quad \text{use generalized power rule}$$

$$\begin{aligned} \Rightarrow f'(x) &= \frac{1}{2}(15x^2 + 1)^{\frac{1}{2}-1} \frac{d}{dx}(15x^2 + 1) \\ &= \frac{1}{2}(15x^2 + 1)^{-\frac{1}{2}}(30x) \\ &= \frac{15x}{\sqrt{15x^2 + 1}} \quad \text{Ans.} \end{aligned}$$

Example 11: The revenue realized by a small city from the collection of fines from parking tickets is given by

$$R(x) = \frac{8000x}{x+2}$$

Where x is the number of work hours each day that can be devoted to parking patrol. At the outbreak of a flu epidemic, 30 work hours are used daily in parking patrol, but during the epidemic that number is decreasing at the rate of 6 work hours per day. How fast is revenue from parking fines decreasing during the epidemic.

Solution: Given $R(x) = \frac{8000x}{x+2}$ (i)

Here the number of work hours each day = $x = 30$ work hours

Day = t

The rate of decreasing work hour per day = 6

i.e. $\frac{dx}{dt} = -6$... (ii) negative sign shows that rate of decreasing

we will find rate of change of revenue $\frac{d}{dt}(R(x))$

As by chain rule $\frac{dR}{dt} = \frac{dR}{dx} \cdot \frac{dx}{dt}$ (iii)

Differentiate (i) w.r.t. to x

$$\frac{dR}{dx} = \frac{d}{dx} \left(\frac{8000x}{x+2} \right) \quad \text{use quotient rule}$$

$$\begin{aligned}
 \Rightarrow \frac{dR}{dx} &= \frac{(x+2) \frac{d}{dx}(8000x) - (8000x) \frac{d}{dx}(x+2)}{(x+2)^2} \\
 &= \frac{(x+2)(8000) - (8000x)(1)}{(x+2)^2} \\
 &= \frac{8000x + 16000 - 8000x}{(x+2)^2} \\
 \Rightarrow R'(x) &= \frac{16000}{(x+2)^2} \dots\dots\dots(iv)
 \end{aligned}$$

Put $x = 30$ in (iv)

$$\Rightarrow \frac{dR}{dx} = R'(30) = \frac{16000}{(30+2)^2} = \frac{16000}{1024} = 15.625 \dots\dots(v)$$

Put (ii) and (v) in (iii)

$$\frac{dR}{dt} = (15.625)(-6) = -93.75$$

This shows that the revenue is lost by approximately 94 rupees per day.

Note: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ then $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are reciprocal of each other.

Example .12. Verify result $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ for the following

Problems .

$$(a) f(x) = (4x-3)^3 \quad (b) f(x) = \sqrt{15x^2+1}$$

Solution: (a) $\frac{d}{dx}(y) = \frac{d}{dx}[(4x-3)^3]$

$$\Rightarrow \frac{dy}{dx} = 3(4x-3)^2 \cdot \frac{d}{dx}(4x-3)$$

$$\Rightarrow \frac{dy}{dx} = 3(4x-4)^2(4)$$

$$\Rightarrow \frac{dy}{dx} = 12(3x-4)^2 \dots\dots(i)$$

$$\text{Now from (i)} \Rightarrow \frac{dx}{dy} = \frac{1}{12(4x-3)^2}$$

$$\Rightarrow \frac{\frac{1}{dx}}{\frac{dy}{12(4x-3)^2}} = \frac{1}{1} = 12(4x-3)^2 \dots\dots(ii)$$

$$\text{From (i) and (ii) it is proved } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

$$\begin{aligned} \text{(b)} \quad \frac{d}{dx}(y) &= \frac{d}{dx}(15x^2+1)^{\frac{1}{2}} \\ &= \frac{1}{2}(15x^2+1)^{\frac{1}{2}-1} \frac{d}{dx}(15x^2+1) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{2}(15x^2+1)^{-\frac{1}{2}}(30x) \\ \Rightarrow \frac{dy}{dx} &= \frac{15x}{(15x^2+1)^{\frac{1}{2}}} = \frac{15}{\sqrt{15x^2+1}} \dots\dots(i) \end{aligned}$$

$$\text{Now from (i)} \Rightarrow \frac{dx}{dy} = \frac{\sqrt{15x^2+1}}{15}$$

$$\frac{\frac{1}{dx}}{\frac{dy}{\frac{\sqrt{15x^2+1}}{15}}} = \frac{1}{1} = \frac{15}{\sqrt{15x^2+1}} \dots\dots(ii)$$

$$\text{From (i) and (ii) it is proved } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Example 13: Differentiate the following functions
w . r . t . x

(a) $f(x) = (11x^2 - 7)^8$ (b) $f(x) = \sqrt{2x^3 + 11}$

Solution: (a) $f(x) = (11x^2 - 7)^8$ Differentiate w. r. t. x .

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}((11x^2 - 7)^8) \text{ use generalized power rule}$$

$$= 8(11x^2 - 7)^{8-1} \frac{d}{dx}(11x^2 - 7)$$

$$= 8(11x^2 - 7)^7 (22x)$$

$$\Rightarrow f'(x) = 176x(11x^2 - 7) \text{ Ans.}$$

(b) $f(x) = \sqrt{2x^3 + 11}$, Differentiate w. r. t. x .

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(2x^3 + 11)^{\frac{1}{2}} \text{ use generalized power rule}$$

$$\Rightarrow f'(x) = \frac{1}{2}(2x^3 + 11)^{\frac{1}{2}-1} \frac{d}{dx}(2x^3 + 11)$$

$$= \frac{1}{2}(2x^3 + 11)^{-\frac{1}{2}}(6x^2)$$

$$= \frac{3x^2}{\sqrt{2x^3 + 11}} \text{ Ans.}$$

Example 14: Differentiate the implicit equation

$$x^2y + 2y^3 = 3x + 2y$$

Solution: $x^2y + 2y^3 = 3x + 2y$ Differentiate w. r. t. to x

$$\frac{d}{dx}(x^2y + 2y^3) = \frac{d}{dx}(3x + 2y)$$

$$\Rightarrow \frac{d}{dx}(x^2y) + 2\frac{d}{dx}(y^3) = 3\frac{d}{dx}(x) + 2\frac{d}{dx}(y)$$

$$\Rightarrow x^2 \frac{dy}{dx} + y \frac{d}{dx}(x^2) + (2)(3)y^2 \frac{dy}{dx} = 3(1) + 2 \frac{dy}{dx}$$

$$\Rightarrow x^2 \frac{dy}{dx} + y(2x) + 6y^2 \frac{dy}{dx} - 2 \frac{dy}{dx} = 3$$

$$\Rightarrow (x^2 + 6y^2 - 2) \frac{dy}{dx} = 3 - 2xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{3 - 2xy}{x^2 + 6y^2 - 2} \quad \text{Ans.}$$

Example 15: Find the slope of a tangent line to the circle $x^2 + y^2 = 5x + 4y$ at a particular point $P(5, 4)$.

Solution: $x^2 + y^2 = 5x + 4y$ (i)

Since the slope of the tangent line to the curve (i) is $\frac{dy}{dx}$.

Then differentiate (i) w.r.t. to x .

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(5x + 4y)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 5 \frac{d}{dx}(x) + 4 \frac{d}{dx}(y)$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = 5(1) + 4 \frac{dy}{dx}$$

$$\Rightarrow 2y \frac{dy}{dx} - 4 \frac{dy}{dx} = 5 - 2x$$

$$\Rightarrow (2y - 4) \frac{dy}{dx} = 5 - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{5 - 2x}{2y - 4} \quad \text{.....(ii)}$$

Now the slope of the tangent line at point $P(5, 4)$ is

Put $x = 5$ and $y = 4$ in (ii)

$$\frac{dy}{dx} = \frac{5 - 2(5)}{2(4) - 4} = \frac{5 - 10}{8 - 4}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{5}{4} \quad \text{Ans.}$$

Exercise 3.3:

Q.1: Determine the indicated derivative in each case:

(a) $w = 4(x^3 - 4x + 2)^5$, $\frac{dw}{dx} = ?$

Solution: $w = 4(x^3 - 4x + 2)^5$

Differentiate w.r.t. to x by generalized power rule

$$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1} \cdot \frac{d}{dx} f(x) \text{ . Then}$$

$$\begin{aligned} \frac{dw}{dx} &= \frac{d}{dx} (4(x^3 - 4x + 2)^5) \\ &= (5)(4)(x^3 - 4x + 2)^{5-1} \frac{d}{dx} (x^3 - 4x + 2) \\ &= 20(x^3 - 4x + 2)^4 (3x^2 - 4) \\ &= 20(x^3 - 4x + 2)^4 (3x^2 - 4) \quad \text{Ans.} \end{aligned}$$

(b) $y = \frac{(4x - x^3)^{11}}{5}$, $\frac{dy}{dx} = ?$

Solution: $y = \frac{(4x - x^3)^{11}}{5}$

Differentiate w.r.t. to x by generalized power rule

$$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1} \cdot \frac{d}{dx} f(x) \text{ . Then}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{(4x - x^3)^{11}}{5} \right) \\ &= \frac{1}{5} (11)(4x - x^3)^{11-1} \frac{d}{dx} (4x - x^3) \\ &= \frac{11}{5} (4x - x^3)^{10} (4 - 3x^2) \\ &= \frac{44}{5} (4x - x^3)^{10} - \frac{33x^2}{5} (4x - x^3)^{10} \quad \text{Ans.} \end{aligned}$$

(c) $u = \sqrt[3]{1 - 3t^2}$, $\frac{du}{dt} = ?$

Solution: $u = \sqrt[3]{1 - 3t^2} = (1 - 3t^2)^{\frac{1}{3}}$ Differentiate w.r.t. to t

$$\frac{du}{dt} = \frac{d}{dt} (1 - 3t^2)^{\frac{1}{3}}$$

$$\begin{aligned}
 &= \frac{1}{3} (1-3t^2)^{\frac{1}{3}-1} \frac{d}{dt} (1-3t^2) \\
 \Rightarrow \frac{du}{dt} &= \frac{1}{3} (1-3t^2)^{-\frac{2}{3}} (-6t) \\
 &= -2t (1-3t^2)^{-\frac{2}{3}} \\
 &= -\frac{2t}{(1-3t^2)^{\frac{2}{3}}} = -\frac{2t}{\sqrt[3]{(1-3t^2)^2}} \quad \text{Ans.}
 \end{aligned}$$

(d) $s = \frac{1}{(3t+1)^7}$, $\frac{ds}{dt} = ?$

Solution: $s = \frac{1}{(3t+1)^7}$ Differentiate w.r.t. to t

$$\begin{aligned}
 \frac{ds}{dt} &= \frac{d}{dt} \left(\frac{1}{(3t+1)^7} \right) = \frac{d}{dt} (3t+1)^{-7} \\
 &= (-7)(3t+1)^{-7-1} \frac{d}{dt} (3t+1) \\
 &= (-7)(3t+1)^{-8} (3) \\
 &= -\frac{21}{(3t+1)^8} \quad \text{Ans.}
 \end{aligned}$$

Q.2: Determine the derivative $f'(x)$ in each case :,

(a) $f(x) = (2x-5)^3 (5x-7)$

Solution: $f(x) = (2x-5)^3 (5x-7)$, Differentiate w.r.t. to x .

$$\begin{aligned}
 \frac{d}{dx} (f(x)) &= \frac{d}{dx} ((2x-5)^3 (5x-7)) \quad \text{use product rule} \\
 &= (2x-5)^3 \frac{d}{dx} (5x-7) + (5x-7) \frac{d}{dx} (2x-5)^3 \\
 &= (2x-5)^3 (5) + (5x-7)(3)(2x-5)^2 \frac{d}{dx} (2x-5)
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow f'(x) &= 5(2x-5) + 5(5x-7)(2x-5)^2(2) \\
 &= (2x-5)^2(5(2x-5) + 6(5x-7)) \\
 &= (2x-5)^2(10x-25+30x-42) \\
 &= (2x-5)^2(40x-67) \quad \text{Ans.}
 \end{aligned}$$

$$(b) \quad f(x) = \frac{(x+2)^2}{x-1}$$

Solution: $f(x) = \frac{(x+2)^2}{x-1}$, Differentiate w.r.t. to x .

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{(x+2)^2}{x-1}\right) \quad \text{use quotient rule}$$

$$\begin{aligned}
 \Rightarrow f'(x) &= \frac{(x-1)\frac{d}{dx}(x+2)^2 - (x+2)^2\frac{d}{dx}(x-1)}{(x-1)^2} \\
 &= \frac{(x-1)(2)(x+2)^{2-1}\frac{d}{dx}(x+2) - (x+2)^2(1)}{(x-1)^2} \\
 &= \frac{2(x-1)(x+2) - (x+2)^2}{(x-1)^2} \\
 &= \frac{(x+2)(2x-2-(x+2))}{(x-1)^2} \\
 &= \frac{(x+2)(2x-2-x-2)}{(x-1)^2} \\
 \Rightarrow f'(x) &= \frac{(x+2)(x-4)}{(x-1)^2} \quad \text{Ans.}
 \end{aligned}$$

$$(c) \quad f(x) = \left(\frac{2x-5}{x-4}\right)^4$$

Solution: $f(x) = \left(\frac{2x-5}{x-4}\right)^4$, Differentiate w.r.t. to x .

$$\frac{d}{dx}(f(x)) = \frac{d}{dx} \left(\frac{2x-5}{x-4}\right)^4 \quad \text{use generalize power rule}$$

$$= 4 \left(\frac{2x-5}{x-4}\right)^{4-1} \frac{d}{dx} \left(\frac{2x-5}{x-4}\right) \quad \text{by quotient rule}$$

$$\Rightarrow f'(x) = 4 \left(\frac{2x-5}{x-4}\right)^3 \left(\frac{(x-4) \frac{d}{dx}(2x-5) - (2x-5) \frac{d}{dx}(x-4)}{(x-4)^2} \right)$$

$$= \frac{4(2x-5)^3}{(x-4)^3} \left(\frac{(x-4)(2) - (2x-5)(1)}{(x-4)^2} \right)$$

$$= \frac{4(2x-5)^3 (2x-8-2x+5)}{(x-4)^5}$$

$$\Rightarrow f'(x) = \frac{4(2x-5)^3 (-3)}{(x-4)^5} = -\frac{12(2x-5)^3}{(x-4)^5} \quad \text{Ans.}$$

(d) $f(x) = x\sqrt{2x^2+11}$

Solution: $f(x) = x\sqrt{2x^2+11}$, Differentiate w.r.t. to x .

$$\frac{d}{dx}(f(x)) = \frac{d}{dx} \left(x(2x^2+11)^{\frac{1}{2}} \right)$$

$$= x \cdot \frac{d}{dx} (2x^2+11)^{\frac{1}{2}} + (2x^2+11)^{\frac{1}{2}} \frac{d}{dx} (x)$$

$$= (x) \left(\frac{1}{2} \right) (2x^2+11)^{\frac{1}{2}-1} \frac{d}{dx} (2x^2+11) + (2x^2+11)^{\frac{1}{2}} (1)$$

$$= \frac{x}{2} (2x^2+11)^{-\frac{1}{2}} (4x) + (2x^2+11)^{\frac{1}{2}}$$

$$= \frac{2x^2}{(2x^2+11)^{\frac{1}{2}}} + (2x^2+11)^{\frac{1}{2}}$$

$$= \frac{2x^2 + (2x^2 + 11)^{\frac{1}{2}}(2x^2 + 11)^{\frac{1}{2}}}{(2x^2 + 11)^{\frac{1}{2}}} = \frac{2x^2 + (2x^2 + 11)^{\frac{1}{2} + \frac{1}{2}}}{(2x^2 + 11)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{2x^2 + 2x^2 + 11}{(2x^2 + 11)^{\frac{1}{2}}} = \frac{4x^2 + 11}{\sqrt{2x^2 + 11}} \text{ Ans.}$$

Q.3: Find $\frac{dy}{dx}$ of the following functions in term of parameter t .

(a) $x = 1 + t^2$, $y = t^3 + 2t^2 + 1$

Solution: Given $x = 1 + t^2$...(i) and $y = t^3 + 2t^2 + 1$ (ii)

Since $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ (A)

Differentiate (i) and (ii) w.r.t. to t .

$$\frac{dx}{dt} = \frac{d}{dt}(1 + t^2) = 2t \Rightarrow \frac{dt}{dx} = \frac{1}{2t} \text{(iii)}$$

$$\frac{dy}{dt} = \frac{d}{dt}(t^3 + 2t^2 + 1) = 3t^2 + 4t \text{(iv)}$$

Put (iii) and (iv) in (A)

$$\frac{dy}{dx} = (3t^2 + 4t) \cdot \frac{1}{2t} = \frac{3t + 4}{2} \text{ Ans.}$$

(b) $x = 3at^2 + 2$, $y = 6t^4 + 9$

Solution: Given $x = 3at^2 + 2$...(i) and $y = 6t^4 + 9$ (ii)

Since $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ (A)

Differentiate (i) and (ii) w.r.t. to t

$$\frac{dx}{dt} = \frac{d}{dt}(3at^2 + 2) = 6at \Rightarrow \frac{dt}{dx} = \frac{1}{6at} \text{(iii)}$$

$$\frac{dy}{dt} = \frac{d}{dt}(6t^4 + 9) = 24t^3 \text{(iv)}$$

Put (iii) and (iv) in (A)

$$\frac{dy}{dx} = (24t^3) \frac{1}{6at} = \frac{4t^2}{a} \quad \text{Ans.}$$

$$(c) \quad x = \frac{a(1-t^2)}{1+t^2}, \quad y = \frac{2bt}{1+t^2}$$

Solution: $x = \frac{a(1-t^2)}{1+t^2}$ (i) and $y = \frac{2bt}{1+t^2}$ (ii)

Since $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ (A)

Differentiate (i) and (ii) w.r.t. to t .

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt} \left(\frac{a(1-t^2)}{1+t^2} \right) \\ &= a \left(\frac{(1+t^2) \frac{d}{dt}(1-t^2) - (1-t^2) \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right) \\ \Rightarrow \frac{dx}{dt} &= a \left(\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} \right) \\ &= a \left(\frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2} \right) \\ \Rightarrow \frac{dx}{dt} &= -\frac{4at}{(1+t^2)^2} \\ \Rightarrow \frac{dt}{dx} &= -\frac{(1+t^2)^2}{4at} \quad \text{.....(iii)} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt} \left(\frac{2bt}{1+t^2} \right) \\ &= 2b \frac{d}{dt} \left(\frac{t}{1+t^2} \right) \end{aligned}$$

$$= 2b \left(\frac{(1+t^2) \frac{d}{dt}(t) - (t) \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right)$$

$$= 2b \left(\frac{1+t^2 - (t)(2t)}{(1+t^2)^2} \right) = 2b \left(\frac{1+t^2 - 2t^2}{(1+t^2)^2} \right)$$

$$\Rightarrow \frac{dy}{dt} = \frac{2b(1-t^2)}{(1+t^2)^2} \dots\dots(iv)$$

Put (iii) and (iv) in (A)

$$\frac{dy}{dx} = \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{-\frac{1}{4at}(1+t^2)^2}{1}$$

$$= -\frac{b(1-t^2)}{2at} = \frac{b(t^2-1)}{2at} \quad \text{Ans.}$$

(d) $x = \frac{3at}{1+t^3}$, $y = \frac{3at^2}{1+t^3}$

Solution: $x = \frac{3at}{1+t^3} \dots\dots(i)$ and $y = \frac{3at^2}{1+t^3} \dots\dots(ii)$

Since $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \dots\dots(A)$

Differentiate (i) and (ii) w.r.t. to t .

$$\frac{dx}{dt} = \frac{d}{dt} \left(\frac{3at}{1+t^3} \right)$$

$$= 3a \frac{d}{dt} \left(\frac{t}{1+t^3} \right)$$

$$= 3a \left(\frac{(1+t^3) \frac{d}{dt}(t) - (t) \frac{d}{dt}(1+t^3)}{(1+t^3)^2} \right)$$

$$= 3a \left(\frac{1+t^3 - (t)(3t^2)}{(1+t^3)^2} \right) = 3a \left(\frac{1+t^3 - 3t^3}{(1+t^3)^2} \right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{3a(1-2t^3)}{(1+t^3)^2}.$$

$$\Rightarrow \frac{dt}{dx} = \frac{(1+t^3)^2}{3a(1-2t^3)} \dots\dots (iii)$$

$$\frac{dy}{dt} = \frac{d}{dt} \left(\frac{3at^2}{1+t^3} \right)$$

$$= 3a \frac{d}{dt} \left(\frac{t^2}{1+t^3} \right)$$

$$= 3a \left(\frac{(1+t^3) \frac{d}{dt}(t^2) - (t^2) \frac{d}{dt}(1+t^3)}{(1+t^3)^2} \right)$$

$$= 3a \left(\frac{(1+t^3)(2t) - (t^2)(3t^2)}{(1+t^3)^2} \right) = 3a \left(\frac{2t + 2t^4 - 3t^4}{(1+t^3)^2} \right)$$

$$\Rightarrow \frac{dy}{dt} = \frac{3a(2t - t^4)}{(1+t^3)^2} = \frac{3at(2-t^3)}{(1+t^3)^2} \dots\dots (iv)$$

Put (iii) and (iv) in (A).

$$\frac{dy}{dx} = \frac{3at(2-t^3)}{(1+t^3)^2} \times \frac{(1+t^3)^2}{3a(1-2t^3)}$$

$$\frac{dy}{dx} = \frac{t(2-t^3)}{1-2t^3} = \frac{2t-t^4}{1-2t^3} \quad \text{Ans.}$$

Q.4: At a certain factor, the total cost of manufacturing q units during the daily production run is

$C(q) = 0.2q^2 + q + 900$ dollars. From experience, it has been determined that approximately

$q(t) = t^2 + 100t$ units are manufactured during the first t hours of a production run. Compute the rate at which the total manufacturing cost is changing with respect to time one hour after production begins.

