Unit -12: Introduction to Numerical

Methods :

Bisection Method : If $f(x)$ is a continuous function on close interval $[a, b]$ and $f(a)$ *and* $f(b)$ have opposite in signs then by bisection method we find the root of $f(x)$. i.e. we find the value of x for which $f(x)=0$.

Procedure : (i) The given interval $[a_o, b_o]$ is the initial interval at which the function $f(x)$ must be opposite in signs. Then the actual root r is lie in $[a_o, b_o]$.

Find $f(a_o)$ *and* $f(b_o)$.

Find the mid point $c_o = \dfrac{a_o + b_o}{2}$.

Then we have two intervals $[a_o \;\; c_o]$ *and* $[c_o \;\; b_o]$

Take this interval in which $f(x)$ have in opposite in signs

Let $f(x)$ be in opposite in signs in $[a_o, c_o]$. Then the actual root of $f(x)$ is r lies in $[a_o, c_o]$.

Find $f(c_o)$ .This is the first iterate c_o.

Let $[a_o \;, c_o] = [a_1 \,, b_1]$

(ii) Similarly take mid point $c_1 = \dfrac{a_1 + b_1}{2}$

Then we have two intervals $[a_1 \;\; c_1]$ *and* $[c_1 \;\; b_1]$

Take this interval in which $f(x)$ have in opposite in signs .

Let $f(x)$ be in opposite in signs in $[a_1 , c_1]$. Then the actual root of $f(x)$ is r lies in $[a_1 , c_1]$.

Find $f(c_1)$ .This is the 2nd iterate c_1.

Let $[a_1 , c_1] = [a_2 , b_2]$

Similarly proceed in the same way we get nth interval $[a_n , b_n]$ whose mid point $c_n = \dfrac{a_n + b_n}{2}$ and we find the desired approximation to the actual root r of $f(x)$.

Example 1: Perform two iterations of the bisection method to approximate the actual root r of the non linear equation $f(x) = \sin x - e^{-x}$ (x is in radians) in the interval $[0.5 , 0.7]$.

Solution: $f(x) = \sin x - e^{-x}$ .....(i)

Let $[a_o, b_o] = [0.5 , 0.7]$, then $a_o = 0.5$ and $b_o = 0.7$

Now put values $x = a_o = 0.5$, $x = b_o = 0.7$ in(i)

$f(a_o) = f(0.5) = \sin(0.5) - e^{-0.5} = 0{,}479 - 0.607 = -0.127 < 0$

$f(b_o) = f(0.7) = \sin(0.7) - e^{-0.7} = 0.644 - 0.496 = 0.147 > 0$

Since $f(a_o)$ and $f(b_o)$ have opposite signs , then the actual root r lie in the interval $[0.5, 0.7]$

(i) Now the mid point of interval $[a_o, b_o]$ is

$$c_o = \frac{a_o + b_o}{2} = \frac{0.5 + 0.7}{2} = \frac{1.2}{2} = 0.6$$

Now put values $x = a_o = 0.5$, $x = c_o = 0.6$, $x = b_o = 0.7$ in(i)

$f(a_o) = f(0.5) = \sin(0.5) - e^{-0.5} = 0{,}479 - 0.607 = -0.127 < 0$

$f(c_o) = f(0.6) = \sin(0.6) - e^{-0.6} = 0{,}565 - 0.549 = 0.016 > 0$

$f(b_o) = f(0.7) = \sin(0.7) - e^{-0.7} = 0.644 - 0.496 = 0.147 > 0$

Since $f(a_o)$ and $f(c_o)$ have opposite signs , then the actual root r lie in the interval $[0.5, 0.6]$. Then the initial interval is rest to obtain the first interval $[a_1,\ b_1]=[0.5,\ ,0.6]$

(ii) Now the mid point of interval $[a_1,\ b_1]$ is

$$c_1=\frac{a_1+b_1}{2}=\frac{0.5+0.6}{2}=\frac{1.1}{2}=0.55$$

Now put values $x=a_1=0.5$,$x=c_1=0.55$, $x=b_1=0.6$, in(i)