Solution: $C(q) = 0.2q^2 + q + 900$ (i)

$$q(t) = t^2 + 100t \text{(ii)}$$

We will find $\frac{dC}{dt}$.

As by chain rule $\frac{dC}{dt} = \frac{dC}{dq} \times \frac{dq}{dt}$ (A)

Differentiate (i) w.r.t. to q .

$$\begin{aligned} \frac{dC}{dq} &= \frac{d}{dq} (0.2q^2 + q + 900) \\ &= (0.2)(2)q + 1 = 0.4q + 1 \text{(iii)} \end{aligned}$$

Differentiate (ii) w.r.t. to t

$$\frac{dq}{dt} = \frac{d}{dt} (t^2 + 100t) = 2t + 100 \text{(iv)}$$

Put (iii) and (iv) in (A)

$$\frac{dC}{dt} = (0.4q + 1)(2t + 100) \text{(B)}$$

After one hour : Put $t = 1$ in (ii)

$$q(1) = (1)^2 + 100(1)$$

$$\Rightarrow q(1) = 1 + 100 = 1001$$

Put $t = 1$ and $q = 1001$ in (B)

$$\begin{aligned} \frac{dC}{dt} &= ((0.4)(1001) + 1)(2(1) + 100) \\ &= (401.4)(102) = 4222.80 \text{ Ans.} \end{aligned}$$

Q.5: Use implicit differentiation to perform $\frac{dy}{dx}$ for the

Following functions. (a) $x^2 + y^2 = 25$

Solution: $x^2 + y^2 = 25$...(i) differentiate (i) w.r.t. to x .

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow 2y \frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y} \quad \text{Ans}$$

(b) $xy(2x + 3y) = 2$

Solution: $xy(2x + 3y) = 2$

$$2x^2y + 3xy^2 = 2$$

Differentiate w. r. t. to x

$$\frac{d}{dx}(2x^2y + 3xy^2) = \frac{d}{dx}(2)$$

$$\Rightarrow 2 \frac{d}{dx}(x^2y) + 3 \frac{d}{dx}(xy^2) = 0$$

$$\Rightarrow 2 \left(x^2 \frac{d}{dx}(y) + y \frac{d}{dx}(x^2) \right) + 3 \left(x \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(x) \right) = 0$$

$$\Rightarrow 2 \left(x^2 \frac{dy}{dx} + 2xy \right) + 3 \left(2xy \frac{dy}{dx} + y^2 \right) = 0$$

$$\Rightarrow 2x^2 \frac{dy}{dx} + 4xy + 6xy \frac{dy}{dx} + 3y^2 = 0$$

$$\Rightarrow (2x^2 + 6xy) \frac{dy}{dx} = -4xy - 3y^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(4xy + 3y^2)}{2x^2 + 6xy} \quad \text{Ans.}$$

(c) $(x + y)^3 + 3y = 3$

Solution: $(x + y)^3 + 3y = 3$

Differentiate (i) w r . t. to x .

$$\frac{d}{dx}((x+y)^3 + 3y) = \frac{d}{dx}(3)$$

$$\Rightarrow \frac{d}{dx}(x+y)^3 + 3 \frac{d}{dx}(y) = 0$$

$$\Rightarrow 3(x+y)^2 \frac{d}{dx}(x+y) + 3 \frac{dy}{dx} = 0$$

$$\Rightarrow 3(x+y)^2 \left(\frac{d}{dx}(x) + \frac{dy}{dx} \right) + 3 \frac{dy}{dx} = 0$$

$$\Rightarrow 3(x+y)^2 \left(1 + \frac{dy}{dx} \right) + 3 \frac{dy}{dx} = 0$$

$$\Rightarrow 3(x+y)^2 + 3(x+y)^2 \frac{dy}{dx} + 3 \frac{dy}{dx} = 0$$

$$\Rightarrow 3((x+y)^2 + 1) \frac{dy}{dx} = -3(x+y)^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{3(x+y)^2}{3((x+y)^2 + 1)}$$

$$= -\frac{(x+y)^2}{((x+y)^2 + 1)} \text{ Ans.}$$

(d) $\frac{1}{y} + \frac{1}{x} = 1$

Solution : $\frac{1}{y} + \frac{1}{x} = 1$

Differentiate (i) w r . t. to x .

$$\frac{d}{dx} \left(\frac{1}{y} + \frac{1}{x} \right) = \frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}(y^{-1}) + \frac{d}{dx}(x^{-1}) = 0$$

$$\Rightarrow (-1)y^{-2} \frac{d}{dx}(y) + (-1)x^{-2} = 0$$

$$\Rightarrow -y^{-2} \frac{dy}{dx} = x^{-2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-2}}{y^{-2}} = -\frac{y^2}{x^2} \quad \text{Ans.}$$

Q.6: Arrange the following functions explicitly to perform

$$\frac{dy}{dx}: \quad (\text{a}) \quad x^2 y^3 + y^3 = 12$$

Solution: $x^2 y^3 + y^3 = 12$

Now differentiate w . r . t . to x ,

$$\frac{d}{dx}(x^2 y^3 + y^3) = \frac{d}{dx}(12)$$

$$\Rightarrow \frac{d}{dx}(x^2 y^3) + \frac{d}{dx}(y^3) = 0$$

$$\Rightarrow x^2 \frac{d}{dx}(y^3) + y^3 \frac{d}{dx}(x^2) + 3y^2 \frac{d}{dx}(y) = 0$$

$$\Rightarrow 3x^2 y^2 \frac{d}{dx}(y) + 2xy^3 + 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow 3y^2(x^2 + 1) \frac{dy}{dx} = -2xy^3$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2xy^3}{3y^2(x^2 + 1)} = -\frac{2x}{3(x^2 + 1)} \quad \text{Ans.}$$

(b) $xy + 2y = x^2$

Solution: $xy + 2y = x^2$ (i)

$$\Rightarrow y(x + 2) = x^2$$

$$\Rightarrow y = \frac{x^2}{x + 2} \quad \text{.....(ii)}$$

Eq(i) is implicit form and (ii) is explicit form .

Now differentiate implicit function (i) w . r . t. to x

$$\begin{aligned}\frac{d}{dx}(xy + 2y) &= \frac{d}{dx}(x^2) \\ \Rightarrow \frac{d}{dx}(xy) + 2 \frac{d}{dx}(y) &= 2x \\ \Rightarrow x \frac{d}{dx}(y) + y \frac{d}{dx}(x) + 2 \frac{dy}{dx} &= 2x \\ \Rightarrow x \frac{dy}{dx} + y + 2 \frac{dy}{dx} &= 2x \\ \Rightarrow (x+2) \frac{dy}{dx} &= 2x - y \\ \Rightarrow \frac{dy}{dx} &= \frac{2x - y}{x + 2}\end{aligned}$$

Now differentiate explicit function (ii) w . r . t. x

$$\begin{aligned}\frac{d}{dx}(y) &= \frac{d}{dx}\left(\frac{x^2}{x+2}\right) \\ \Rightarrow \frac{dy}{dx} &= \frac{(x+2) \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(x+2)}{(x+2)^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{(x+2)(2x) - x^2}{(x+2)^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{2x^2 + 4x - x^2}{(x+2)^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{x^2 + 4x}{(x+2)^2}\end{aligned}$$

$$(c) \quad x + \frac{1}{y} = 5$$

Solution: $x + \frac{1}{y} = 5 \quad \dots\dots(i)$

Eq(i) is implicit form

$$\text{Now } x + \frac{1}{y} = 5$$

$$\Rightarrow \frac{1}{y} = 5 - x$$

$$\Rightarrow y = \frac{1}{5-x} \dots\dots(ii)$$

Eq(ii) is explicit form .

Now differentiate implicit function (i) w . r . t. to x

$$\frac{d}{dx} \left(x + \frac{1}{y} \right) = \frac{d}{dx} (5)$$

$$\Rightarrow \frac{d}{dx} (x) + \frac{d}{dx} (y^{-1}) = 0$$

$$\Rightarrow 1 + (-1)y^{-2} \frac{d}{dx} (y) = 0$$

$$\Rightarrow -y^{-2} \frac{dy}{dx} = -1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{y^{-2}} = y^2$$

Now differentiate explicit function (ii) w . r . t. x

$$\frac{d}{dx} (y) = \frac{d}{dx} \left(\frac{1}{5-x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} (5-x)^{-1}$$

$$= (-1)(5-x)^{-2} \frac{d}{dx} (5-x)$$

$$= -(5-x)^{-2} (-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(5-x)^2}$$

(d) $xy - x = y + 2$

Solution : $xy - x = y + 2 \dots\dots(i)$

Eq(i) is implicit form.

$$xy - x = y + 2$$

$$\Rightarrow xy - y = x + 2$$

$$\Rightarrow y(x-1) = 2+x$$

$$\Rightarrow y = \frac{2+x}{x-1} \dots\dots(ii)$$

Eq(ii) is explicit form .

Now differentiate implicit function (i) w . r . t. to x

$$\frac{d}{dx}(xy-x) = \frac{d}{dx}(y+2)$$

$$\Rightarrow \frac{d}{dx}(xy) - \frac{d}{dx}(x) = \frac{d}{dx}(y) + \frac{d}{dx}(2)$$

$$\Rightarrow x \frac{d}{dx}(y) + y \frac{d}{dx}(x) - 1 = \frac{dy}{dx} + 0$$

$$\Rightarrow x \frac{dy}{dx} + y - 1 - \frac{dy}{dx} = 0$$

$$\Rightarrow (x-1) \frac{dy}{dx} = 1-y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-y}{x-1}$$

Now differentiate explicit function (ii) w . r . t. x

$$\frac{d}{dx}(y) = \frac{d}{dx} \left(\frac{2+x}{x-1} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x-1) \frac{d}{dx}(2+x) - (2+x) \frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)(1) - (2+x)(1)}{(x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-1-2-x}{(x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{3}{(x-1)^2}$$

Q.7: Let $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$, where a and b are non zero

constants . Find : (a) $\frac{du}{dv}$ (b) $\frac{dv}{du}$

Solution: $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$ (i)

(a) Differentiate w . r . t. to u

$$\begin{aligned}\frac{d}{du} \left(\frac{u^2}{a^2} + \frac{v^2}{b^2} \right) &= \frac{d}{du} (1) \\ \Rightarrow \frac{d}{du} \left(\frac{u^2}{a^2} \right) + \frac{d}{du} \left(\frac{v^2}{b^2} \right) &= 0 \\ \Rightarrow \frac{1}{a^2} \frac{d}{du} (u^2) + \frac{1}{b^2} \frac{d}{du} (v^2) &= 0 \\ \Rightarrow \frac{2u}{a^2} + \frac{2v}{b^2} \frac{d}{du} (v) &= 0 \\ \Rightarrow \frac{2v}{b^2} \frac{dv}{du} &= -\frac{2u}{a^2} \\ \Rightarrow \frac{dv}{du} &= -\frac{2u}{a^2} \times \frac{b^2}{2v} \\ \Rightarrow \frac{dv}{du} &= -\frac{b^2 u}{a^2 v}\end{aligned}$$

(b) Differentiate w . r . t. to v

$$\begin{aligned}\frac{d}{dv} \left(\frac{u^2}{a^2} + \frac{v^2}{b^2} \right) &= \frac{d}{dv} (1) \\ \Rightarrow \frac{d}{dv} \left(\frac{u^2}{a^2} \right) + \frac{d}{dv} \left(\frac{v^2}{b^2} \right) &= 0 \\ \Rightarrow \frac{1}{a^2} \frac{d}{dv} (u^2) + \frac{1}{b^2} \frac{d}{dv} (v^2) &= 0 \\ \Rightarrow \frac{2u}{a^2} \frac{d}{dv} (u) + \frac{2v}{b^2} &= 0\end{aligned}$$

$$\Rightarrow \frac{2u}{a^2} \frac{du}{dv} = -\frac{2v}{b^2}$$

$$\Rightarrow \frac{du}{dv} = -\frac{2v}{b^2} \times \frac{a^2}{2u}$$

$$\Rightarrow \frac{du}{dv} = -\frac{a^2 v}{b^2 u}$$

Q.8: Determine the slope of the tangent line to the curve

$$3x^2 - 7y^2 + 14y = 27 \text{ at the point } P(-3, 0)$$

Solution: $3x^2 - 7y^2 + 14y = 27$ (i)

Since the slope of the tangent line to the curve (i) is $\frac{dy}{dx}$.

Then differentiate (i) w r . t. to x .

$$\frac{d}{dx}(3x^2 - 7y^2 + 14y) = \frac{d}{dx}(27)$$

$$\Rightarrow 3 \frac{d}{dx}(x^2) - 7 \frac{d}{dx}(y^2) + 14 \frac{d}{dx}(y) = 0$$

$$\Rightarrow 6x - 14y \frac{dy}{dx} + 14 \frac{dy}{dx} = 0$$

$$\Rightarrow (14 - 14y) \frac{dy}{dx} = -6x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-6x}{14 - 14y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-6x}{-14(y-1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x}{7(y-1)} \text{(ii)}$$

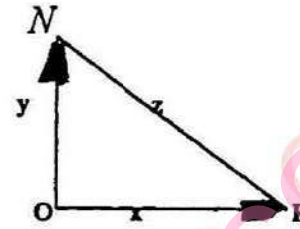
Now the slope of the tangent line at point $P(-3, 0)$ is

Put $x = -3$ and $y = 0$ in (ii)

$$\frac{dy}{dx} = \frac{3(-3)}{7(0-1)} = \frac{-9}{-7}$$

$$\Rightarrow \frac{dy}{dx} = \frac{9}{7} \quad \text{Ans.}$$

Q.9: Suppose two moter boats leave from the same point at the same time. If one boat travels north at 15 miles per hour and the other boat travels east at 20 miles per hour. How fast will the distance between them be changing after 2 hours.



Solution: Let both boats start from point O. Let x is the distance of one boat from point O towards east. Let y is the distance of another boat from point O towards north. z is the distance between the two boats.

From the figure we see that y is perpendicular, x is the base and z is the hypotenuse.

Then by Pythagorean theorem

$$z^2 = x^2 + y^2 \quad \dots\dots(i)$$

Since distance is a function of " t ". i.e. x and y are the functions of " t ". Then from eq(i) z also the function of " t ". Given that

$$\text{one boat travels east at 15 miles per hour} = \frac{dx}{dt} = 20$$

$$\text{other boat travels north at 15 miles per hour} = \frac{dy}{dt} = 15$$

$$\text{Total distance toward east after 2 hours} = x = (20)(2)$$

$$\Rightarrow x = 40$$

$$\text{Total distance toward north after 2 hours} = y = (15)(2)$$

$$\Rightarrow y = 30 \quad \text{put } x = 40 \text{ and } y = 30 \text{ in (i)}$$

$$\Rightarrow z^2 = (40)^2 + (30)^2 = 1600 + 900$$

$$\Rightarrow z^2 = 2500 \quad \text{Take square root of both sides}$$

$$\Rightarrow z = \pm 50 \quad \text{Since } z \text{ is a distance then it is positive.}$$

Thus $z = 50$

Now we will find how fast will the distance between the boats be changing. i.e. we find $\frac{dz}{dt}$.

Differentiate (i) w.r.t. "t".

$$\frac{d}{dt}(z^2) = \frac{d}{dt}(x^2 + y^2) \Rightarrow 2z \frac{dz}{dt} = \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2)$$

$$\Rightarrow 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \quad \text{both sides divided by 2}$$

$$\Rightarrow z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} \quad \dots\dots\dots (ii)$$

Put vlues in (ii)

$$50 \cdot \frac{dz}{dt} = (40)(20) + (30)(15)$$

$$\Rightarrow 50 \frac{dz}{dt} = 800 + 450$$

$$\Rightarrow 50 \frac{dz}{dt} = 1250 \Rightarrow \frac{dz}{dt} = \frac{1250}{50}$$

$$\Rightarrow \frac{dz}{dt} = 25 \text{ miles per hour.}$$

Derivatives of Trigonometric Functions:

(i) $\frac{d}{dx}(\sin x) = \cos x$

Proof: As $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(i)$

Since $f(x) = \sin x \dots\dots(ii)$

$\Rightarrow f(x + \Delta x) = \sin(x + \Delta x) \dots\dots(iii)$

Put (ii) and (iii) in (i)

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

Use $\sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right)$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{2 \cos\left(\frac{x + \Delta x + x}{2}\right) \cdot \sin\left(\frac{x + \Delta x - x}{2}\right)}{\Delta x}$$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{2 \cos\left(\frac{2x + \Delta x}{2}\right) \cdot \sin \frac{\Delta x}{2}}{\Delta x}$$

Denominator multiplied and divided by 2

$$\Rightarrow \frac{d}{dx} f(x) = \frac{1}{2} \lim_{\Delta x \rightarrow 0} 2 \cos\left(x + \frac{\Delta x}{2}\right) \cdot \lim_{\Delta x \rightarrow 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$\Rightarrow \frac{d}{dx} f(x) = \cos(x + 0) \cdot 1 \quad \boxed{\text{As } \lim_{\Delta x \rightarrow 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1}$$

$$\Rightarrow \frac{d}{dx} (\sin x) = \cos x \quad \text{Proved}$$

(ii) $\frac{d}{dx} (\cos x) = -\sin x$

Proof: Let $f(x) = \cos x \Rightarrow f(x + \Delta x) = \cos(x + \Delta x)$

$$\text{Since } f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{--- (i)}$$

Put values in (i)

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{-2 \sin\left(\frac{x + \Delta x + x}{2}\right) \cdot \sin\left(\frac{x + \Delta x - x}{2}\right)}{\Delta x}$$

$$\Rightarrow f'(x) = -2 \lim_{\Delta x \rightarrow 0} \sin\left(\frac{2x + \Delta x}{2}\right) \cdot \frac{\sin \frac{\Delta x}{2}}{\Delta x}$$

Denominator multiplied and divided by 2

$$\Rightarrow f'(x) = -2 \lim_{\Delta x \rightarrow 0} \sin\left(\frac{2x}{2} + \frac{\Delta x}{2}\right) \cdot \frac{\sin \frac{\Delta x}{2}}{2 \frac{\Delta x}{2}}$$

$$\Rightarrow f'(x) = - \lim_{\Delta x \rightarrow 0} \sin\left(x + \frac{\Delta x}{2}\right) \cdot \lim_{\Delta x \rightarrow 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$\text{As } \lim_{\Delta x \rightarrow 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1$$

$$\Rightarrow f'(x) = -\sin(x+0) \cdot 1$$

$$\Rightarrow \frac{d}{dx}(\cos x) = -\sin x \quad \text{Proved}$$

(iii) $\frac{d}{dx}(\tan x) = \sec^2 x$

Proof: Let $f(x) = \tan x$

$$\Rightarrow f(x + \Delta x) = \tan(x + \Delta x)$$

Since $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(i)$

Put values in (i)

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\tan(x + \Delta x) - \tan x}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin(x + \Delta x)}{\cos(x + \Delta x)} - \frac{\sin x}{\cos x} \right]$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin(x + \Delta x) \cos x - \cos(x + \Delta x) \sin x}{\cos(x + \Delta x) \cos x} \right]$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x - x)}{\Delta x \cos(x + \Delta x) \cos x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x} \cdot \lim_{\Delta x \rightarrow 0} \frac{1}{\cos(x + \Delta x) \cos x}$$

$$\Rightarrow \frac{d}{dx} f(x) = 1 \cdot \frac{1}{\cos x \cdot \cos x}$$

$$\Rightarrow \frac{d}{dx} (\tan x) = \frac{1}{\cos^2 x} = \sec^2 x \quad \text{Proved}$$

$$(iv) \quad \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$$

Proof: Let $f(x) = \cot x$

$$\Rightarrow f(x + \Delta x) = \cot(x + \Delta x)$$

$$\text{Since } f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{--- (i)}$$

Put values in (i)

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} [\cot(x + \Delta x) - \cot x]$$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\cos(x + \Delta x)}{\sin(x + \Delta x)} - \frac{\cos x}{\sin x} \right]$$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\cos(x + \Delta x) \sin x - \sin(x + \Delta x) \cos x}{\sin(x + \Delta x) \sin x} \right]$$

$$\Rightarrow \frac{d}{dx} f(x) = - \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin(x + \Delta x) \cos x - \cos(x + \Delta x) \sin x}{\sin(x + \Delta x) \sin x} \right]$$

$$\Rightarrow \frac{d}{dx} f(x) = - \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin(x + \Delta x - x)}{\sin(x + \Delta x) \sin x} \right]$$

$$\Rightarrow \frac{d}{dx} f(x) = - \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x (\sin(x + \Delta x) \sin x)}$$

$$\Rightarrow \frac{d}{dx} f(x) = - \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x} \cdot \lim_{\Delta x \rightarrow 0} \frac{1}{\sin(x + \Delta x) \sin x}$$

$$\begin{aligned} \Rightarrow \frac{d}{dx} (\cot x) &= (1) \cdot \frac{-1}{\sin x \cdot \sin x} \\ &= -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x \end{aligned}$$

$$(v) \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

Proof: Let $f(x) = \sec x$

$$\Rightarrow f(x + \Delta x) = \sec(x + \Delta x)$$

$$\text{Since } f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ -----(i)}$$

Put values in (i)

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{\sec(x + \Delta x) - \sec x}{\Delta x}$$

$$= \lim_{h \rightarrow 0} \frac{1}{\Delta x} \left[\frac{1}{\cos(x + \Delta x)} - \frac{1}{\cos x} \right]$$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\cos x - \cos(x + \Delta x)}{\cos(x + \Delta x) \cos x} \right]$$

$$\Rightarrow \frac{d}{dx} f(x) = - \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\cos(x + \Delta x) - \cos x}{\cos(x + \Delta x) \cos x} \right]$$

$$\Rightarrow \frac{d}{dx} f(x) = - \lim_{\Delta x \rightarrow 0} \frac{-2 \sin\left(\frac{x + \Delta x + x}{2}\right) \sin\left(\frac{x + \Delta x - x}{2}\right)}{\Delta x \cos(x + \Delta x) \cos x}$$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{2 \sin\left(\frac{2x + \Delta x}{2}\right) \sin \Delta x / 2}{\cos(x + \Delta x) \cos x \Delta x}$$

Denominator multiplied and divided by 2

$$\Rightarrow f'(x) = -2 \lim_{\Delta x \rightarrow 0} \sin\left(\frac{2x + \Delta x}{2}\right) \cdot \frac{\sin \Delta x / 2}{2 \Delta x / 2}$$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{\sin\left(x + \frac{\Delta x}{2}\right)}{\cos(x + \Delta x) \cos x} \times \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x / 2}{\Delta x / 2}$$

$$\Rightarrow \frac{d}{dx} f(x) = \frac{\sin(x + 0)}{\cos(x + 0) \cos x} \cdot 1$$

$$\Rightarrow \frac{d}{dx} f(x) = \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$$

$$\Rightarrow \frac{d}{dx} (\sec x) = \tan x \cdot \sec x \quad \text{Proved}$$

$$\text{(vi)} \quad \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

Proof: Let $f(x) = \operatorname{cosec} x$

$$\Rightarrow f(x + \Delta x) = \operatorname{cosec}(x + \Delta x)$$

$$\text{As } f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{--- (i)}$$

Put values in (i)

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{\operatorname{cosec}(x + \Delta x) - \operatorname{cosec} x}{\Delta x}$$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{1}{\sin(x + \Delta x)} - \frac{1}{\sin x} \right]$$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin x - \sin(x + \Delta x)}{\sin(x + \Delta x) \sin x} \right]$$

$$\Rightarrow \frac{d}{dx} f(x) = -\lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin(x + \Delta x) - \sin x}{\sin(x + \Delta x) \sin x} \right]$$

$$\Rightarrow \frac{d}{dx} f(x) = -\lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{2 \cos\left(\frac{x + \Delta x + x}{2}\right) \sin\left(\frac{x + \Delta x - x}{2}\right)}{\sin(x + \Delta x) \sin x} \right]$$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{-2 \cos\left(\frac{2x + \Delta x}{2}\right) \sin \frac{\Delta x}{2}}{\sin(x + \Delta x) \sin x} \times \frac{\sin \frac{\Delta x}{2}}{\Delta x}$$

Denominator multiplied and divided by 2

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{-2 \cos\left(x + \frac{\Delta x}{2}\right) \sin \frac{\Delta x}{2}}{\sin(x + \Delta x) \sin x} \times \frac{\sin \frac{\Delta x}{2}}{2 \frac{\Delta x}{2}}$$

$$\begin{aligned} \Rightarrow \frac{d}{dx} f(x) &= \lim_{\Delta x \rightarrow 0} \frac{-\cos\left(x + \frac{\Delta x}{2}\right)}{\sin(x + \Delta x) \sin x} \times \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x/2}{\Delta x/2} \\ \Rightarrow \frac{d}{dx} f(x) &= -\frac{\cos(x+0)}{\sin(x+0) \sin x} \cdot 1 \\ \Rightarrow \frac{d}{dx} f(x) &= -\frac{\cos x}{\sin x \cdot \sin x} = -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} \\ \Rightarrow \frac{d}{dx} (\operatorname{cosec} x) &= -\cot x \cdot \operatorname{cosec} x \quad \text{Proved} \end{aligned}$$

Note : (i) $\frac{d}{dx} (\sin u) = \cos u \frac{du}{dx}$ (ii) $\frac{d}{dx} (\cos u) = -\sin u \frac{du}{dx}$
 (iii) $\frac{d}{dx} (\tan u) = \sec^2 u \frac{du}{dx}$ (iv) $\frac{d}{dx} (\cot u) = -\operatorname{cosec}^2 u \frac{du}{dx}$
 (v) $\frac{d}{dx} (\sec u) = \sec u \cdot \tan u \frac{du}{dx}$
 (vi) $\frac{d}{dx} (\operatorname{cosec} u) = -\operatorname{cosec} u \cot u \frac{du}{dx}$

Example 17: Differentiate the following trigonometric functions .

(a) $f(x) = \sec x \tan x$ (b) $f(x) = \frac{x^2 + \tan x}{3x + 2 \tan x}$

Solution: (a) $f(x) = \sec x \tan x$, Differentiate w.r.t to x

$$\frac{d}{dx} (f(x)) = \frac{d}{dx} (\sec x \tan x) \quad \text{use product rule}$$

$$\begin{aligned} \Rightarrow f'(x) &= \sec x \frac{d}{dx} (\tan x) + \tan x \frac{d}{dx} (\sec x) \\ &= \sec x \cdot \sec^2 x + \tan x \cdot \tan x \cdot \sec x \\ &= \sec^3 x + \tan^2 x \sec x \quad \text{Ans.} \end{aligned}$$

b) $f(x) = \frac{x^2 + \tan x}{3x + 2 \tan x}$, Differentiate w.r.t to x

$$\begin{aligned}\frac{d}{dx}(f(x)) &= \frac{d}{dx} \left(\frac{x^2 + \tan x}{3x + 2 \tan x} \right) \text{ use quotient rule} \\ f'(x) &= \frac{(3x + 2 \tan x) \frac{d}{dx}(x^2 + \tan x) - (x^2 + \tan x) \frac{d}{dx}(3x + 2 \tan x)}{(3x + 2 \tan x)^2} \\ &= \frac{(3x + 2 \tan x)(2x + \sec^2 x) - (x^2 + \tan x)(3 + 2 \sec^2 x)}{(3x + 2 \tan x)^2} \\ &= \frac{1}{(3x + 2 \tan x)^2} \left(6x^2 + 3x \sec^2 x + 4x \tan x + 2 \tan x \sec^2 x - 3x^2 - 2x^2 \sec^2 x - 3 \tan x - 2 \tan x \sec^2 x \right) \\ \Rightarrow f'(x) &= \frac{3x^2 + x(3 - 2x) \sec^2 x + (4x - 3) \tan x}{(3x + 2 \tan x)^2} \text{ Ans.}\end{aligned}$$

Derivatives of Inverse Trigonometric Functions

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad x \in (-1, 1)$

Proof: Let $y = \sin^{-1} x$

$\Rightarrow \sin y = x$ differentiate w.r.t "x"

$\Rightarrow \frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$

$\Rightarrow \cos y \cdot \frac{d}{dx}(y) = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

$\Rightarrow \frac{dy}{dx} = \pm \frac{1}{\sqrt{1-\sin^2 y}} \text{ -----(i)}$

Put $\sin y = x$ in (i)

$\Rightarrow \frac{dy}{dx} = \pm \frac{1}{\sqrt{1-x^2}} \text{ -----(ii)}$

As $\cos^2 y + \sin^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \pm \sqrt{1 - \sin^2 y}$

As $y = \sin^{-1} x$ and $x \in (-1, 1)$

Then the value of y is also $y = \sin^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

And we know that $\cos y$ is +ve for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

So we take +ve sign of (ii)

$$\Rightarrow \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad \text{Proved}$$

$$(ii) \quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

Proof: Let $y = \cos^{-1} x \Rightarrow \cos y = x$

$$\Rightarrow \frac{d}{dx}(\cos y) = \frac{d}{dx}(x) \Rightarrow -\sin y \frac{d}{dx}(y) = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} \Rightarrow \frac{dy}{dx} = \pm \frac{-1}{\sqrt{1-\cos^2 y}}$$

By definition of $\cos^{-1} x$, $0 \leq \cos^{-1} x \leq \pi$
or $0 \leq y \leq \pi$ then $\sin y$ is positive.

Then we take positive sign.

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}} \quad \text{put } \cos y = x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad \text{Proved}$$

$$(iii) \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Proof: Let $y = \tan^{-1} x \Rightarrow \tan y = x$

$$\Rightarrow \frac{d}{dx}(\tan y) = \frac{d}{dx}(x) \Rightarrow \sec^2 y \frac{d}{dx}(y) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y} \quad \text{As } \sec^2 \theta = 1 + \tan^2 \theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2} \quad \text{Put } \tan y = x$$

$$(iv) \quad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\text{Proof: Let } y = \cot^{-1} x \Rightarrow \cot y = x$$

$$\Rightarrow \frac{d}{dx}(\cot y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\operatorname{cosec}^2 y \frac{d}{dx}(y) = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{cosec}^2 y} \quad \text{As } \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{1 + \cot^2 y} \quad \text{Put } \cot^2 y = x^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{1+x^2} \quad \text{Proved}$$

$$(v) \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\text{Proof: Let } y = \sec^{-1} x \Rightarrow \sec y = x$$

$$\Rightarrow \frac{d}{dx}(\sec y) = \frac{d}{dx}(x)$$

$$\Rightarrow \sec y \tan y \frac{d}{dx}(y) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \sqrt{\sec^2 y - 1}}$$

$$\text{Put } \sec y = x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

Proved

$$\text{As } 1 + \tan^2 \theta = \sec^2 \theta$$

$$\tan^2 \theta = \sec^2 \theta - 1$$

$$\text{So } \tan \theta = \sqrt{\sec^2 \theta - 1}$$

$$(vi) \quad \frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

Proof: Let $y = \operatorname{cosec}^{-1}x \Rightarrow \operatorname{cosec}y = x$

$$\Rightarrow \frac{d}{dx}(\operatorname{cosec}y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\operatorname{cosec}y \cot y \frac{d}{dx}(y) = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{cosec}y \cot y}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{cosec}y \sqrt{\operatorname{cosec}^2y - 1}}$$

Put $\operatorname{cosec}y = x$

$$\text{As } 1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

$$\cot^2 \theta = \operatorname{cosec}^2 \theta - 1$$

$$\text{So } \cot \theta = \sqrt{\operatorname{cosec}^2 \theta - 1}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}} \quad \text{Proved}$$

Example 18: Differentiate the following inverse

trigonometric functions.

$$(a) \quad y = \tan^{-1} \sqrt{x} \quad (b) \quad f(x) = \cos^{-1} \left(\frac{x - x^{-1}}{x + x^{-1}} \right)$$

Solution: (a) $y = \tan^{-1} \sqrt{x} \dots (i)$

Let $u = \sqrt{x} \dots (ii)$ put (ii) in (i)

Then $y = \tan^{-1} u \dots (iii)$

Now by chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \dots (A)$

Differentiate (ii) w.r.t. to x

$$\frac{du}{dx} = \frac{d}{dx} \left(x^{\frac{1}{2}} \right) = \frac{1}{2} x^{\frac{1}{2}-1}$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \dots\dots(iv)$$

Differentiate (iii) w.r.t. to u

$$\frac{d}{du}(y) = \frac{d}{du}(\tan^{-1} u) = \frac{1}{1+u^2} \dots\dots(v)$$

Put (iii) and (v) in (A)

$$\frac{dy}{dx} = \frac{1}{1+u^2} \times \frac{1}{2\sqrt{x}} \quad \text{put } u = \sqrt{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+(\sqrt{x})^2} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)} \quad \text{Ans.}$$

$$(b) \quad f(x) = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) \dots\dots(i)$$

$$\text{Let } u = \frac{x-x^{-1}}{x+x^{-1}} = \frac{x - \frac{1}{x}}{x + \frac{1}{x}}$$

$$\Rightarrow u = \frac{x^2 - 1}{x^2 + 1} \dots\dots(ii) \quad \text{put (ii) in (i)}$$

$$f(u) = \cos^{-1} u \dots\dots(iii)$$

$$\text{Now by chain rule } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \dots\dots(A)$$

Differentiate (ii) w.r.t. to x

$$\frac{d}{dx}(u) = \frac{d}{dx}\left(\frac{x^2 - 1}{x^2 + 1}\right)$$

$$\Rightarrow \frac{du}{dx} = \frac{(x^2 + 1) \frac{d}{dx}(x^2 - 1) - (x^2 - 1) \frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2}$$

$$= \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2}$$

$$\Rightarrow \frac{du}{dx} = \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2} \dots\dots(iv)$$

Differentiate (iii) w . r . t . to u

$$\frac{dy}{du} = \frac{d}{du}(\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}} \text{ put } u = \frac{x^2 - 1}{x^2 + 1}$$

$$\Rightarrow \frac{dy}{du} = -\frac{1}{\sqrt{1 - \left(\frac{x^2 - 1}{x^2 + 1}\right)^2}} = -\frac{1}{\sqrt{\frac{(x^2 + 1)^2 - (x^2 - 1)^2}{(x^2 + 1)^2}}}$$

$$\Rightarrow \frac{dy}{du} = -\frac{1}{\frac{\sqrt{x^4 + 2x^2 + 1 - x^4 + 2x^2 - 1}}{(x^2 + 1)}}$$

$$\Rightarrow \frac{dy}{du} = -\frac{x^2 + 1}{\sqrt{4x^2}} = -\frac{x^2 + 1}{2x} \dots\dots(v)$$

Put (iv) and (v) in (A)

$$\frac{dy}{dx} = \frac{x^2 + 1}{2x} \times \frac{4x}{(x^2 + 1)^2} = -\frac{2}{x^2 + 1} \text{ Ans.}$$

Note: (a) $\frac{d}{dx}(e^x) = e^x$

(b) $\frac{d}{dx}(\ln x) = \frac{1}{x}$

The proof of (a) and (b) are in the next exercise .

Exercise 3.4:

Q.1: Use the rule of first principle to differentiate the following functions .

(a) $y = \sin 2x$

Solution: As $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(i)$

Since $f(x) = \sin 2x$ --- (ii)

$$\Rightarrow f(x + \Delta x) = \sin 2(x + \Delta x) \text{ --- (iii)}$$

Put (ii) and (iii) in (i)

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sin(2x + 2\Delta x) - \sin 2x}{\Delta x}$$

Use $\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{2 \cos\left(\frac{2x + 2\Delta x + 2x}{2}\right) \sin\left(\frac{2x + 2\Delta x - 2x}{2}\right)}{\Delta x}$$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{2 \cos\left(\frac{4x + 2\Delta x}{2}\right) \sin \frac{2\Delta x}{2}}{\Delta x}$$

$$\Rightarrow \frac{d}{dx} f(x) = \frac{1}{2} \lim_{\Delta x \rightarrow 0} 2 \cos(2x + \Delta x) \cdot \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x}$$

$$\Rightarrow \frac{d}{dx} f(x) = 2 \cos(2x + 0) \cdot 1$$

$$\Rightarrow \frac{d}{dx} (\sin 2x) = 2 \cos 2x$$

(b) $y = \cot 3x$

Solution: Let $f(x) = \cot 3x$

$$\Rightarrow f(x + \Delta x) = \cot(3x + 3\Delta x)$$

$$\text{Since } f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ --- (i)}$$

Put values in (i)

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} [\cot(3x + 3\Delta x) - \cot 3x]$$

$$\Rightarrow \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\cos(3x + 3\Delta x)}{\sin(3x + 3\Delta x)} - \frac{\cos 3x}{\sin 3x} \right]$$

$$\begin{aligned}
\Rightarrow \frac{d}{dx} f(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\cos(3x + 3\Delta x) \sin 3x - \sin(3x + 3\Delta x) \cos 3x}{\sin(3x + 3\Delta x) \sin 3x} \right] \\
\Rightarrow \frac{d}{dx} f(x) &= - \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin(3x + 3\Delta x) \cos 3x - \cos(3x + 3\Delta x) \sin 3x}{\sin(3x + 3\Delta x) \sin 3x} \right] \\
\Rightarrow \frac{d}{dx} f(x) &= - \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin(3x + 3\Delta x - 3x)}{\sin(3x + 3\Delta x) \sin 3x} \right] \\
\Rightarrow \frac{d}{dx} f(x) &= - \lim_{\Delta x \rightarrow 0} \frac{\sin 3\Delta x}{\Delta x (\sin(3x + 3\Delta x) \sin 3x)} \\
\Rightarrow \frac{d}{dx} f(x) &= -3 \lim_{\Delta x \rightarrow 0} \frac{\sin 3\Delta x}{3\Delta x} \cdot \lim_{\Delta x \rightarrow 0} \frac{1}{\sin(3x + 3\Delta x) \sin 3x} \\
\Rightarrow \frac{d}{dx} (\cot x) &= -3(1) \cdot \frac{1}{\sin 3x \cdot \sin 3x} \\
&= -\frac{3}{\sin^2 3x} = -3 \operatorname{cosec}^2 3x
\end{aligned}$$

(c) $y = \cos 3x + \tan 3x$

Solution: Let $f(x) = \cos 3x + \tan 3x$

$$\Rightarrow f(x + \Delta x) = \cos(3x + 3\Delta x) + \tan(3x + 3\Delta x)$$

$$\text{Since } f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \dots (i)$$

Put values in (i)