$f(a_1)=f(0.5)=\sin(0.5)-e^{-0,5}=0,479-0.607=-0.127<0$

$f(c_1)=f(0.55)=\sin(0.55)-e^{-0,55}=0.523-0.577=-0.054<0$

$f(b_1)=f(0.6)=\sin(0.6)-e^{-0,6}=0,565-0.549=0.016>0$

Since $f(c_1)$ and $f(b_1)$ have opposite signs , then the actual root r lie in the interval $[0.55, 0.6]$.Then the initial interval is rest to obtain the second interval $[a_2,\ b_2]=[0.55,\ ,0.6]$.

After the second iteration of the bisection method the mid Point $c_1=0.55$ is the approximation root to actual root r.

<u>Regula – falsi method</u> : Let $f(x)$ is continuous on $[a\ ,\ b]$ and $f(a)$ *and* $f(b)$ have opposites in signs then $c=a-\dfrac{(a-b)f(a)}{f(a)-f(b)}$. Where $(c,\ 0)$ is the intersection point of the straight line formed by $(a,\ f(a))$ *and* $(b,\ f(b))$.

<u>Note</u>: (i) If function values $f(a)$ and $f(c)$ at $x=a$ and $x=c$ have opposite signs . Then the root lie in the interval $[a,\ c]$ and discard b.

(ii) If function values $f(c)$ and $f(b)$ at $x=c$ and $x=b$ have opposite signs . Then the root lie in the interval $[c,\ b]$ and discard a.

(iii) If $f(c)=0$, then c is our approximate root.

Example 2: Perform two iterations of the regula – false method to approximate the actual root r of the non linear equation $f(x)=\sin x-e^{-x}$ (x is in radians) in the interval $[0.5\ ,0.7]$.

Solution: $f(x)=\sin x-e^{-x}$ .....(i) . Reset the given Interval to obtain the initial interval

$[a_o,\ b_o]=[0.5\ ,0.7]$, then $a_o=0.5$ and $b_o=0.7$

Now put values $x=a_o=0.5$, $x=b_o=0.7$ in(i)

$f(a_o)=f(0,5)=\sin(0.5)-e^{-0,5}=0,479-0.607=-0.127<0$

$f(b_o)=f(0.7)=\sin(0.7)-e^{-0.7}=0.644-0.496=0.148>0$

Since $f(a_o)$ and $f(b_o)$ have opposite signs , then the actual root r lie in the interval $[0.5,\ 0.7]$.

As by regula – false method $c_o=a_o-\dfrac{(a_o-b_o)f(a_o)}{f(a_o)-f(b_o)}$ ...(ii)

Put values in (ii)

=> $c_o=0.5-\dfrac{(0.5-0.7)(-0.127)}{-0.127-0.148}=0.592364$

(i) Now put values $x=a_{o1}=0.5$, $x=c_o=0.592364$, $x=b_o=0.7$ in(i)

$f(a_o)=f(0,5)=\sin(0.5)-e^{-0,5}=0,479-0.607=-0.127<0$

$f(c_o)=f(0.592364)=\sin(0.592364)-e^{-0,592364}=0.81>0$

$f(b_o)=f(0.7)=\sin(0.7)-e^{-0.7}=0.644-0.496=0.147>0$

Since $f(a_o)$ and $f(c_o)$ have opposite signs , then the actual root r lie in the interval $[0.5,\ 0.592364]$.

Then the initial interval is rest to obtain the first interval

$[a_1,\ b_1]=[0.5,\ ,0.592364]$

Then by regula – false method $c_1=a_1-\dfrac{(a_1-b_1)f(a_1)}{f(a_1)-f(b_1)}$

$$=> \quad c_1 = 0.5 - \frac{(0.5-0.592364)(-0.127)}{-0.127-0.81} = 0.755$$

Now put values $x = a_1 = 0.5$, $x = c_1 = 0.755$, $x = b_1 = 0.592364$, in(i)

$$f(a_1) = f(0,5) = \sin(0.5) - e^{-0,5} = 0,479 - 0.607 = -0.127 < 0$$

$$f(c_1) = f(0.755) = \sin(0.755) - e^{-0,755} = 0.215 > 0$$

$$f(b_1) = f(0.592364) = \sin(0.592364) - e^{-0,592364} = 0.81 > 0$$

Since $f(a_1)$ and $f(c_1)$ have opposite signs , then the actual root r lie in the interval $[0.5,\ 0.755]$.