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\cos(3x + 3\Delta x) - \cos 3x + \tan(3x + 3\Delta x) - \tan 3x}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\cos(3x + 3\Delta x) - \cos 3x}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{\tan(3x + 3\Delta x) - \tan 3x}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{-2 \sin\left(\frac{3x + 3\Delta x + 3x}{2}\right) \cdot \sin\left(\frac{3x + 3\Delta x - 3x}{2}\right)}{\Delta x}$$

$$+ \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin(3x + 3\Delta x)}{\cos(3x + 3\Delta x)} - \frac{\sin 3x}{\cos 3x} \right]$$

$$\Rightarrow f'(x) = -2 \lim_{\Delta x \rightarrow 0} \sin\left(\frac{6x + 3\Delta x}{2}\right) \cdot \frac{\sin \frac{3\Delta x}{2}}{\Delta x} \\ + \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin(3x + 3\Delta x) \cos 3x - \cos(3x + 3\Delta x) \sin 3x}{\cos(3x + 3\Delta x) \cos 3x} \right]$$

$$\Rightarrow f'(x) = -2 \cdot \frac{3}{2} \lim_{\Delta x \rightarrow 0} \sin\left(\frac{6x}{2} + \frac{3\Delta x}{2}\right) \cdot \frac{\sin \frac{3\Delta x}{2}}{\frac{3\Delta x}{2}} \\ + \lim_{\Delta x \rightarrow 0} \frac{\sin(3x + 3\Delta x - 3x)}{\Delta x \cos(3x + 3\Delta x) \cos 3x}$$

$$\Rightarrow f'(x) = -3 \lim_{\Delta x \rightarrow 0} \sin\left(3x + \frac{\Delta x}{2}\right) \cdot \lim_{\Delta x \rightarrow 0} \frac{\sin \frac{3\Delta x}{2}}{\frac{3\Delta x}{2}} \\ + 3 \lim_{\Delta x \rightarrow 0} \frac{\sin 3\Delta x}{3\Delta x} \cdot \lim_{\Delta x \rightarrow 0} \frac{1}{\cos(3x + 3\Delta x) \cos 3x}$$

$$\text{As } \lim_{\Delta x \rightarrow 0} \frac{\sin \frac{3\Delta x}{2}}{\frac{3\Delta x}{2}} = 1$$

$$\Rightarrow f'(x) = -3 \sin(3x + 0) \cdot 1 + 3 \cdot 1 \cdot \frac{1}{\cos 3x \cdot \cos 3x}$$

$$\Rightarrow f'(x) = -3 \sin 3x + \frac{1}{\cos^2 3x} = -3 \sin 3x + 3 \sec^2 3x$$

(d) $y = \cot^2 x$

Solution: Let $f(x) = \cot^2 x$

$$\Rightarrow f(x + \Delta x) = \cot^2(x + \Delta x)$$

$$\text{Since } f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ --- (i)}$$

Put values in (i)

$$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{\cot^2(x + \Delta x) - \cot^2 x}{\Delta x}$$

$$\begin{aligned}
\Rightarrow f'(x) &= \lim_{h \rightarrow 0} \frac{[\cot(x + \Delta x) + \cot x][\cot(x + \Delta x) - \cot x]}{\Delta x} \\
\Rightarrow f'(x) &= \lim_{h \rightarrow 0} (\cot(x + \Delta x) + \cot x) \cdot \lim_{h \rightarrow 0} \frac{1}{\Delta x} [\cot(x + \Delta x) - \cot x] \\
\Rightarrow \frac{d}{dx} f(x) &= (\cot(x + 0) + \cot x) \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\cos(x + \Delta x)}{\sin(x + \Delta x)} - \frac{\cos x}{\sin x} \right] \\
&= (\cot x + \cot x) \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\cos(x + \Delta x) \sin x - \sin(x + \Delta x) \cos x}{\sin(x + \Delta x) \sin x} \right] \\
\Rightarrow \frac{d}{dx} f(x) &= -2 \cot x \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin(x + \Delta x) \cos x - \cos(x + \Delta x) \sin x}{\sin(x + \Delta x) \sin x} \right] \\
\Rightarrow \frac{d}{dx} f(x) &= -2 \cot x \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin(x + \Delta x - x)}{\sin(x + \Delta x) \sin x} \right] \\
\Rightarrow \frac{d}{dx} f(x) &= -2 \cot x \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x (\sin(x + \Delta x) \sin x)} \\
\Rightarrow \frac{d}{dx} f(x) &= -2 \cot x \cdot \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x} \cdot \lim_{\Delta x \rightarrow 0} \frac{1}{\sin(x + \Delta x) \sin x} \\
\Rightarrow \frac{d}{dx} (\cot^2 x) &= -2 \cot x \cdot (1) \cdot \frac{1}{\sin x \cdot \sin x} \\
&= -\frac{2 \cot x}{\sin^2 x} = -2 \cot x \operatorname{cosec}^2 x
\end{aligned}$$

(e) $y = \tan \sqrt{x}$

Solution: Let $f(x) = \tan \sqrt{x}$

$$\Rightarrow f(x + \Delta x) = \tan \sqrt{x + \Delta x}$$

Since $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots \dots (i)$

Put values in (i)

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\tan \sqrt{x + \Delta x} - \tan \sqrt{x}}{\Delta x}$$

$$\begin{aligned} \Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin \sqrt{x + \Delta x}}{\cos \sqrt{x + \Delta x}} - \frac{\sin \sqrt{x}}{\cos \sqrt{x}} \right] \\ \Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{\sin \sqrt{x + \Delta x} \cos \sqrt{x} - \cos \sqrt{x + \Delta x} \sin \sqrt{x}}{\cos \sqrt{x + \Delta x} \cos \sqrt{x}} \right] \\ \Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\sin(\sqrt{x + \Delta x} - \sqrt{x})}{\Delta x \cos \sqrt{x + \Delta x} \cos \sqrt{x}} \\ \Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\sin((x + \Delta x)^{\frac{1}{2}} - x^{\frac{1}{2}})}{\Delta x \cos \sqrt{x + \Delta x} \cos \sqrt{x}} \\ &\quad \sin \left(x^{\frac{1}{2}} \left(1 + \frac{\Delta x}{x} \right)^{\frac{1}{2}} - x^{\frac{1}{2}} \right) \\ \Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\sin \left(x^{\frac{1}{2}} \left(1 + \frac{\Delta x}{x} \right)^{\frac{1}{2}} - x^{\frac{1}{2}} \right)}{\Delta x \cos \sqrt{x + \Delta x} \cos \sqrt{x}} \end{aligned}$$

Expand by binomial expansion .

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sin \left(x^{\frac{1}{2}} \left(1 + \frac{1}{2} \cdot \frac{\Delta x}{x} + \dots \right) - x^{\frac{1}{2}} \right)}{\Delta x \cos \sqrt{x + \Delta x} \cos \sqrt{x}}$$

Since $\Delta x \rightarrow 0$ then neglect those terms which contains $(\Delta x)^2$ and high powers of Δx .

$$\begin{aligned} \Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\sin \left(x^{\frac{1}{2}} + \frac{1}{2} x^{\frac{1}{2}} \cdot \frac{\Delta x}{x} - x^{\frac{1}{2}} \right)}{\Delta x \cos \sqrt{x + \Delta x} \cos \sqrt{x}} \\ &\quad \sin \left(\frac{\Delta x}{2x^{\frac{1}{2}}} \right) \\ \Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\sin \left(\frac{\Delta x}{2x^{\frac{1}{2}}} \right)}{\Delta x \cos \sqrt{x + \Delta x} \cos \sqrt{x}} \end{aligned}$$

$$\begin{aligned}
 \Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\sin\left(\frac{\Delta x}{2\sqrt{x}}\right)}{\Delta x \cos\sqrt{x} + \Delta x \cos\sqrt{x}} \\
 \Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\sin\left(\frac{\Delta x}{2\sqrt{x}}\right)}{2\sqrt{x} \frac{\Delta x}{2\sqrt{x}}} \cdot \lim_{\Delta x \rightarrow 0} \frac{1}{\cos\sqrt{x} + \Delta x \cos\sqrt{x}} \\
 \Rightarrow f'(x) &= \frac{1}{2\sqrt{x}} \cdot \lim_{\Delta x \rightarrow 0} \frac{\sin\left(\frac{\Delta x}{2\sqrt{x}}\right)}{\frac{\Delta x}{2\sqrt{x}}} \cdot \frac{1}{\cos\sqrt{x} + 0 \cos\sqrt{x}} \\
 &= \frac{1}{2\sqrt{x}} \cdot 1 \cdot \frac{1}{\cos\sqrt{x} \cdot \cos\sqrt{x}} = \frac{1}{\sqrt{x}} \cdot \frac{1}{\cos^2\sqrt{x}} \\
 &= \frac{1}{2\sqrt{x}} \cdot \sec^2\sqrt{x} \quad \text{Ans.}
 \end{aligned}$$

(f) $y = \sin^3 x$

Solution: As $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(i)$

Since $f(x) = \sin^3 x \dots\dots(ii)$

$\Rightarrow f(x + \Delta x) = \sin^3(x + \Delta x) \dots\dots(iii)$

Put (ii) and (iii) in (i)

$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sin^3(x + \Delta x) - \sin^3 x}{\Delta x}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[(\sin(x + \Delta x) - \sin x) \left(\sin^2(x + \Delta x) + \sin(x + \Delta x) \sin x + \sin^2 x \right) \right] \\
 f'(x) &= \lim_{\Delta x \rightarrow 0} \left[\frac{(\sin(x + \Delta x) - \sin x)}{\Delta x} \right] \cdot \lim_{\Delta x \rightarrow 0} \left(\sin^2(x + \Delta x) + \sin(x + \Delta x) \sin x + \sin^2 x \right)
 \end{aligned}$$

$$\text{Use } \sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right)$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\left(2 \cos \left(\frac{x + \Delta x + x}{2} \right) \sin \left(\frac{x + \Delta x - x}{2} \right) \right)}{\Delta x} \cdot \left[\sin^2(x+0) + \sin(x+0)\sin x + \sin^2 x \right]$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{2 \cos \left(\frac{2x + \Delta x}{2} \right) \sin \frac{\Delta x}{2}}{\Delta x} \right) \cdot (\sin^2 x + \sin^2 x + \sin^2 x)$$

Denominator multiplied and divided by 2

$$\Rightarrow f'(x) = \left(\frac{1}{2} \lim_{\Delta x \rightarrow 0} 2 \cos \left(x + \frac{\Delta x}{2} \right) \cdot \lim_{\Delta x \rightarrow 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \right) \cdot (3 \sin^2 x)$$

$$\Rightarrow f'(x) = (\cos(x+0) \cdot 1) \cdot (3 \sin^2 x)$$

$$\text{As } \lim_{\Delta x \rightarrow 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1$$

$$\Rightarrow f'(x) = 3 \sin^2 x \cos x$$

Q.2: Differentiate the following trigonometric functions by using any suitable rule.

(a) $y = x^2 \cot 3x$

Solution: $y = x^2 \cot 3x$, differentiate w. r. t. to x .

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^2 \cot 3x)$$

$$\Rightarrow \frac{dy}{dx} = x^2 \cdot \frac{d}{dx}(\cot 3x) + \left(\frac{d}{dx}(x^2) \right) \cdot \cot 3x$$

$$\begin{aligned}
 \Rightarrow \frac{dy}{dx} &= x^2(-\operatorname{cosec}^2 3x) \frac{d}{dx}(3x) + 2x \cot 3x \\
 \Rightarrow \frac{dy}{dx} &= -x^2(\operatorname{cosec}^2 3x)(3) + 2x \cot 3x \\
 \Rightarrow \frac{dy}{dx} &= 2x \cot 3x - 3x^2 \operatorname{cosec}^2 3x \quad \text{Ans.}
 \end{aligned}$$

(b) $y = (\sin 2x + \cot 3x)^2$

Solution: $y = (\sin 2x + \cot 3x)^2$, differentiate w. r. t. to x .

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx}(\sin 2x + \cot 3x)^2 \\
 &= 2(\sin 2x + \cot 3x)^{2-1} \frac{d}{dx}(\sin 2x + \cot 3x) \\
 \Rightarrow \frac{dy}{dx} &= 2(\sin 2x + \cot 3x) \left(\frac{d}{dx} \sin 2x + \frac{d}{dx} \cot 3x \right) \\
 &= 2(\sin 2x + \cot 3x) \left(\cos 2x \cdot \frac{d}{dx}(2x) + (-\operatorname{cosec}^2 3x) \frac{d}{dx}(3x) \right) \\
 &= 2(\sin 2x + \cot 3x) (2 \cos 2x - 3 \operatorname{cosec}^2 3x)
 \end{aligned}$$

(c) $y = 4 \operatorname{cosec} 2x$

Solution: $y = 4 \operatorname{cosec} 2x$, differentiate w. r. t. to x .

$$\begin{aligned}
 \frac{d}{dx}(y) &= \frac{d}{dx}(4 \operatorname{cosec} 2x) \\
 \Rightarrow \frac{dy}{dx} &= 4(-\operatorname{cosec} 2x \cot 2x) \frac{d}{dx}(2x) \\
 \Rightarrow \frac{dy}{dx} &= -(4 \operatorname{cosec} 2x \cot 2x)(2) \\
 \Rightarrow \frac{dy}{dx} &= -8 \operatorname{cosec} 2x \cot 2x \quad \text{Ans.}
 \end{aligned}$$

(d) $y = 2 \tan(x+3)^2$

Solution: $y = 2 \tan(x+3)^2$ (i)

Let $u = (x + 3)^2$ (ii)

Put (ii) in (i) $y = 2 \tan u$ (iii)

As by chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (A)

Differentiate (ii) w. r. t. to x .

$$\frac{d}{dx}(u) = \frac{d}{dx}(x + 3)^2 = 2(x + 3)^{2-1} \frac{d}{dx}(x + 3)$$

$$\Rightarrow \frac{du}{dx} = 2(x + 3)(1) = 2(x + 3) \text{(iv)}$$

Differentiate (iii) w. r. t. to u

$$\frac{d}{du}(y) = \frac{d}{du}(2 \tan u) = 2 \sec^2 u$$

put $u = (x + 3)^2$

$$\Rightarrow \frac{dy}{du} = 2 \sec^2(x + 3)^2 \text{(v)}$$

Put (iv) and (v) in (A)

$$\begin{aligned} \frac{dy}{dx} &= 2 \sec^2(x + 3)^2 \times 2(x + 3) \\ &= 4(x + 3) \sec^2(x + 3)^2 \text{ Ans.} \end{aligned}$$

(e) $y = \cos \sqrt{x} + \sqrt{\tan x}$

Solution: $y = \cos \sqrt{x} + \sqrt{\tan x}$, differentiate w. r. t. to x .

$$\frac{d}{dx}(y) = \frac{d}{dx}(\cos \sqrt{x} + \sqrt{\tan x})$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\cos \sqrt{x}) + \frac{d}{dx}(\tan x)^{\frac{1}{2}}$$

$$= -\sin \sqrt{x} \cdot \frac{d}{dx}\left(x^{\frac{1}{2}}\right) + \frac{1}{2}(\tan x)^{\frac{1}{2}-1} \frac{d}{dx}(\tan x)$$

$$= (-\sin \sqrt{x}) \frac{1}{2} x^{\frac{1}{2}-1} + \frac{1}{2}(\tan x)^{-\frac{1}{2}}(\sec^2 x)$$

$$= -\frac{1}{2} \sin \sqrt{x} x^{-\frac{1}{2}} + \frac{\sec^2 x}{2(\tan x)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sin \sqrt{x}}{2\sqrt{x}} + \frac{\sec^2 x}{2\sqrt{\tan x}}$$

$$(f) \quad y = \frac{\sqrt{\tan x}}{\cos \sqrt{x}}$$

Solution : $y = \frac{\sqrt{\tan x}}{\cos \sqrt{x}}$, Differentiate w. r. t. to x .

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{(\tan x)^{\frac{1}{2}}}{\cos \sqrt{x}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos \sqrt{x} \frac{d}{dx} (\tan x)^{\frac{1}{2}} - (\tan x)^{\frac{1}{2}} \frac{d}{dx} (\cos \sqrt{x})}{(\cos \sqrt{x})^2}$$

$$= \frac{1}{\cos^2 \sqrt{x}} \left[\cos \sqrt{x} \cdot \frac{1}{2} (\tan x)^{\frac{1}{2}-1} \cdot \frac{d}{dx} (\tan x) - (\tan x)^{\frac{1}{2}} (-\sin \sqrt{x}) \frac{d}{dx} x^{\frac{1}{2}} \right]$$

$$= \frac{1}{\cos^2 \sqrt{x}} \left[\cos \sqrt{x} \cdot \frac{1}{2} (\tan x)^{-\frac{1}{2}} (\sec^2 x) - (\tan x)^{\frac{1}{2}} (-\sin \sqrt{x}) \frac{1}{2} x^{-\frac{1}{2}} \right]$$

$$= \frac{1}{\cos^2 \sqrt{x}} \left[\frac{\cos \sqrt{x} \cdot \sec^2 x}{2\sqrt{\tan x}} + \frac{\sqrt{\tan x} \cdot \sin \sqrt{x}}{2\sqrt{x}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x} \cos \sqrt{x} \cdot \sec^2 x + \tan x \cdot \sin \sqrt{x}}{2\sqrt{x} \cos \sqrt{x} \cdot \sqrt{\tan x}} \text{ Ans.}$$

Q.3: Use any suitable rule of differentiation to perform

$\frac{dy}{dx}$ for the following functions .

$$(a) \quad y = \cos^{-1} \left(\frac{x}{a} \right)$$

Solution : $y = \cos^{-1}\left(\frac{x}{a}\right)$ (i)

Let $u = \frac{x}{a}$ (ii)

Put (ii) in (i) $y = \cos^{-1} u$ (iii)

As by chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (A)

Differentiate (ii) w . r . t . to x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x}{a}\right) = \frac{1}{a} \text{(iv)}$$

$$\frac{dy}{du} = \frac{d}{du} \cos^{-1} u$$

$$\Rightarrow \frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}} \quad \text{Put } u = \frac{x}{a}$$

$$\frac{dy}{du} = -\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \text{(v)}$$

Put (iv) and (v) in (A)

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \times \frac{1}{a} = -\frac{1}{a\sqrt{1-\frac{x^2}{a^2}}} \text{ Ans.}$$

(b) $y = \tan^{-1}\left(\frac{x}{p}\right)$

Solution : $y = \tan^{-1}\left(\frac{x}{p}\right)$, Differentiate w . r . t . to x .

$$\frac{dy}{dx} = \frac{d}{dx} \left(\tan^{-1}\left(\frac{x}{p}\right) \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\left(\frac{x}{p}\right)^2} \frac{d}{dx} \left(\frac{x}{p} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1 + \frac{x^2}{p^2}} \cdot \frac{1}{p}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{p^2 + x^2}{p^2}} \cdot \frac{1}{p} = \frac{p^2}{p^2 + x^2} \cdot \frac{1}{p} = \frac{p}{p^2 + x^2} \quad \text{Ans.}$$

(c) $y = \cot^{-1}\left(\frac{a}{x}\right)$

Solution : $y = \cot^{-1}\left(\frac{a}{x}\right)$, Differentiate w . r . t . to x .

$$\frac{dy}{dx} = \frac{d}{dx} \cot^{-1}\left(\frac{a}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{1 + \left(\frac{a}{x}\right)^2} \frac{d}{dx} \left(\frac{a}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{1 + \frac{a^2}{x^2}} \cdot a \cdot \frac{d}{dx} (x^{-1})$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\frac{x^2 + a^2}{x^2}} \cdot a(-1)x^{-2}$$

$$= -\frac{x^2}{x^2 + a^2} \cdot \left(\frac{-a}{x^2}\right) = \frac{a}{x^2 + a^2} \quad \text{Ans.}$$

(d) $y = \operatorname{cosec}^{-1}(\sqrt{1+x^2})$

Solution: $y = \operatorname{cosec}^{-1}(\sqrt{1+x^2})$ (i)