Then $b = 0.592364$ is discard .

Then the initial interval is rest to obtain the second interval $[a_2,\ b_2] = [0.5,\ 0.755]$.

After the second iteration of the regula- falsi method the Point $c_1 = 0.755$ is the approximation root to actual root r.

Newton raphson – method :

If $f(x)$ *and* $f'(x)$ are continues near the actual roots

Then $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ ....(i) for $i = 0, 1, 2, 3$,............

Then eq(i) is called Newton raphson method .

Example .3: Use Newton- Raphson iterative method to approximate the actual root $r = 0.438447$ of the non – linear equation $f(x) = x^2 - 5x + 2$ with initial start $x_o = 0.4$ that must be accurate to six decimal places .

Solution: $f(x) = x^2 - 5x + 2$ .....(i)

Differentiate (i) w . r . t. to x .

$$f'(x) = 2x - 5 \quad ......(ii)$$

As we know by Newton – Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad .....(A) \quad \text{for } i = 0, 1, 2, 3, ...$$

Put $i = 0$ *in (A)* $x_1 = x_o - \dfrac{f(x_o)}{f'(x_o)} = x_o - \dfrac{x_o^2 - 5x_o + 2}{2x_o - 5}$ ....*(a)*

Put $x_o = 0.4$ **in(a)**

$$x_1 = 0.4 - \frac{(0.4)^2 - 5(0.4) + 2}{2(0.4) - 5} = 0.438095$$

Put $i = 0$ **in (A)** $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = x_1 - \dfrac{x_1^2 - 5x_1 + 2}{2x_1 - 5}$ ....**(b)**

Put $x_1 = 0.438095$ **in(b)**

$$\Rightarrow x_2 = 0.438095 - \frac{(0.438095)^2 - 5(0.438095) + 2}{2(0.438095) - 5}$$

$$\Rightarrow x_2 = 0.438095 - \frac{0.00145229}{-4.12381} = 0.438095 + 0.00035217$$

$$\Rightarrow x_2 = 0.438447$$

Which is convergent to the actual root $r = 0.438447$.

Thus the second iterative agrees to six decimal accuracy of actual root.

<u>Example 4</u>: Use MAPLE command "fsolve" to solve

(a) Linear equation $x^2 - 5x + 6$ **with initial start** $x_o = 1.8$

(b) Non linear equation $\sin x - e^{-x}$

<u>Solution</u>:

> $fsolve(x^2 - 5\cdot x + 6);$

$$2.000000000, 3.000000000$$

$polynomial := x^2 - 5\cdot x + 6$

$$x^2 - 5x + 6$$

$fsolve(polynomial)$

$$2.000000000, 3.000000000$$

(b)

$fsolve(\sin(x) - \exp(-x))$

$$\mathbf{0.5885327440}$$

Exercise 12.1:

Q.1. Find an interval $a \le x \le b$ at which $f(a)$ and $f(b)$ have opposite signs for the following functions.

(a) $f(x) = e^x - 2 - x$

Solution: $f(x) = e^x - 2 - x$ ……(i)

Put $x = 0$ **in (i)** => $f(0) = e^0 - 2 - 0 = 1 - 2 = -1$

Put $x = 1$ **in (i)** => $f(1) = e - 2 - 1 = 2.7183 - 3 = -0.2817$

Put $x = 2$ **in (i)** => $f(2) = e^2 - 2 - 1 = 4.3891 > 0$

Thus $f(x)$ has opposite signs in the interval $[1 \quad 2]$.