Let $u = (1+x^2)^{\frac{1}{2}}$ (ii) put (ii) in (i)

$$y = \operatorname{cosec}^{-1}u \quad \text{.....(iii)}$$

As by chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (A)

use formula

$$\frac{d}{dx} \cot^{-1} u = -\frac{1}{1+u^2} \frac{d}{dx} (u)$$

Differentiate (ii) w . r . t . to x ,

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx} (1+x^2)^{\frac{1}{2}} \\ &= \frac{1}{2} (1+x^2)^{\frac{1}{2}-1} \frac{d}{dx} (1+x^2) \\ \Rightarrow \frac{du}{dx} &= \frac{1}{2} (1+x^2)^{-\frac{1}{2}} (2x) = \frac{x}{\sqrt{1+x^2}} \dots\dots(iv)\end{aligned}$$

Differentiate (iii) w . r . t . to u ,

$$\begin{aligned}\frac{dy}{du} &= \frac{d}{du} (\operatorname{cosec}^{-1} u) \\ \Rightarrow \frac{dy}{du} &= -\frac{1}{u\sqrt{u^2-1}} \quad \text{put } u = \sqrt{1+x^2} \\ \Rightarrow \frac{dy}{du} &= -\frac{1}{\sqrt{1+x^2} \sqrt{(\sqrt{1+x^2})^2-1}} = -\frac{1}{\sqrt{1+x^2} \sqrt{1+x^2-1}} \\ &= -\frac{1}{\sqrt{1+x^2} \cdot \sqrt{x^2}} = -\frac{1}{x\sqrt{1+x^2}} \dots\dots(v)\end{aligned}$$

Put (iv) and (v) in (A)

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{x\sqrt{1+x^2}} \times \frac{x}{\sqrt{1+x^2}} = -\frac{1}{(\sqrt{1+x^2})(\sqrt{1+x^2})} \\ \Rightarrow \frac{dy}{dx} &= -\frac{1}{1+x^2} \text{ Ans.}\end{aligned}$$

(e) $y = \operatorname{cosec}^{-1}(x+3)$

Solution : $y = \operatorname{cosec}^{-1}(x+3) \dots\dots(i)$

Let $u = x+3 \dots\dots(ii)$ put (ii) in (i)

$y = \operatorname{cosec}^{-1} u \dots\dots(ii)$

As by chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \dots\dots(A)$

Differentiate (ii) w . r . t . to x .

$$\frac{du}{dx} = \frac{d}{dx} (x+3) = 1 \dots\dots(iv)$$

Differentiate (iii) w . r . t . to u ,

$$\frac{dy}{du} = \frac{d}{du} (\operatorname{cosec}^{-1} u)$$

$$\Rightarrow \frac{dy}{du} = -\frac{1}{u\sqrt{u^2-1}} \quad \text{put } u = x+3$$

$$\Rightarrow \frac{dy}{du} = -\frac{1}{(x+3)\sqrt{(x+3)^2-1}} \dots\dots\dots (v)$$

Put (iv) and (v) in (A)

$$\frac{dy}{dx} = -\frac{1}{(x+3)\sqrt{(x+3)^2-1}} \times 1$$

$$= -\frac{1}{(x+3)\sqrt{x^2+6x+9-1}} = -\frac{1}{(x+3)\sqrt{x^2+6x+8}} \quad \text{Ans.}$$

$$(f) \quad y = \frac{1}{x} \tan^{-1} \frac{x+1}{x-1}$$

Solution: $y = \frac{1}{x} \tan^{-1} \frac{x+1}{x-1}$, Differentiate (ii) w . r . t . to x .

$$\frac{d}{dx} (y) = \frac{d}{dx} \left(\frac{1}{x} \cdot \tan^{-1} \frac{x+1}{x-1} \right)$$

$$= \frac{1}{x} \frac{d}{dx} \left(\tan^{-1} \frac{x+1}{x-1} \right) + \left(\frac{d}{dx} \left(\frac{1}{x} \right) \right) \left(\tan^{-1} \frac{x+1}{x-1} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{1 + \left(\frac{x+1}{x-1} \right)^2} \cdot \frac{d}{dx} \left(\frac{x+1}{x-1} \right) + \left(\frac{d}{dx} x^{-1} \right) \left(\tan^{-1} \frac{x+1}{x-1} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{1 + \frac{(x+1)^2}{(x-1)^2}} \cdot \frac{(x-1) \frac{d}{dx} (x+1) - (x+1) \frac{d}{dx} (x-1)}{(x-1)^2}$$

$$+ (-1)x^{-2} \left(\tan^{-1} \frac{x+1}{x-1} \right)$$

$$\begin{aligned}
 \Rightarrow \frac{dy}{dx} &= \frac{1}{x} \cdot \frac{1}{(x-1)^2 + (x+1)^2} \cdot \frac{(x-1) \cdot 1 - (x+1) \cdot 1}{(x-1)^2} - \frac{1}{x^2} \left(\tan^{-1} \frac{x+1}{x-1} \right) \\
 \Rightarrow \frac{dy}{dx} &= \frac{1}{x} \cdot \frac{(x-1)^2}{(x-1)^2 + (x+1)^2} \cdot \frac{x-1-x-1}{(x-1)^2} - \frac{1}{x^2} \left(\tan^{-1} \frac{x+1}{x-1} \right) \\
 \Rightarrow \frac{dy}{dx} &= \frac{1}{x} \left(\frac{1}{x^2 - 2x + 1 + x^2 + 2x + 1} \right) \cdot \frac{-2}{1} - \frac{1}{x^2} \left(\tan^{-1} \frac{x+1}{x-1} \right) \\
 \Rightarrow \frac{dy}{dx} &= \frac{1}{x} \left(-\frac{2}{2x^2 + 2} \right) - \frac{1}{x^2} \left(\tan^{-1} \frac{x+1}{x-1} \right) \\
 \Rightarrow \frac{dy}{dx} &= -\frac{2}{2x(x^2 + 1)} - \frac{1}{x^2} \left(\tan^{-1} \frac{x+1}{x-1} \right) \\
 \Rightarrow \frac{dy}{dx} &= -\frac{\left(\tan^{-1} \frac{x+1}{x-1} \right)}{x^2} - \frac{1}{x(x^2 + 1)} \quad \text{Ans.}
 \end{aligned}$$

Q.4: Suppose profits on the sale of swimming suits in a Departmental store are given approximately by

$$p(t) = 5 - 5 \cos \frac{\pi t}{26}, \quad 0 \leq t \leq 104$$

Where $P(t)$ is profit (in hundreds of dollars) for a week of sales t weeks after January first.

- What is the rate of change of profit t weeks after the first year?
- What is the rate of change of profit 8 weeks after the first year? 26 weeks after the first year? 50 weeks after the first year?

Solution : $p(t) = 5 - 5 \cos \frac{\pi t}{26} \dots (i)$, $0 \leq t \leq 104$

(a) Differentiate (i) w.r.t. to t .

$$\frac{d}{dt} P(t) = \frac{d}{dt} \left(5 - 5 \cos \frac{\pi t}{26} \right)$$

$$\begin{aligned}\Rightarrow P'(t) &= \frac{d}{dt}(5) - 5 \frac{d}{dt} \left(\cos \frac{\pi t}{26} \right) \\ &= 0 - 5 \left(-\sin \frac{\pi t}{26} \right) \frac{d}{dt} \left(\frac{\pi t}{26} \right)\end{aligned}$$

$$\Rightarrow P'(t) = \left(5 \sin \frac{\pi t}{26} \right) \left(\frac{\pi}{26} \right)$$

$$\Rightarrow P'(t) = \frac{5\pi}{26} \sin \frac{\pi t}{26} \dots\dots(A) \quad , \quad 0 \leq t \leq 104$$

Eq(A) shows that the rate of change of profit t weeks after the first year .

(b) The rate of change of profit 8 weeks after the first year .

put $t = 8$ in (A)

$$\begin{aligned}P'(8) &= \frac{5\pi}{26} \sin \frac{8\pi}{26} \\ &= \frac{5(3.14)}{26} \sin \left(\frac{8 \times 180^\circ}{26} \right) \\ &= (0.604) \sin 55.88^\circ \\ &= (0.604)(0.8279)\end{aligned}$$

As π radian $= 180^\circ$

$$\pi = 3.14$$

$$\Rightarrow P'(8) = 0.50 \text{ hundred dollors / week}$$

$$\text{or } P'(8) = (0.50)(100) \text{ dollors / week}$$

$$\Rightarrow P'(8) = 50 \text{ dollors / week}$$

The rate of change of profit 26 weeks after the first year .

put $t = 26$ in (A)

$$\begin{aligned}P'(26) &= \frac{5\pi}{26} \sin \left(\frac{26\pi}{26} \right) \\ P'(26) &= \frac{5(3.14)}{26} \sin \left(\frac{26 \times 180^\circ}{26} \right)\end{aligned}$$

$$\Rightarrow P'(26) = (0.604) \sin 180^\circ$$

$$\Rightarrow P'(26) = (0.604)(0) = 0 \text{ dollors / week}$$

=====

The rate of change of profit 50 weeks after the first year
 put $t = 50$ in (A)

$$\Rightarrow P'(50) = \frac{5\pi}{26} \sin\left(\frac{50\pi}{26}\right)$$

$$P'(50) = \frac{5(3.14)}{26} \sin\left(\frac{50 \times 180^\circ}{26}\right)$$

$$\Rightarrow P'(50) = (0.604) \sin(346.15^\circ)$$

$$= (0.604)(-0.239)$$

$$\Rightarrow P'(50) = -0.14 \text{ hundred dollors / week}$$

$$= (-0.14)(100) \text{ dollors / week}$$

$$\Rightarrow P'(50) = -14 \text{ dollors / week}$$

Q.5: A normal seated adult breaths in and exhales about 0.8 litre of air every 4 seconds. The volume of air $V(t)$ in the lungs t seconds after exhaling is given approximately

$$\text{by } V(t) = 0.45 - 0.35 \cos \frac{\pi t}{2}, \quad 0 \leq t \leq 8$$

(a) What is the rate of flow of air t seconds after exhaling ?

(b) What is the rate of flow of air 3 seconds after exhaling ?
 4 seconds after exhaling ? 5 seconds after exhaling ?

Solution : $V(t) = 0.45 - 0.35 \cos \frac{\pi t}{2} \dots (i), \quad 0 \leq t \leq 8$

(a) Differentiate w. r. t. to t .

$$\frac{d}{dt} V(t) = \frac{d}{dt} \left(0.45 - 0.35 \cos \frac{\pi t}{2} \right)$$

$$\Rightarrow V'(t) = \frac{d}{dt} (0.45) - 0.35 \frac{d}{dt} \left(\cos \frac{\pi t}{2} \right)$$

$$= 0 - 0.35 \left(-\sin \frac{\pi t}{2} \right) \frac{d}{dt} \left(\frac{\pi t}{2} \right)$$

$$\Rightarrow V'(t) = (0.35 \sin \frac{\pi t}{2}) \left(\frac{\pi}{2} \right)$$

$$\Rightarrow V'(t) = \frac{0.35\pi}{2} \sin \frac{\pi t}{2} \dots\dots(A) , \quad 0 \leq t \leq 8$$

Eq(A) shows that the rate of flow of air t seconds after exhaling

(b) The rate of flow of air 3 seconds after exhaling :

Put $t = 3$ in (A)

$$\begin{aligned} V'(3) &= \frac{0.35\pi}{2} \sin \frac{3\pi}{2} & \text{As } \sin \frac{3\pi}{2} &= -1 \\ &= \frac{(0.35)(3.14)}{2} (-1) \end{aligned}$$

$$\Rightarrow V'(3) = (0.55)(-1) = -0.55 \text{ Liter / second}$$

The rate of flow of air 4 seconds after exhaling :

Put $t = 4$ in (A)

$$\begin{aligned} V'(4) &= \frac{0.35\pi}{2} \sin \frac{4\pi}{2} & \text{As } \sin 2\pi &= 0 \\ &= \frac{0.35\pi}{2} \sin 2\pi \\ &= \frac{(0.35)(3.14)}{2} (0) \end{aligned}$$

$$\Rightarrow V'(4) = 0 \text{ Liter / second}$$

The rate of flow of air 5 seconds after exhaling :

Put $t = 5$ in (A)

$$\begin{aligned} V'(5) &= \frac{0.35\pi}{2} \sin \frac{5\pi}{2} & \text{As } \sin \frac{5\pi}{2} &= 1 \\ &= \frac{(0.35)(3.14)}{2} (1) \end{aligned}$$

$$\Rightarrow V'(5) = (0.55)(1) = 0.55 \text{ Liter / second}$$

Differentiation of exponential and Logarithmic functions :

(i) Derivative of e^x and a^x by first principle :

As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(A)$$

Let $y = f(x) = e^x \dots\dots(i)$

$f(x + \Delta x) = e^{x+\Delta x} \dots\dots(ii)$

Put (i) and (ii) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$\Rightarrow \frac{d}{dx}(y) = \lim_{\Delta x \rightarrow 0} \frac{e^x e^{\Delta x} - e^x}{\Delta x}$$

$$\begin{aligned} \Rightarrow \frac{d}{dx}(e^x) &= \lim_{\Delta x \rightarrow 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x} \\ &= e^x \lim_{\Delta x \rightarrow 0} \frac{e^{\Delta x} - 1}{\Delta x} \end{aligned}$$

$$\text{As } \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$$

$$\Rightarrow \frac{d}{dx}(e^x) = e^x \ln e \quad \text{but } \ln e = 1$$

$$\Rightarrow \frac{d}{dx}(e^x) = e^x$$

Derivative a^x :

Let $y = f(x) = a^x \dots\dots(iii)$

$f(x + \Delta x) = a^{x+\Delta x} \dots\dots(iv)$

Put (iii) and (iv) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

$$\Rightarrow \frac{d}{dx}(y) = \lim_{\Delta x \rightarrow 0} \frac{a^x a^{\Delta x} - a^x}{\Delta x}$$

$$\begin{aligned} \Rightarrow \frac{d}{dx}(a^x) &= \lim_{\Delta x \rightarrow 0} \frac{a^x(a^{\Delta x} - 1)}{\Delta x} \\ &= a^x \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x} \end{aligned}$$

$$\text{As } \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$$

$$\Rightarrow \frac{d}{dx}(a^x) = a^x \ln a \quad \text{As } \ln a = \log_e a$$

$$\Rightarrow \frac{d}{dx}(a^x) = a^x \log_e a$$

(ii) Derivative of $\ln x$ and $\log_a x$ by first principle :

As by first principle rule

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(A)$$

$$\text{Let } y = f(x) = \ln x \dots\dots(i)$$

$$f(x + \Delta x) = \ln(x + \Delta x) \dots\dots(ii)$$

Put (i) and (ii) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\ln(x + \Delta x) - \ln x}{\Delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \ln \left(\frac{x + \Delta x}{x} \right)$$

R . H . S . multiplied and divided by x .

$$\Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x} \lim_{\Delta x \rightarrow 0} \frac{x}{\Delta x} \ln \left(\frac{x + \Delta x}{x} \right)$$

$$= \frac{1}{x} \lim_{\Delta x \rightarrow 0} \ln \left(1 + \frac{\Delta x}{x} \right)^{\frac{x}{\Delta x}}$$

Use log rule

$$\log m^n = n \log m$$

$$\Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x} \ln \left(\lim_{\Delta x \rightarrow 0} \left(1 + \frac{\Delta x}{x} \right)^{\frac{x}{\Delta x}} \right)$$

$$\text{As } \lim_{\Delta x \rightarrow 0} \left(1 + \frac{\Delta x}{x} \right)^{\frac{x}{\Delta x}} = e$$

$$\Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x} \ln e \quad \text{As } \ln e = 1$$

$$\Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Derivative of $\log_a x$:

Let $y = f(x) = \log_a x$ (i)

$$f(x + \Delta x) = \log_a (x + \Delta x) \dots (ii)$$

Put (i) and (ii) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\log_a (x + \Delta x) - \log_a x}{\Delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \log_a \left(\frac{x + \Delta x}{x} \right)$$

R . H . S. multiplied and divided by x .

$$\Rightarrow \frac{d}{dx} (\log_a x) = \frac{1}{x} \lim_{\Delta x \rightarrow 0} \frac{x}{\Delta x} \log_a \left(\frac{x + \Delta x}{x} \right)$$

Use log rule

$$= \frac{1}{x} \lim_{\Delta x \rightarrow 0} \log_a \left(1 + \frac{\Delta x}{x} \right)^{\frac{x}{\Delta x}}$$

$$\log m^n = n \log m$$

$$\Rightarrow \frac{d}{dx} (\log_a x) = \frac{1}{x} \log_a \left(\lim_{\Delta x \rightarrow 0} \left(1 + \frac{\Delta x}{x} \right)^{\frac{x}{\Delta x}} \right)$$

$$\text{As } \lim_{\Delta x \rightarrow 0} \left(1 + \frac{\Delta x}{x} \right)^{\frac{x}{\Delta x}} = e$$

$$\Rightarrow \frac{d}{dx} (\log_a x) = \frac{1}{x} \log_a e \quad \text{As } \log_a e = \frac{1}{\log_e a}$$

$$\Rightarrow \frac{d}{dx} (\log_a x) = \frac{1}{x \log_e a} = \frac{1}{x \ln a} \text{ Ans.}$$

Note: (a) $\frac{d}{dx} e^u = e^u \frac{du}{dx}$ (b) $\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$

(c) $\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$ (d) $\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}$

Example 19: Differentiate the following functions .

(a) $f(x) = 7^{(4-3x^5)}$

Solution: $y = 7^{(4-3x^5)}$ (i)

Let $u = 4 - 3x^5$ (ii) put (ii) in (i)

$$y = 7^u \dots (iii)$$

As by chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (A)

Differentiate (ii) w . r . t . to x .

$$\frac{du}{dx} = \frac{d}{dx}(4 - 3x^5) = -15x^4 \text{(iv)}$$

Differentiate (iii) w . r . t . to u

$$\frac{dy}{du} = \frac{d}{du} 7^u = 7^u \ln 7 \quad \text{put } u = 4 - 3x^5$$

$$\Rightarrow \frac{dy}{du} = 7^{4-3x^5} \text{(v)}$$

Put (iv) and (v) in (A)

$$\begin{aligned} \frac{dy}{dx} &= (7^{4-3x^5} \ln 7)(-15x^4) \\ &= -15x^4 \ln 7 \cdot 7^{4-3x^5} \end{aligned}$$

(b) $y = \log_{10} \sqrt{x^2 - 7x} + x^3$

Solution: $y = \log_{10} \sqrt{x^2 - 7x} + x^3$

$$\Rightarrow y = \log_{10} (x^2 - 7x)^{\frac{1}{2}} + x^3$$

$$\Rightarrow y = \frac{1}{2} \log_{10} (x^2 - 7x) + x^3 \quad \text{Differentiate w.r.t. to } x$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{2} \log_{10} (x^2 - 7x) + x^3 \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \frac{d}{dx} \log_{10} (x^2 - 7x) + \frac{d}{dx} (x^3)$$

$$= \frac{1}{2} \cdot \frac{1}{(x^2 - 7x)} \log_{10} e \frac{d}{dx} (x^2 - 7x) + 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(x^2 - 7x)} (\log_{10} e)(2x - 7) + 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x - 7}{2(x^2 - 7x)} \log_{10} e + 3x^2$$

(c) $y = \ln(e^{mx} + e^{-mx})$

Solution : $y = \ln(e^{mx} + e^{-mx}) \dots\dots(i)$

Let $u = e^{mx} + e^{-mx} \dots\dots(ii)$ put (ii) in (i)

$y = \ln u \dots\dots(iii)$

As by chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \dots\dots(A)$

Differentiate (ii) w . r . t . to x .

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx}(e^{mx} + e^{-mx}) \\ &= \frac{d}{dx}(e^{mx}) + \frac{d}{dx}(e^{-mx}) \end{aligned}$$

$$\Rightarrow \frac{du}{dx} = e^{mx} \frac{d}{dx}(mx) + e^{-mx} \frac{d}{dx}(-mx)$$

$$\Rightarrow \frac{du}{dx} = me^{mx} - me^{-mx} = m(e^{mx} - e^{-mx}) \dots\dots(iv)$$

Differentiate (iii) w . r . t . to u

$$\frac{dy}{du} = \frac{d}{du} \ln u = \frac{1}{u} \quad \text{put } u = e^{mx} + e^{-mx}$$

$$\frac{dy}{du} = \frac{1}{e^{mx} + e^{-mx}} \dots\dots(v)$$

Put (iv) and (v) in (A)

$$\frac{dy}{dx} = \left(\frac{1}{e^{mx} + e^{-mx}} \right) \times m(e^{mx} - e^{-mx})$$

$$\Rightarrow \frac{dy}{dx} = \frac{m(e^{mx} - e^{-mx})}{(e^{mx} + e^{-mx})} \text{ Ans.}$$

(d) $y = \frac{e^{2x}}{\ln x}$

Solution : $y = \frac{e^{2x}}{\ln x}$, Differentiate w . r . t to x

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{e^{2x}}{\ln x} \right) \\
 \Rightarrow \frac{dy}{dx} &= \frac{\ln x \frac{d}{dx} (e^{2x}) - (e^{2x}) \frac{d}{dx} \ln x}{(\ln x)^2} \\
 &= \frac{(\ln x) e^{2x} \frac{d}{dx} (2x) - e^{2x} \left(\frac{1}{x} \right)}{(\ln x)^2} \\
 &= \frac{2(\ln x) e^{2x} - e^{2x} \left(\frac{1}{x} \right)}{(\ln x)^2} \\
 \Rightarrow \frac{dy}{dx} &= \frac{e^{2x} (2x \ln x - 1)}{x (\ln x)^2} \quad \text{Ans.}
 \end{aligned}$$

Example 20: Differentiate the following functions .

(a) $y = \ln \left(\frac{x(x^2 - 3)^2}{(x^2 - 4)^{\frac{1}{2}}} \right)$

Solution : $y = \ln \left(\frac{x(x^2 - 3)^2}{(x^2 - 4)^{\frac{1}{2}}} \right)$ use logarithm law

$$\Rightarrow y = \ln(x(x^2 - 3)^2) - \ln(x^2 - 4)^{\frac{1}{2}}$$

$$\Rightarrow y = \ln x + \ln(x^2 - 3)^2 - \frac{1}{2} \ln(x^2 - 4)$$

$$\Rightarrow y = \ln x + 2 \ln(x^2 - 3) - \frac{1}{2} \ln(x^2 - 4)$$

Differentiate w . r . t . to x ,

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(\ln x + 2 \ln(x^2 - 3) - \frac{1}{2} \ln(x^2 - 4) \right) \\
 &= \frac{d}{dx} \ln x + 2 \frac{d}{dx} \ln(x^2 - 3) - \frac{1}{2} \frac{d}{dx} \ln(x^2 - 4) \\
 \Rightarrow \frac{dy}{dx} &= \frac{1}{x} + \frac{2}{x^2 - 3} \frac{d}{dx} (x^2 - 3) - \frac{1}{2(x^2 - 4)} \frac{d}{dx} (x^2 - 4) \\
 \Rightarrow \frac{dy}{dx} &= \frac{1}{x} + \frac{4x}{x^2 - 3} - \frac{2x}{2(x^2 - 4)} \\
 \Rightarrow \frac{dy}{dx} &= \frac{(x^2 - 3)(x^2 - 4) + 4x^2(x^2 - 4) - x^2(x^2 - 3)}{x(x^2 - 3)(x^2 - 4)} \\
 &= \frac{x^4 - 4x^2 - 3x^2 + 12 + 4x^4 - 16x^2 - x^4 + 3x^2}{x(x^2 - 3)(x^2 - 4)} \\
 \Rightarrow \frac{dy}{dx} &= \frac{4x^4 - 20x^2 + 12}{x(x^2 - 3)(x^2 - 4)} \quad \text{Ans.}
 \end{aligned}$$

(b) $y = x^{\sin x}$

Solution : $y = x^{\sin x}$ (i) taking log of both sides

$$\ln y = \ln x^{\sin x}$$

$$\Rightarrow \ln y = \sin x \ln x, \text{ differentiate w.r.t.to } x$$

$$\frac{d}{dx} (\ln y) = \frac{d}{dx} (\sin x \ln x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \sin x \frac{d}{dx} \ln x + \ln x \frac{d}{dx} \sin x$$

$$\Rightarrow \frac{dy}{dx} = y \left((\sin x) \left(\frac{1}{x} \right) + (\ln x) \cos x \right) \text{(ii)}$$

Put (i) in (ii)

$$\frac{dy}{dx} = x^{\sin x} \left(\frac{\sin x}{x} + \cos x \ln x \right) \quad \text{Ans.}$$

Hyperbolic Functions:

$$\begin{aligned}
 \text{(i)} \quad \sinh x &= \frac{e^x - e^{-x}}{2} & \text{(ii)} \quad \cosh x &= \frac{e^x + e^{-x}}{2} \\
 \text{(iii)} \quad \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} & \text{(iv)} \quad \coth x &= \frac{e^x + e^{-x}}{e^x - e^{-x}}, x \neq 0 \\
 \text{(v)} \quad \operatorname{cosech} x &= \frac{2}{e^x - e^{-x}} & \text{(vi)} \quad \operatorname{sech} x &= \frac{2}{e^x + e^{-x}}
 \end{aligned}$$

Derivatives of Hyperbolic Functions:

$$\text{(i)} \quad \frac{d}{dx}(\sinh x) = \cosh x$$

$$\text{Proof: } \frac{d}{dx}(\sinh x) = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right)$$

$$\frac{d}{dx}(\sinh x) = \frac{e^x + e^{-x}}{2} = \cosh x \quad \text{Proved}$$

$$\text{(ii)} \quad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\text{Proof: } \frac{d}{dx}(\cosh x) = \frac{d}{dx} \left(\frac{e^x + e^{-x}}{2} \right)$$

$$\frac{d}{dx}(\cosh x) = \frac{e^x - e^{-x}}{2}$$

$$\Rightarrow \frac{d}{dx}(\cosh x) = \sinh x \quad \text{Proved}$$

$$\text{(iii)} \quad \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\text{Proof: } \frac{d}{dx}(\tanh x) = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)$$

$$= \frac{(e^x + e^{-x}) \frac{d}{dx}(e^x - e^{-x}) - (e^x - e^{-x}) \frac{d}{dx}(e^x + e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$\begin{aligned}\Rightarrow \frac{d}{dx} \tanh x &= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x + e^{-x})^2} \\&= \frac{4}{(e^x + e^{-x})^2} \\&= \left(\frac{2}{(e^x + e^{-x})} \right)^2\end{aligned}$$

$$\Rightarrow \frac{d}{dx} (\tanh x) = \operatorname{sech}^2 x \quad \text{Proved}$$

$$(iv) \quad \frac{d}{dx} (\coth x) = -\operatorname{cosech}^2 x$$

$$\begin{aligned}\text{Proof: } \frac{d}{dx} (\coth x) &= \frac{d}{dx} \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} \right) \\&= \frac{(e^x - e^{-x}) \frac{d}{dx} (e^x + e^{-x}) - (e^x + e^{-x}) \frac{d}{dx} (e^x - e^{-x})}{(e^x - e^{-x})^2} \\&= \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2} \\&= \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2}\end{aligned}$$

$$\begin{aligned}\Rightarrow \frac{d}{dx} (\coth x) &= \frac{e^{2x} + e^{-2x} - 2 - e^{2x} - e^{-2x} - 2}{(e^x - e^{-x})^2} \\&= -\frac{4}{(e^x - e^{-x})^2} \\&= -\left(\frac{2}{(e^x - e^{-x})} \right)^2\end{aligned}$$

$$\Rightarrow \frac{d}{dx} (\coth x) = -\operatorname{cosech}^2 x \quad \text{Proved}$$

$$(v) \quad \frac{d}{dx} (\sec hx) = -\sec hx \tanh x$$

$$\begin{aligned}
 \text{Proof: } \frac{d}{dx}(\sec hx) &= \frac{d}{dx} \left(\frac{2}{e^x + e^{-x}} \right) \\
 \Rightarrow \frac{d}{dx}(\sec hx) &= 2 \frac{d}{dx} (e^x + e^{-x})^{-1} \\
 &= 2(-1)(e^x + e^{-x})^{-2} \frac{d}{dx} (e^x + e^{-x}) \\
 &= -\frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
 \Rightarrow \frac{d}{dx}(\sec hx) &= -\left(\frac{2}{e^x + e^{-x}} \right) \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right) \\
 &= -\sec hx \tanh x \quad \text{Proved}
 \end{aligned}$$

$$(vi) \quad \frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

$$\begin{aligned}
 \text{Proof: } \frac{d}{dx}(\operatorname{cosech} x) &= \frac{d}{dx} \left(\frac{2}{e^x - e^{-x}} \right) \\
 \Rightarrow \frac{d}{dx}(\operatorname{cosech} x) &= 2 \frac{d}{dx} (e^x - e^{-x})^{-1} \\
 &= 2(-1)(e^x - e^{-x})^{-2} \frac{d}{dx} (e^x - e^{-x}) \\
 &= -\frac{2(e^x + e^{-x})}{(e^x - e^{-x})^2} \\
 \Rightarrow \frac{d}{dx}(\operatorname{cosech} x) &= -\left(\frac{2}{e^x - e^{-x}} \right) \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} \right) \\
 &= -\operatorname{cosech} x \coth x \quad \text{Proved}
 \end{aligned}$$

Example 21: Differentiate the following functions .

$$(a) \quad y = \cosh(2x^2 - 1) \quad (b) \quad y = \sec h \left(\frac{1-x}{1+x} \right)$$

Solution : (a) $y = \cosh(2x^2 - 1)$, differentiate w . r . t . to x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \cosh(2x^2 - 1) \\ \Rightarrow \frac{dy}{dx} &= \sinh(2x^2 - 1) \frac{d}{dx} (2x^2 - 1) \\ \Rightarrow \frac{dy}{dx} &= (\sinh(2x^2 - 1))(4x) \\ \Rightarrow \frac{dy}{dx} &= 4x \sinh(2x^2 - 1) \quad \text{Ans.}\end{aligned}$$

$$(b) \quad y = \sec h \left(\frac{1-x}{1+x} \right) \dots\dots(i)$$

$$\text{Let } u = \frac{1-x}{1+x} \dots\dots(ii) \quad \text{put (ii) in (i)}$$

$$y = \sec hu \dots\dots(iii)$$

$$\text{As by chain rule } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \dots\dots(A)$$

Differentiate (ii) w . r . t . to x ,

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx} \left(\frac{1-x}{1+x} \right) \\ \Rightarrow \frac{du}{dx} &= \frac{(1+x) \frac{d}{dx} (1-x) - (1-x) \frac{d}{dx} (1+x)}{(1+x)^2} \\ \Rightarrow \frac{du}{dx} &= \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2} \\ \Rightarrow \frac{du}{dx} &= \frac{-1-x-1+x}{(1+x)^2} \\ \Rightarrow \frac{du}{dx} &= -\frac{2}{(1+x)^2} \dots\dots(iv)\end{aligned}$$

$$\text{Now } \frac{dy}{du} = \frac{d}{du} (\sec hu)$$

$$\Rightarrow \frac{dy}{du} = -\sec hu \tanh u \quad \text{put } u = \frac{1-x}{1+x}$$

$$\Rightarrow \frac{dy}{du} = -\sec h\left(\frac{1-x}{1+x}\right) \tanh\left(\frac{1-x}{1+x}\right) \dots\dots\dots(v)$$

Put (iv) and (v) in (A) .

$$\frac{dy}{dx} = -\sec h\left(\frac{1-x}{1+x}\right) \tanh\left(\frac{1-x}{1+x}\right) \times \frac{-2}{(1+x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{(1+x)^2} \sec h\left(\frac{1-x}{1+x}\right) \tanh\left(\frac{1-x}{1+x}\right) \quad \text{Ans.}$$

- Note:** (i) $\cosh^2 x - \sinh^2 x = 1$
(ii) $\cosh^2 x + \sinh^2 x = \cosh 2x$
(iii) $\sec^2 x = 1 + \tan^2 x$
(iv) $\operatorname{cosec}^2 x = \coth^2 x + 1$
(v) $\sinh 2x = 2 \sinh x \cosh x$

Proof: (i) Take L.H.S

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4} \\ &= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4} \\ &= \frac{4}{4} = 1 \quad \text{Proved} \end{aligned}$$

Proof: (ii) Take L.H.S

$$\begin{aligned} \cosh^2 x + \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + e^{-2x} + 2}{4} + \frac{e^{2x} + e^{-2x} - 2}{4} \end{aligned}$$

$$\begin{aligned}
 &= \frac{e^{2x} + e^{-2x} + 2 + e^{2x} + e^{-2x} - 2}{4} \\
 &= \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2} \\
 \cosh^2 x + \sinh^2 x &= \frac{e^{2x} + e^{-2x}}{2} \text{ --- (i)} \\
 \text{But } \cosh 2x &= \frac{e^{2x} + e^{-2x}}{2} \text{ --- (ii)}
 \end{aligned}$$

From (i) and (ii)

$$\Rightarrow \cosh^2 x + \sinh^2 x = \cosh 2x$$

Proof: (iii) Take L.H.S

$$\sec^2 x = \left(\frac{2}{e^x + e^{-x}} \right)^2 \text{ --- (i)}$$

$$\text{Take R.H.S } 1 - \tanh^2 x = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)^2$$

$$\begin{aligned}
 1 - \tanh^2 x &= 1 - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
 &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
 &= \frac{e^{2x} + e^{-2x} + 2 - (e^{2x} + e^{-2x} - 2)}{(e^x + e^{-x})^2} \\
 &= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x + e^{-x})^2} \\
 &= \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^x + e^{-x}} \right)^2 \text{ --- (ii)}
 \end{aligned}$$

$$\text{As from (i) and (ii)} \Rightarrow \sec^2 x = 1 - \tanh^2 x$$

Proof: (iv) Take L.H.S

$$\operatorname{cosech}^2 x = \left(\frac{2}{e^x - e^{-x}} \right)^2 \text{ --- (i)}$$

Take R.H.S

$$\coth^2 x - 1 = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} \right)^2 - 1$$

$$\begin{aligned} \Rightarrow \coth^2 x - 1 &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x - e^{-x})^2} \\ &= \frac{e^{2x} + e^{-2x} + 2 - (e^{2x} + e^{-2x} - 2)}{(e^x - e^{-x})^2} \\ &= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x - e^{-x})^2} \end{aligned}$$

$$\Rightarrow \coth^2 x - 1 = \frac{4}{(e^x - e^{-x})^2} = \left(\frac{2}{e^x - e^{-x}} \right)^2 \quad \text{---(ii)}$$

From (i) and (ii) $\operatorname{cosech}^2 x = \coth^2 x - 1$ Proved

Derivatives of Inverse Hyperbolic Functions:

(i) $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

Proof: Let $y = \sinh^{-1} x \Rightarrow \sinh y = x$

$$\Rightarrow \frac{d}{dx}(\sinh y) = \frac{d}{dx}(x)$$

$$\Rightarrow \cosh y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}}$$

We take positive sign put $\sinh y = x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \text{ Ans.}$$

(ii) $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, \quad |x| > 1$

Proof: Let $y = \cosh^{-1} x \Rightarrow \cosh y = x$

$$\Rightarrow \frac{d}{dx}(\cosh y) = \frac{d}{dx}(x)$$

$$\Rightarrow \sinh y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y - 1}} \quad \text{Put } \cosh y = x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}} \quad \text{Proved}$$

$$(iii) \quad \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$$

$$\text{Proof: Let } y = \tanh^{-1} x \Rightarrow \tanh y = x$$

$$\Rightarrow \frac{d}{dx}(\tanh y) = \frac{d}{dx}(x)$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1 - \tanh^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1 - x^2}$$

$$(iv) \quad \frac{d}{dx}(\coth^{-1} x) = -\frac{1}{x^2 - 1}$$

$$\text{Proof: Let } \coth^{-1} x = y$$

$$\Rightarrow x = \coth y \quad \text{Differentiate w.r.t "x"}$$

$$\Rightarrow \frac{d}{dx}(\coth y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\operatorname{cosech}^2 y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{Cosech}^2 y} = -\frac{1}{\operatorname{Coth}^2 y - 1}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x^2 - 1} \text{ Proved}$$

$$(v) \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$$

Proof: Let $y = \operatorname{sech}^{-1} x$

$$\Rightarrow \operatorname{sech} y = x$$

$$\Rightarrow \frac{d}{dx}(\operatorname{sech} y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\operatorname{sech} y \tanh y \frac{dy}{dx} = 1$$

$$\text{As } \operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{sech} y \tanh y}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{sech} y \sqrt{1 - \operatorname{sech}^2 y}} \quad \text{Put } \operatorname{sech} y = x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}} \quad \text{Proved}$$

$$(vi) \frac{d}{dx}(\operatorname{cosech}^{-1} x) = \frac{-1}{x\sqrt{1+x^2}}$$

Proof: Let $y = \operatorname{cosech}^{-1} x \Rightarrow \operatorname{cosech} y = x$

$$\Rightarrow \frac{d}{dx}(\operatorname{cosech} y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\operatorname{cosech} y \coth y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{cosech} y \coth y}$$

$$\text{As } \operatorname{cosech}^2 x = \coth^2 x - 1$$

$$\operatorname{cosech}^2 x + 1 = \coth^2 x$$

$$\Rightarrow \frac{dy}{dx} = \pm \frac{-1}{\operatorname{cosech} y \sqrt{\operatorname{cosech}^2 y + 1}}$$

Put $\operatorname{cosech} y = x$

$$\Rightarrow \frac{dy}{dx} = \pm \frac{-1}{x \sqrt{x^2 + 1}}$$

Here, the sign of the radical is the same as that of $\operatorname{coth} y$ which. Here $\operatorname{coth} y$ is positive or negative according as x is positive or negative. Then

$$\frac{dy}{dx} = \frac{-1}{x \sqrt{x^2 + 1}} \text{ if } x > 0 \text{ and } \frac{dy}{dx} = \frac{-1}{-x \sqrt{x^2 + 1}} \text{ if } x < 0$$

$$\text{Thus } \frac{d}{dx}(\operatorname{cosech} x) = -\frac{1}{|x| \sqrt{x^2 + 1}} \text{ for all values of } x.$$

$$\text{Note : (a) } \frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx}$$

$$(b) \frac{d}{dx} \cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx} \quad (c) \frac{d}{dx} \tanh^{-1} u = \frac{1}{1 - u^2} \frac{du}{dx}$$

$$(d) \frac{d}{dx} (\operatorname{coth}^{-1} u) = -\frac{1}{u^2 - 1} \frac{du}{dx} \quad (e) \frac{d}{dx} \operatorname{sech}^{-1} u = -\frac{1}{u \sqrt{1 - u^2}} \frac{du}{dx}$$

$$(f) \frac{d}{dx} (\operatorname{cosech}^{-1} u) = -\frac{1}{|u| \sqrt{u^2 + 1}} \frac{du}{dx}$$

Example 22: Differentiate the following functions .

$$(a) y = \sinh^{-1}(x^3) \quad (b) y = \frac{\sinh^{-1} x}{\cosh^{-1} x}$$

$$\text{Solution : (a) } y = \sinh^{-1}(x^3) \dots(i)$$

$$\text{Let } u = x^3 \dots(ii) \text{ put (ii) in (i)}$$

$$y = \sinh^{-1} u \dots(iii)$$

As by chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (A)

Differentiate (ii) w . r . t . to x ,

$$\frac{du}{dx} = \frac{d}{dx}(x^3) = 3x^2 \text{(iv)}$$

$$\frac{dy}{du} = \frac{d}{du} \sinh^{-1} u = \frac{1}{\sqrt{1+u^2}} \quad \text{put } u = x^3$$

$$\Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1+(x^3)^2}} = \frac{1}{\sqrt{1+x^6}} \text{(v)}$$

Put (iv) and (v) in (A)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^6}} \times 3x^2 = \frac{3x^2}{\sqrt{1+x^6}} \text{ Ans.}$$