(b) $f(x) = \cos x + 1 - x$, x **is in radian.**

Solution: $f(x) = \cos x + 1 - x$ ……*(i)*

Put $x = 0$ **in (i)** , $f(0) = \cos 0 + 1 - 0 = 1 + 1 = 2 > 0$

Put $x = \frac{\pi}{2}$ **in (i)** => $f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + 1 - \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = 0 + 1 - 1.57 = -0.57 < 0$$

Thus $f(x)$ **has opposite signs in the interval** $\left[0 \quad \frac{\pi}{2}\right]$.

(c) $f(x)=\ln x-5+x$

Solution: $f(x)=\ln x-5+x$ ......(i)

Put $x=3$ **in(i)** => $f(3)=\ln 3-5+3=1.098-5+3=-0.901$

Put $x=4$ **in(i)** => $f(4)=\ln 4-5+4=1.39-5+4=0.386$

Thus $f(x)$ **has opposite signs in the interval** $[3 \quad 4]$.

(d) $f(x)=x^2-10x+23$

Solution: $f(x)=x^2-10x+23$ ......*(i)*

Put $x=3$ **in (i)** => $f(3)=(3)^2-10(3)+23$

=> $f(3)=9-30+23=2>0$

Put $x=4$ **in (i)** => $f(4)=(4)^2-10(4)+23$

=> $f(4)=16-40+23=-1<0$

Thus $f(x)$ **has opposite signs in the interval** $[3 \quad 4]$.

Q.2: Compute four iterates of the bisection method for the following functions with indicated interval $[a_o\ ,\ b_o]$.

(a) $f(x)=e^x-2-x$, $[1, \quad 1.8]$

Solution: $f(x)=e^x-2-x$ ....(i)

Put $x=1$ **in (i)** => $f(1)=e-2-1=2.7183-3=-0.2817<0$

Put $x=2$ **in (i)** => $f(2)=e^2-2-1=4.3891>0$

Thus $f(x)$ ***has opposite signs in the interval*** $[1 \quad 1.8]$.

Then the root is lying in the interval $[1 \quad 1.8]$.

Now mid point of the interval $[1 \quad 1.8]$ ***is***

$$c_o=\frac{1+1.8}{2}=\frac{2.8}{2}=1.4$$

First iteration :

Now $f(1.4)=e^{1.4}-2-1.4=4.0552-3.4=0.6552>0$

and $f(1) = e - 2 - 1 = 2.7183 - 3 = -0.2817 < 0$

Since $f(x)$ *has opposite signs in the interval* $[1\ ,\ 1.4]$.

Then the root is lying in the interval $[1\ ,\ 1.4]$.

Now mid point of the interval $[1\ ,\ 1.4]$ *is*

$$c_1 = \frac{1+1.4}{2} = \frac{2.4}{2} = 1.2$$

2^{nd} iteration:

$$f(1.2) = e^{1.2} - 2 - 1.2 = 3.3201 - 3.2 = 0.12 > 0$$

and $f(1) = e - 2 - 1 = 2.7183 - 3 = -0.2817 < 0$

Then the root is lying in the interval $[1\ ,\ 1.2]$.

Now mid point of the interval $[1\ ,\ 1.2]$ is

$$c_2 = \frac{1+1.2}{2} = \frac{2.2}{2} = 1.1$$

3^{rd} iteration :

$$f(1.1) = e^{1.1} - 2 - 1.1 = 3.0042 - 3.1 = -0.0958 < 0$$

and $f(1.2) = e^{1.2} - 2 - 1.2 = 3.3201 - 3.2 = 0.12 > 0$

Since $f(x)$ *has opposite signs in the interval* $[1.1\ \ 1.2]$

Then the root is lying in interval $[1.1\ \ 1.2]$.

Now mid point of the interval $[1.1\ \ 1.2]$ is

$$c_3 = \frac{1.1+1.2}{2} = \frac{2.3}{2} = 1.15$$

Fourth iteration:

$$f(1.15) = e^{1.15} - 2 - 1.15 = 3.1582 - 2 - 1.15 = 0.00938 > 0$$

And $f(1.1) = e^{1.1} - 2 - 1.1 = 3.0042 - 3.1 = -0.0958 < 0$

Then the root is lying in the interval $[1.1\ ,\ 1.15]$.