(b) $y = \frac{\sinh^{-1} x}{\cosh^{-1} x}$ differentiate w . r . t . to x ,

$$\frac{d}{dx}(y) = \frac{d}{dx} \left(\frac{\sinh^{-1} x}{\cosh^{-1} x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cosh^{-1} x \frac{d}{dx}(\sinh^{-1} x) - (\sinh^{-1} x) \frac{d}{dx}(\cosh^{-1} x)}{(\cosh^{-1} x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(\cosh^{-1} x)^2} \left(\cosh^{-1} x \cdot \frac{1}{\sqrt{1+x^2}} - \sinh^{-1} x \cdot \frac{1}{\sqrt{x^2-1}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(\cosh^{-1} x)^2} \left(\frac{(\sqrt{x^2-1}) \cosh^{-1} x - (\sqrt{1+x^2}) \sinh^{-1} x}{(\sqrt{1+x^2})(\sqrt{x^2-1})} \right) \text{ Ans.}$$

Example. 23: Use MAPLE command "diff" to

Differentiate. (a) $f(x) = x^5 + 7x + 2$ w.r.t. to x

(b) $f(x) = \frac{(x^4 + 2x + 16)}{(x^3 + 3x - 2)}$ w.r.t. to x

(c) $f(x) = x^3 + (\sin x)^2 + \arccos x$ w.r.t. to x

(d) $f(x) = x^2 \cosh x + \arcsin hx$ w.r.t. to x

Solution: (a) **Note:** Open MAPLE programme .Then open new document in MAPLE . Use keyboard keys and write

$\text{diff}(x^5 + 7x + 2)$ you get $\text{diff}(x^5 + 7x + 2, x);$

Command:

$\text{diff}(x^5 + 7x + 2, x);$ Then press enter key you get result.

$$5x^4 + 7$$

(b) Open MAPLE programme .Then open new document in MAPLE . Use keyboard keys and write

$\text{diff}((x^4 + 2 \cdot x + 16)/(x^3 + 3 \cdot x - 2), x)$ you get

$$\text{diff}\left(\frac{(x^4 + 2 \cdot x + 16)}{(x^3 + 3 \cdot x - 2)}, x\right)$$

Command:

$\text{diff}\left(\frac{(x^4 + 2 \cdot x + 16)}{(x^3 + 3 \cdot x - 2)}, x\right)$ Press enter key you get result.

$$\frac{4x^3 + 2}{x^3 + 3x - 2} - \frac{(x^4 + 2x + 16)(3x^2 + 3)}{(x^3 + 3x - 2)^2}$$

(c)

Open MAPLE programme .Then open new document in MAPLE . Use keyboard keys and write

$\text{diff}(x^3 + \sin(x)^2 + \arccos(x), x)$ you get

$\text{diff}(x^3 + \sin(x)^2 + \arccos(x), x);$

Command :

$\text{diff}(x^3 + \sin(x)^2 + \arccos(x), x);$ Press enter key you get result.

$$3x^2 + 2\sin(x)\cos(x) - \frac{1}{\sqrt{-x^2 + 1}}$$

(d)

Open MAPLE programme .Then open new document in MAPLE . Use keyboard keys and write

$\text{diff}(x^2 \cdot \cosh(x) + \arcsinh(x), x)$ you get

$\text{diff}(x^2 \cdot \cosh(x) + \arcsinh(x), x)$

Command:

$$\text{diff}(x^2 \cdot \cosh(x) + \operatorname{arcsinh}(x), x)$$

press enter key get result .

$$2x \cosh(x) + x^2 \sinh(x) + \frac{1}{\sqrt{x^2 + 1}}$$

Exercise 3.5:

Q.1: Use the rule of first principle to find the derivative of the following functions .

(a) $f(x) = e^{2x}$

Solution: As $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(A)$

Let $y = f(x) = e^{2x} \dots\dots(i)$

$f(x + \Delta x) = e^{2x+2\Delta x} \dots\dots(ii)$

Put (i) and (ii) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{e^{2x+2\Delta x} - e^{2x}}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{e^{2x} e^{2\Delta x} - e^{2x}}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{e^{2x} (e^{2\Delta x} - 1)}{\Delta x}$$

$$= 2e^{2x} \lim_{\Delta x \rightarrow 0} \frac{e^{2\Delta x} - 1}{2\Delta x}$$

As $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$

$$\Rightarrow f'(x) = 2e^{2x} \ln e \quad \text{but } \ln e = 1$$

$$\Rightarrow f'(x) = 2e^{2x}$$

(b) $f(x) = \frac{1}{3} e^{3x}$

Solution: As $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(A)$

Let $y = f(x) = \frac{1}{3} e^{3x} \dots\dots(i)$

$f(x + \Delta x) = \frac{1}{3} e^{3x+3\Delta x} \dots\dots(ii)$

Put (i) and (ii) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{3}e^{3x+3\Delta x} - \frac{1}{3}e^{3x}}{\Delta x}$$

$$\Rightarrow f'(x) = \frac{1}{3} \lim_{\Delta x \rightarrow 0} \frac{e^{3x}e^{3\Delta x} - e^{3x}}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{e^{3x}(e^{3\Delta x} - 1)}{3\Delta x}$$

$$= e^{3x} \lim_{\Delta x \rightarrow 0} \frac{e^{3\Delta x} - 1}{3\Delta x}$$

$$\text{As } \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$$

$$\Rightarrow f'(x) = e^{3x} \ln e \quad \text{but } \ln e = 1$$

$$\Rightarrow f'(x) = e^{3x}$$

c) $f(x) = \frac{5}{8}e^{x^2} + 1$

Solution: As $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(A)$

Let $y = f(x) = \frac{5}{8}e^{x^2} + 1 \dots\dots(i)$

$$f(x + \Delta x) = \frac{5}{8}e^{(x+\Delta x)^2} + 1 = \frac{5}{8}e^{x^2+2x\Delta x+(\Delta x)^2} + 1 \dots\dots(ii)$$

Put (i) and (ii) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\frac{5}{8}e^{x^2+2x\Delta x+(\Delta x)^2} + 1 - \left(\frac{5}{8}e^{x^2} + 1\right)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\frac{5}{8}e^{x^2} \cdot e^{2x\Delta x+(\Delta x)^2} + 1 - \frac{5}{8}e^{x^2} - 1}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\frac{5}{8}e^{x^2}e^{2x\Delta x+(\Delta x)^2} - \frac{5}{8}e^{x^2}}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\frac{5}{8} e^{x^2} (e^{\Delta x(2x+\Delta x)} - 1)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) \cdot \frac{\frac{5}{8} e^{x^2} (e^{\Delta x(2x+\Delta x)} - 1)}{\Delta x((2x + \Delta x))}$$

Multiplied and divided by $2x + \Delta x$

$$f'(x) = \frac{5}{8} e^{x^2} \lim_{\Delta x \rightarrow 0} (2x + \Delta x) \cdot \lim_{\Delta x \rightarrow 0} \frac{(e^{\Delta x(2x+\Delta x)} - 1)}{\Delta x((2x + \Delta x))}$$

$$\Rightarrow f'(x) = \frac{5}{8} e^{x^2} (2x + 0) \ln e$$

$$\Rightarrow f'(x) = \frac{5}{8} e^{x^2} (2x)$$

$$\Rightarrow f'(x) = \frac{5}{4} x e^{x^2}$$

$$(d) f(x) = 2^x$$

Solution: As $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(A)$

$$f(x) = 2^x \dots\dots(i)$$

$$f(x + \Delta x) = 2^{x+\Delta x} \dots\dots(ii)$$

Put (i) and (ii) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{2^{x+\Delta x} - 2^x}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{2^x 2^{\Delta x} - 2^x}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{2^x (2^{\Delta x} - 1)}{\Delta x}$$

$$= 2^x \lim_{\Delta x \rightarrow 0} \frac{2^{\Delta x} - 1}{\Delta x}$$

$$\Rightarrow f'(x) = 2^x \ln 2$$

$$\text{As } \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$$

$$\text{but } \ln e = 1$$

$$\text{As } \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$$

(e) $f(x) = 4^{x+4}$

Solution: As $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(A)$

$f(x) = 4^{x+4} \dots\dots(i)$

$f(x + \Delta x) = 4^{x+\Delta x+4} \dots\dots(ii)$

Put (i) and (ii) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{4^{x+\Delta x+4} - 4^{x+4}}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{4^{x+4} 4^{\Delta x} - 4^{x+4}}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{4^{x+4} (4^{\Delta x} - 1)}{\Delta x}$$

$$= 4^{x+4} \lim_{\Delta x \rightarrow 0} \frac{4^{\Delta x} - 1}{\Delta x}$$

As $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$

$\Rightarrow f'(x) = 4^{x+4} \ln 4$

(f) $f(x) = \log(x+1)$

Solution: As $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(A)$

$f(x) = \log(x+1) \dots\dots(i)$

$f(x + \Delta x) = \log(x + \Delta x + 1) \dots\dots(ii)$

Put (i) and (ii) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\log(x + \Delta x + 1) - \log(x + 1)}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \log\left(\frac{x + \Delta x + 1}{x + 1}\right)$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \log\left(\frac{x + 1 + \Delta x}{x + 1}\right)$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \log\left(\frac{x + 1}{x + 1} + \frac{\Delta x}{x + 1}\right)$$

R . H . S . multiplied and divided by $x + 1$.

$$\Rightarrow f'(x) = \frac{1}{x+1} \lim_{\Delta x \rightarrow 0} \frac{x+1}{\Delta x} \log \left(1 + \frac{\Delta x}{x+1} \right)$$

$$= \frac{1}{x+1} \lim_{\Delta x \rightarrow 0} \log \left(1 + \frac{\Delta x}{x+1} \right)^{\frac{x+1}{\Delta x}}$$

Use log rule

$$\log m^n = n \log m$$

$$\Rightarrow f'(x) = \frac{1}{x+1} \log e$$

$$\text{As } \lim_{\Delta x \rightarrow 0} \left(1 + \frac{\Delta x}{x} \right)^{\frac{x}{\Delta x}} = e$$

$$(g) f(x) = \log_a(x^2)$$

$$f(x + \Delta x) = \log_a(x + \Delta x)^2 = \log_a(x^2 + 2x\Delta x + (\Delta x)^2) \dots (ii)$$

Put (i) and (ii) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\log_a(x^2 + 2x\Delta x + (\Delta x)^2) - \log_a x^2}{\Delta x}$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \log_a \left(\frac{x^2 + 2x\Delta x + (\Delta x)^2}{x^2} \right)$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \log_a \left(\frac{x^2}{x^2} + \frac{2x\Delta x + (\Delta x)^2}{x^2} \right)$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x(2x + \Delta x)}{x^2} \right)$$

R. H. S. multiplied and divided by $\frac{x^2}{2x + \Delta x}$.

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\frac{x^2}{2x + \Delta x}} \left(\frac{x^2}{2x + \Delta x} \right) \cdot \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x(2x + \Delta x)}{x^2} \right)$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{2x + \Delta x}{x^2} \cdot \lim_{\Delta x \rightarrow 0} \frac{x^2}{\Delta x(2x + \Delta x)} \log_a \left(1 + \frac{\Delta x(2x + \Delta x)}{x^2} \right)$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{2x + \Delta x}{x^2} \cdot \lim_{\Delta x \rightarrow 0} \log_a \left(1 + \frac{\Delta x(2x + \Delta x)}{x^2} \right)^{\frac{x^2}{\Delta x(2x + \Delta x)}}$$

$$\Rightarrow f'(x) = \frac{2x + 0}{x^2} \log_a \lim_{\Delta x \rightarrow 0} \left(1 + \frac{\Delta x(2x + \Delta x)}{x^2} \right)^{\frac{x^2}{\Delta x(2x + \Delta x)}}$$

$$\Rightarrow f'(x) = \frac{2x}{x^2} \log_a e$$

As $\lim_{\Delta x \rightarrow 0} \left(1 + \frac{\Delta x}{x} \right)^{\frac{\Delta x}{x}} = e$

$$\Rightarrow \Rightarrow f'(x) = \frac{2}{x} \log_a e \quad \text{As } \log_a e = \frac{1}{\log_e a}$$

$$\Rightarrow \Rightarrow f'(x) = \frac{2}{x \log_e a} = \frac{2}{x \ln a} \text{ Ans.}$$

(h) $f(x) = \sinh 2x$

Solution: As $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots(A)$

$$f(x) = \sinh 2x = \frac{e^{2x} - e^{-2x}}{2} \dots\dots(i)$$

$$f(x + \Delta x) = \sinh(2x + \Delta x) = \frac{e^{2(x + \Delta x)} - e^{-2(x + \Delta x)}}{2} \dots\dots(ii)$$

Put (i) and (ii) in (A)

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{e^{2(x + \Delta x)} - e^{-2(x + \Delta x)}}{2} - \frac{e^{2x} - e^{-2x}}{2} \right]$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{e^{2x + 2\Delta x} - e^{-2x - 2\Delta x} - e^{2x} + e^{-2x}}{2} \right]$$

$$\Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(\frac{e^{2x + 2\Delta x} - e^{2x}}{2} - \frac{e^{-2x - 2\Delta x} - e^{-2x}}{2} \right)$$

$$\begin{aligned}\Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(\frac{e^{2x} \cdot e^{2\Delta x} - e^{2x}}{2} - \frac{e^{-2x} e^{-2\Delta x} - e^{-2x}}{2} \right) \\ \Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[e^{2x} \left(\frac{e^{2\Delta x} - 1}{2} \right) - e^{-2x} \left(\frac{e^{-2\Delta x} - 1}{2} \right) \right] \\ \Rightarrow f'(x) &= e^{2x} \lim_{\Delta x \rightarrow 0} \left(\frac{e^{2\Delta x} - 1}{2\Delta x} \right) + e^{-2x} \lim_{\Delta x \rightarrow 0} \left(\frac{e^{-2\Delta x} - 1}{-2\Delta x} \right) \dots (B)\end{aligned}$$

We have a formula $\lim_{y \rightarrow 0} \frac{e^y - 1}{y} = 1$. Use this formula

Then $\lim_{\Delta x \rightarrow 0} \left(\frac{e^{2\Delta x} - 1}{2\Delta x} \right) = 1$,,,,,(iii)

And $\lim_{\Delta x \rightarrow 0} \left(\frac{e^{-2\Delta x} - 1}{-2\Delta x} \right) = 1$,,,,,(iv)

let $y = 2\Delta x$

if $y \rightarrow 0$, then $\Delta x \rightarrow 0$

Put (iii) and (iv) in (B)

$$\Rightarrow f'(x) = e^{2x} \cdot 1 + e^{-2x} \cdot 1$$

$$\Rightarrow f'(x) = 2 \left(\frac{e^{2x} + e^{-2x}}{2} \right) = 2 \cosh 2x \quad \text{Ans.}$$

Q.2: Find $f'(x)$ if $f(x)$ is :

(a) $y = 11^{(3-4x^5)}$

Solution: $y = 11^{(3-4x^5)}$, differentiate w . r . t . to x

$$\frac{dy}{dx} = \frac{d}{dx} 11^{(3-4x^5)}$$

$$\Rightarrow \frac{dy}{dx} = 11^{(3-4x^5)} \ln 11 \frac{d}{dx} (3-4x^5) \quad \left\{ \text{use } \frac{d}{dx} a^u = a^u \ln a \frac{du}{dx} \right\}$$

$$\Rightarrow \frac{dy}{dx} = 11^{(3-4x^5)} \ln 11 \cdot (0 - 20x^4)$$

$$\Rightarrow \frac{dy}{dx} = -20x^4 11^{(3-4x^5)} \ln 11 \quad \text{Ans.}$$

(b) $y = e^{\sqrt{x}-5}$

Solution : $y = e^{\sqrt{x}-5}$, differentiate w.r.t. to x ,

$$\frac{dy}{dx} = \frac{d}{dx} e^{\sqrt{x}-5} \quad \text{use } \frac{d}{dx} e^u = e^u \frac{d}{dx}(u)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sqrt{x}-5} \frac{d}{dx} \left(x^{\frac{1}{2}} - 5 \right)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sqrt{x}-5} \left(\frac{1}{2} \right) x^{\frac{1}{2}-1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} e^{\sqrt{x}-5} \left(x^{-\frac{1}{2}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}-5}}{2\sqrt{x}} \text{ Ans.}$$

(c) $y = x^5 e^{\frac{1}{x}}$

Solution : $y = x^5 e^{\frac{1}{x}}$, differentiate w.r.t. to x ,

$$\frac{dy}{dx} = \frac{d}{dx} \left(x^5 e^{\frac{1}{x}} \right) = x^5 \cdot \frac{d}{dx} \left(e^{\frac{1}{x}} \right) + \left(\frac{d}{dx} x^5 \right) e^{\frac{1}{x}}$$

$$= x^5 \cdot e^{\frac{1}{x}} \frac{d}{dx} \left(\frac{1}{x} \right) + 5x^4 e^{\frac{1}{x}}$$

$$= x^5 e^{\frac{1}{x}} \left(\frac{d}{dx} x^{-1} \right) + 5x^4 e^{\frac{1}{x}}$$

$$\Rightarrow \frac{dy}{dx} = x^5 e^{\frac{1}{x}} (-x^{-2}) + 5x^4 e^{\frac{1}{x}}$$

$$= -\frac{x^5 e^{\frac{1}{x}}}{x^2} + 5x^4 e^{\frac{1}{x}} = -x^3 e^{\frac{1}{x}} + 5x^4 e^{\frac{1}{x}} \text{ Ans.}$$

(d) $y = \frac{e^{2x}}{e^{-2x} + 1}$

Solution: $y = \frac{e^{2x}}{e^{-2x} + 1}$, differentiate w . r . t . to x ,

$$\begin{aligned}\frac{d}{dx}(y) &= \frac{d}{dx} \left(\frac{e^{2x}}{e^{-2x} + 1} \right) \\ \Rightarrow \frac{dy}{dx} &= \frac{(e^{-2x} + 1) \frac{d}{dx}(e^{2x}) - e^{2x} \frac{d}{dx}(e^{-2x} + 1)}{(e^{-2x} + 1)^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{(e^{-2x} + 1)2(e^{2x}) - e^{2x}(-2e^{-2x})}{(e^{-2x} + 1)^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{(e^{-2x} + 1)2(e^{2x})}{(e^{-2x} + 1)^2} - \frac{e^{2x}(-2e^{-2x})}{(e^{-2x} + 1)^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{2e^{2x}}{(e^{-2x} + 1)} + \frac{2e^{-2x}e^{2x}}{(e^{-2x} + 1)^2} \quad \text{Ans.}\end{aligned}$$

(e) $y = \ln(e^{mx} - e^{-mx})$

Solution: $y = \ln(e^{mx} - e^{-mx})$, differentiate w . r . t . to x ,

$$\begin{aligned}\frac{d}{dx} y &= \frac{d}{dx} \ln(e^{mx} - e^{-mx}) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{e^{mx} - e^{-mx}} \cdot \frac{d}{dx}(e^{mx} - e^{-mx}) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{e^{mx} - e^{-mx}} \left(\frac{d}{dx} e^{mx} - \frac{d}{dx} e^{-mx} \right) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{e^{mx} - e^{-mx}} \left(e^{mx} \cdot \frac{d}{dx}(mx) - e^{-mx} \frac{d}{dx}(-mx) \right) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{e^{mx} - e^{-mx}} (me^{mx} - e^{-mx}(-m)) \\ &= \frac{me^{mx} + me^{-mx}}{e^{mx} - e^{-mx}} \quad \text{Ans.}\end{aligned}$$

(f) $y = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

Solution: $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, differentiate w. r. t. to x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^x + e^{-x}) \frac{d}{dx} (e^x - e^{-x}) - (e^x - e^{-x}) \frac{d}{dx} (e^x + e^{-x})}{(e^x + e^{-x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^x + e^{-x})(ae^x + ae^{-x}) - (e^x - e^{-x})(ae^x - ae^{-x})}{(e^x + e^{-x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^x + e^{-x})(ae^x + ae^{-x}) - (e^x - e^{-x})(ae^x - ae^{-x})}{(e^x + e^{-x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^x + e^{-x})(ae^x + ae^{-x})}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(ae^x - ae^{-x})}{(e^x + e^{-x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(ae^x + ae^{-x})}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(ae^x - ae^{-x})}{(e^x + e^{-x})^2} \quad \text{Ans.}$$

Q.3: Find $\frac{dy}{dx}$ if:

(a) $y = x^3 \ln \sqrt{x}$

Solution: $y = x^3 \ln \sqrt{x} \Rightarrow y = x^3 \ln x^{\frac{1}{2}}$

$$\Rightarrow y = \frac{1}{2} x^3 \ln x, \text{ differentiate w.r.t. to } x.$$

$$\Rightarrow \frac{d}{dx} y = \frac{d}{dx} \left(\frac{1}{2} x^3 \ln x \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(x^3 \cdot \frac{d}{dx} \ln x + \left(\frac{d}{dx} x^3 \right) \ln x \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(x^3 \cdot \frac{1}{x} + (3x^2) \ln x \right) = \frac{x^3}{2x} + \frac{3x^2 \ln x}{2}$$

$$= \frac{x^2}{2} + \frac{3x^2 \ln x}{2} \text{ Ans.}$$

(b) $y = x^2 \sqrt{\ln x}$

Solution: $y = x^2 \sqrt{\ln x} \Rightarrow y = x^2 (\ln x)^{\frac{1}{2}}$

Differentiate w.r.t. to x .

$$\Rightarrow \frac{d}{dx} y = \frac{d}{dx} \left(x^2 (\ln x)^{\frac{1}{2}} \right)$$

$$\Rightarrow \frac{dy}{dx} = x^2 \frac{d}{dx} (\ln x)^{\frac{1}{2}} + \left(\frac{d}{dx} x^2 \right) (\ln x)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = x^2 \frac{1}{2} (\ln x)^{\frac{1}{2}-1} \frac{d}{dx} (\ln x) + (2x) (\ln x)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = x^2 \frac{1}{2} (\ln x)^{-\frac{1}{2}} \cdot \frac{1}{x} + (2x) (\ln x)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{2(\ln x)^{\frac{1}{2}}} + 2x \sqrt{\ln x} = \frac{x}{2\sqrt{\ln x}} + 2x \sqrt{\ln x} \text{ Ans.}$$

(c) $y = \ln \sqrt{\frac{x^2+1}{x^2-1}}$

Solution: $y = \ln \sqrt{\frac{x^2+1}{x^2-1}}$

$$\Rightarrow y = \ln \left(\frac{x^2+1}{x^2-1} \right)^{\frac{1}{2}}$$

$$\Rightarrow y = \frac{1}{2} \ln \left(\frac{x^2+1}{x^2-1} \right) = \frac{1}{2} \ln(x^2+1) - \frac{1}{2} \ln(x^2-1)$$

Differentiate w.r.t. to x ,

$$\frac{dy}{dx} = \frac{d}{dx} \left[\frac{1}{2} \ln(x^2+1) - \frac{1}{2} \ln(x^2-1) \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \frac{d}{dx} \ln(x^2+1) - \frac{1}{2} \frac{d}{dx} \ln(x^2-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(x^2+1)} \frac{d}{dx}(x^2+1) - \frac{1}{2(x^2-1)} \frac{d}{dx}(x^2-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{2(x^2+1)} - \frac{2x}{2(x^2-1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{x^2+1} - \frac{x}{x^2-1}$$

(d) $y = \ln(x - \sqrt{x^2+1})$

Solution: $y = \ln(x - \sqrt{x^2+1})$. Differentiate w.r.t. to x ,

$$\frac{dy}{dx} = \frac{d}{dx}(\ln(x - \sqrt{x^2+1}))$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x - \sqrt{x^2+1}} \cdot \frac{d}{dx}(x - \sqrt{x^2+1})$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x - \sqrt{x^2+1}} \left(\frac{d}{dx}(x) - \frac{d}{dx}(x^2+1)^{\frac{1}{2}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x - \sqrt{x^2+1}} \left(1 - \frac{1}{2}(x^2+1)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(x^2+1) \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x - \sqrt{x^2+1}} \left(1 - \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot (2x) \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x - \sqrt{x^2+1}} \left(1 - \frac{2x}{2(x^2+1)^{\frac{1}{2}}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x - \sqrt{x^2+1}} \left(1 - \frac{x}{\sqrt{x^2+1}} \right) \quad \text{Ans.}$$

(e) $y = e^{-2x} \cos 2x$

Solution: $y = e^{-2x} \cos 2x$. Differentiate w.r. t. to x .

$$\frac{dy}{dx} = \frac{d}{dx}(e^{-2x} \cos 2x)$$

$$\begin{aligned}
\Rightarrow \frac{dy}{dx} &= e^{-2x} \frac{d}{dx} (\cos 2x) + (\cos 2x) \frac{d}{dx} (e^{-2x}) \\
\Rightarrow \frac{dy}{dx} &= e^{-2x} (-\sin 2x) \frac{d}{dx} (2x) + (\cos 2x) (e^{-2x}) \frac{d}{dx} (-2x) \\
\Rightarrow \frac{dy}{dx} &= e^{-2x} (-\sin 2x) (2) + (\cos 2x) (e^{-2x}) (-2) \\
\Rightarrow \frac{dy}{dx} &= -2e^{-2x} \sin 2x - 2(\cos 2x) (e^{-2x}) \text{ Ans.}
\end{aligned}$$

(f) $y = x^2 e^{\sin x}$

Solution : $y = x^2 e^{\sin x}$. Differentiate w . r . t . to x ,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} (x^2 e^{\sin x}) \\
\Rightarrow \frac{dy}{dx} &= x^2 \frac{d}{dx} (e^{\sin x}) + e^{\sin x} \frac{d}{dx} (x^2) \\
\Rightarrow \frac{dy}{dx} &= x^2 \cdot e^{\sin x} \frac{d}{dx} (\sin x) + e^{\sin x} (2x) \\
\Rightarrow \frac{dy}{dx} &= x^2 \cdot e^{\sin x} \cdot \cos x + 2x e^{\sin x} \text{ Ans.}
\end{aligned}$$

Q.4: Differentiate the following functions .

(a) $y = \log(x+2)^3$

Solutiion : $y = \log(x+2)^3 \Rightarrow y = 3 \log(x+2)$

. Differentiate (ii) w . r . t . to x ,

$$\begin{aligned}
\frac{dy}{dx} &= 3 \frac{d}{dx} (\log(x+2)) \\
&= 3 \cdot \frac{1}{x+2} \cdot \frac{d}{dx} (x+2) = \frac{3}{x+2} \text{ Ans.}
\end{aligned}$$

(b) $y = \cosh(3x)$

Solutiion : $y = \cosh(3x)$. Differentiate (ii) w . r . t . to x ,

$$\frac{dy}{dx} = \frac{d}{dx}(\cosh 3x)$$

$$= \sinh(3x) \frac{d}{dx}(3x) = 3 \sinh 3x \text{ Ans.}$$

(c) $y = \sinh^{-1}(\sin x)$

Solution: $y = \sinh^{-1}(\sin x)$.Differentiate (iii) w . r . t. to x ,

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1}(\cos x))$$

$$= \frac{1}{\sqrt{1 + \cos^2 x}} \cdot \frac{d}{dx}(\cos x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 + \cos^2 x}} \cdot (-\sin x) = -\frac{\sin x}{\sqrt{1 + \cos^2 x}} \text{ Ans.}$$

(d) $y = \tanh^{-1}\left(\frac{x}{2}\right)$

Solution: $y = \tanh^{-1}\left(\frac{x}{2}\right)$. Differentiate (iii) w . r . t. to x .

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tanh^{-1}\left(\frac{x}{2}\right)\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1 - \left(\frac{x}{2}\right)^2} \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{\left(1 - \frac{x^2}{4}\right)} \cdot \frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\left(1 - \frac{x^2}{4}\right)} \text{ Ans.}$$

(e) $y = \ln(\coth x)$

Solution : $y = \ln(\coth x)$.Differentiate (ii) w . r . t. to x .

$$\frac{dy}{dx} = \frac{d}{dx} \ln(\coth x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\coth x} \cdot \frac{d}{dx}(\coth x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\coth x} \cdot (-\operatorname{cosech}^2 x)$$

$$\text{As } 1 - \coth^2 x = -\operatorname{cosech}^2 x$$

$$= \frac{1 - \coth^2 x}{\coth x} \quad \text{Ans}$$

$$(e) \quad y = x \cosh^{-1} x - \sqrt{x^2 - 1}$$

Solution : $y = x \cosh^{-1} x - \sqrt{x^2 - 1}$, differentiate w.r.t. to x .

$$\frac{dy}{dx} = \frac{d}{dx} (x \cosh^{-1} x - \sqrt{x^2 - 1})$$

$$= \frac{d}{dx} (x \cosh^{-1} x) - \frac{d}{dx} (x^2 - 1)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = x \frac{d}{dx} \cosh^{-1} x + \cosh^{-1} x \frac{d}{dx} (x) - \frac{1}{2} (x^2 - 1)^{\frac{1}{2} - 1} \frac{d}{dx} (x^2 - 1)$$

$$\Rightarrow \frac{dy}{dx} = x \cdot \frac{1}{\sqrt{x^2 - 1}} + \cosh^{-1} x - \frac{1}{2} (x^2 - 1)^{-\frac{1}{2}} (2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 1}} + \cosh^{-1} x - \frac{x}{\sqrt{x^2 - 1}}$$

$$\Rightarrow \frac{dy}{dx} = \cosh^{-1} x \quad \text{Ans.}$$

Q.5: A research group (used hospital records) developed the approximate mathematical model related to systolic blood pressure and age is : $p(x) = 40 + 25 \ln(x + 1)$

$$, \quad 0 \leq x \leq 65$$

Where $p(x)$ is the pressure measured in millimeters of mercury and x is age in years . what is the rate of change pressure at the end of 10 years ? at the end of 30 years ? at the end of 60 years ? .

Solution: $p(x) = 40 + 25 \ln(x + 1) \dots\dots(i) , \quad 0 \leq x \leq 65$

Differentiate (i) w.r.t. to x ,

$$\begin{aligned}\frac{d}{dx} p(x) &= \frac{d}{dx} [40 + 25 \ln(x+1)] \\ \Rightarrow p'(x) &= \frac{d}{dx} (40) + 25 \frac{d}{dx} \ln(x+1) \\ \Rightarrow p'(x) &= 0 + \frac{25}{x+1} \frac{d}{dx} (x+1) \\ \Rightarrow p'(x) &= \frac{25}{x+1} \dots\dots\dots(ii)\end{aligned}$$

Eq(ii) shows that the rate of change of pressure in x years.

The rate of change of pressure at the end of 10 years:

Put $x = 10$ in (ii)

$$p'(10) = \frac{25}{10+1} = \frac{25}{11} = 2.27 \text{ mm of mercury / year}$$

The rate of change of pressure at the end of 30 years:

Put $x = 30$ in (ii)

$$p'(30) = \frac{25}{30+1} = \frac{25}{31} = 0.81 \text{ mm of mercury / year}$$

The rate of change of pressure at the end of 60 years:

Put $x = 60$ in (ii)

$$p'(60) = \frac{25}{60+1} = \frac{25}{61} = 0.41 \text{ mm of mercury / year}$$

Q.6: A single cholera bacterium divides every 0.5 hour to produce two complete cholera bacteria. If we start with a colony of 5,000 bacteria, then after t hours there will be a $A(t) = (5000)2^{2t}$. Find $A'(t)$, $A'(1)$ and $A'(5)$.

Interpret the results.

Solution : $A(t) = (5000)2^{2t} \dots\dots(i)$

Differentiate w . r . t . to t .

$$\frac{d}{dt} A(t) = \frac{d}{dt} (5000 \cdot 2^{2t})$$

$$\Rightarrow A'(t) = (5000)2^{2t} \cdot \ln 2 \frac{d}{dt} (2t)$$

$$\Rightarrow A'(t) = (5000)(2) \ln 2 \cdot 2^{2t}$$

$$\Rightarrow A'(t) = 10000 \ln 2 \cdot 2^{2t} \dots\dots\dots(i)$$

Put $t = 1$ in (i)

$$A'(1) = (10000) \ln 2 \cdot 2^2$$

$$= (4)(10000) \ln 2 = 27726 \text{ bacteria / hour}$$

Put $t = 5$ in (i)

$$A'(5) = (10000) \ln 2 \cdot 2^{2(5)}$$

$$= (1024)(10000) \ln 2 = 7097827 \text{ bacteria / hour}$$

Q.7: Use MAPLE command "diff" to differentiate all the Functions given in Q. 3 and Q. 4.

Q.3: (a) $y = x^3 \ln \sqrt{x}$ (b) $y = x^2 \sqrt{\ln x}$ (c) $y = \ln \sqrt{\frac{x^2 + 1}{x^2 - 1}}$

(d) $y = \ln(x - \sqrt{x^2 + 1})$ (e) $y = e^{-2x} \cos 2x$ (f) $y = x^2 e^{\sin x}$

Q.4: (a) $y = \sinh(2x)$ (b) $y = \cosh(3x)$ (c) $y = \sinh^{-1}(\sin x)$

(d) $y = \tanh^{-1}\left(\frac{x}{2}\right)$ (e) $y = \ln(\coth x)$

(f) $y = x \cosh^{-1} x - \sqrt{x^2 - 1}$

Solution: Q. 3: (a) $y = x^3 \ln \sqrt{x}$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write $\text{diff}(x^3 \cdot \ln(\text{sqrt}(x)), x)$: then press enter key and get result .

Command:

> $\text{diff}(x^3 \cdot \ln(\text{sqrt}(x)), x)$;

$$\frac{3}{2} x^2 \ln(x) + \frac{1}{2} x^2$$

(b) $y = x^2 \sqrt{\ln x}$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write $\text{diff}(x^2 \cdot \text{sqrt}(\ln(x)), x)$; then press enter key and get result .

Command :

> diff($x^2 \cdot \text{sqrt}(\ln(x))$), x);

$$2x\sqrt{\ln(x)} + \frac{1}{2} \frac{x}{\sqrt{\ln(x)}}$$

(c) $y = \ln \sqrt{\frac{x^2+1}{x^2-1}}$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write

diff(ln(sqrt(x^2+1/x^2-1))), x);

then press enter key and get result .

Command :

> diff(ln(sqrt($\frac{x^2+1}{x^2-1}$))), x);

$$\frac{1}{2} \frac{\left(\frac{2x}{x^2-1} - \frac{2(x^2+1)x}{(x^2-1)^2} \right) (x^2-1)}{x^2+1}$$

(d) $y = \ln(x - \sqrt{x^2+1})$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write

diff(ln(x - sqrt(x^2+1))), x);

then press enter key and get result .

Command :

> diff(ln(x - sqrt(x^2+1))), x);

$$\frac{1 - \frac{x}{\sqrt{x^2+1}}}{x - \sqrt{x^2+1}}$$

(e) $y = e^{-2x} \cos 2x$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write

$\text{diff}(e^{-2 \cdot x} \cdot \cos(2 \cdot x), x);$.then press enter key and get result

Command :

> $\text{diff}(e^{-2 \cdot x} \cdot \cos(2 \cdot x), x);$
 $-2 e^{-2 x} \ln(e) \cos(2 x) - 2 e^{-2 x} \sin(2 x)$

(f) $y = x^2 e^{\sin x}$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write $\text{diff}(x^2 \cdot e^{\sin(x)}, x);$.then press enter key and get result .

Command :

> $\text{diff}(x^2 \cdot e^{\sin(x)}, x);$
 $2 x e^{\sin(x)} + x^2 e^{\sin(x)} \cos(x) \ln(e)$

Q.4: (a) $y = \sinh(2x)$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write $\text{diff}(\sinh(2 \cdot x), x);$. then press enter key and get result .

Command :

> $\text{diff}(\sinh(2 \cdot x), x);$
 $2 \cosh(2 x)$

(b) $y = \cosh(3x)$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write $\text{diff}(\sinh(2 \cdot x), x);$. then press enter key and get result .

Command :

> $\text{diff}(\cosh(3 \cdot x), x);$
 $3 \sinh(3 x)$

(c) $y = \sinh^{-1}(\sin x)$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write $\text{diff}(\sinh^{-1}(\sin(x)), x);$

then press enter key and get result .

Command :

> diff(sinh⁻¹(sin(x)), x);

$$\frac{\cos(x)}{\sqrt{1 + \sin(x)^2}}$$

(d) $y = \tanh^{-1}\left(\frac{x}{2}\right)$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write diff(tanh⁻¹(x/2), x);

then press enter key and get result .

Command :

> diff(tanh⁻¹($\frac{x}{2}$), x);

$$\frac{1}{2\left(1 - \frac{1}{4}x^2\right)}$$

(e) $y = \ln(\coth x)$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write

then press enter key and get result . diff(ln(coth(x)), x);

Command :

> diff(ln(coth(x)), x);

$$\frac{1 - \coth(x)^2}{\coth(x)}$$

(f) $y = x \cosh^{-1} x - \sqrt{x^2 - 1}$. Context Menu.

Open the MAPLE programme then open new document and use key board keys and write

then press enter key and get result .

diff(x * cosh⁻¹(x) - sqrt(x² - 1), x);

Command :

> diff(x*cosh⁻¹(x) - sqrt(x² - 1), x);

$$\operatorname{arccosh}(x) + \frac{x}{\sqrt{x-1}\sqrt{x+1}} - \frac{x}{\sqrt{x^2-1}}$$

Review Exercise 3:

Q.1: Chose the correct option:

(i) If $f(x) = 2t^2 + 3t + 2$, then $f(-3)$ is

(a) 9 (b) 11 (c) 21 (d) 29

(ii) The average rate of change for $f(x) = x^2 - 6x + 5$ is
if x increase at $x \in [1, 3]$

(a) 1 (b) -1 (c) 2 (d) -2

(iii) If $y = f(x)$ then $f'(x) = \dots\dots\dots$

(a) $\lim_{\Delta x \rightarrow 0} \frac{f(x) - f(\Delta x)}{\Delta x}$

(b) $\Delta \lim_{x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

(c) $\lim_{\Delta x \rightarrow 0} \frac{f(x - \Delta x) + f(x)}{\Delta x}$

(d) $\lim_{\Delta x \rightarrow 0} \frac{f(x - \Delta x) - f(x)}{f(x)}$

(iv) If $f(t) = 2t^3 - 3t^2 + 4$, then $f'(3)$ is

(a) 54 (b) 45 (c) 36 (d) 27

(v) If $f(x) = \frac{h(x)}{g(x)}$ then $f'(x) = \dots\dots\dots$

(a) $\frac{h'(x) - g'(x)}{[g(x)]^2}$ (b) $\frac{f(x)g'(x) - g'(x)f''(x)}{[g(x)]^2}$

(c) $\frac{g(x)h'(x) - g'(x)h(x)}{[g(x)]^2}$ (d) $\frac{f'(x)g'(x) - g''(x)f''(x)}{[g(x)]^2}$

(vi) If $h(t) = \sqrt{t^3}$ then $h'(t)$ is

(a) $\frac{3}{2}\sqrt{t}$ (b) $\frac{2}{3}\sqrt{t}$ (c) $\frac{3t}{2\sqrt{t^2}}$ (d) $\frac{3t^2}{2\sqrt{t^3}}$

(vii) If $f(x) = x \tan x$ then $f'(x) = \dots\dots\dots$

(a) $\tan x + x \sec^2 x$ (b) $x \tan x - x \sec^2 x$
(c) $\sec^2 x$ (d) $\tan x + x(1 + \tan^2 x)$

(viii) $\frac{d}{dy}(\cos^{-1} y) = \dots\dots$

(a) $\frac{1}{\sqrt{y^2-1}}$ (b) $\frac{x}{\sqrt{x^2-y^2}}$ (c) $\frac{-1}{\sqrt{1-y^2}}$ (d) $\frac{x}{\sqrt{1+y^2}}$

(ix) $-\frac{1}{|x|(\sqrt{x^2-1})} = \frac{d}{dx}(\dots\dots)$

(a) $\sec^{-1} x$ (b) $\cos^{-1} x$ (c) $\operatorname{cosec}^{-1} x$ (d) $\tan^{-1} x$

(x) If $f(x) = \ln(x)$, then $f'(x) = \dots\dots\dots$

(a) $\frac{1}{\ln x}$ (b) $\frac{2}{x}$ (c) $\frac{1}{x^2}$ (d) $\frac{1}{x}$

Answers: (i) b (ii) d (iii) b (iv) c (v) c
(vi) a, d (vii) a, d (viii) c (ix) c (x) d