(b) $f(x)=\cos x+1-x$, $[0.8\ ,\ 1.6]$ x is in radian.

Solution: $f(x)=\cos x+1-x$ ......*(i)*

Put $x=0.8$ in (i) ,

$f(0.8)=\cos 0.8+1-0.8=0.6967+1-0.8=0.897 \quad >0$

Put $x=1.6$ in (i) => $f(1.6)=\cos 1.6+1-1.6$

$f(1.6)=-0.0292+1-1.6=-0.63<0$

As $f(x)$ *has opposite signs in the interval* $[0.8\quad 1.6]$.

Then the root is lying in the interval $[0.8\quad 1.6]$.

The mid point of $[0.8\quad 1.6]$ *is* $c_o=\dfrac{0.8+1.6}{2}=\dfrac{2.4}{2}=1.2$

First iteration:

$f(1.2)=\cos(1.2)+1-1.2=0.3623+1-1.2=0.1623>0$

And $f(1.6)=-0.0292+1-1.6=-0.63<0$

As $f(x)$ *has opposite signs in the interval* $[1.2\quad 1.6]$.

Then the root is lying in the interval $[1.2\quad 1.6]$.

The mid point of $[1.2\quad 1.6]$ is $c_o=\dfrac{1.2+1.6}{2}=\dfrac{2.8}{2}$

=> $c_1=1.4$

2^{nd} iteration :

$f(1.4)=\cos 1.4+1-1.4=0.1699-0.4=-0.23<0$

And $f(1.2)=\cos(1.2)+1-1.2=0.3623+1-1.2=0.1623>0$

As $f(x)$ *has opposite signs in the interval* $[1.2 \quad 1.4]$.

Then the root is lying in the interval $[1.2 \quad 1.4]$.

The mid point of $[1.2 \quad 1.4]$ is $c_2 = \dfrac{1.2+1.4}{2} = \dfrac{2.6}{2}$

$\Rightarrow c_2 = 1.3$

3rd iteration :

$f(1.3) = \cos 1.3 + 1 - 1.3 = 0.2674 - 0.3 = -0.032 < 0$

And $f(1.2) = \cos(1.2) + 1 - 1.2 = 0.3623 + 1 - 1.2 = 0.1623 > 0$

As $f(x)$ *has opposite signs in the interval* $[1.2 \quad 1.3]$.

Then the root is lying in the interval $[1.2 \quad 1.3]$.

The mid point of $[1.2 \quad 1.3]$ *is* $c_2 = \dfrac{1.2+1.3}{2} = \dfrac{2.5}{2}$

$\Rightarrow c_3 = 1.25$

4th iteration:

$f(1.25) = \cos 1.25 + 1 - 1.25 = 0.3153 - 0.5 = -0.1846 < 0$

And $f(1.2) = \cos(1.2) + 1 - 1.2 = 0.3623 + 1 - 1.2 = 0.1623 > 0$

As $f(x)$ **has opposite signs in the interval** $[1.2 \quad 1.25]$.

Then the root is lying in the interval $[1.2 \quad 1.25]$.

(c) $f(x) = \ln x - 5 + x \;,\; [3.2 \,,\; 4]$

Solution: $f(x) = \ln x - 5 + x$ ……(i)

Put $x = 3.2$ in(i)

=> $f(3.2) = \ln 3.2 - 5 + 3.2 = 1.163 - 5 + 3.2 = -0.837 < 0$

Put $x = 4$ in(i) => $f(4) = \ln 4 - 5 + 4 = 1.39 - 5 + 4 = 0.386 > 0$

Thus $f(x)$ has opposite signs in the interval $[3.2 \quad 4]$.

Thus the root is lying in the interval $[3.2 \quad 4]$.

The mid point of the interval $[3.2 \quad 4]$ is

$$c_o = \frac{3.2+4}{2} = \frac{7.2}{2} = 3.6$$

First iteration :

Put $x = 3.6$ in (i)

$f(3.6) = \ln(3.6) - 5 + 3.6 = 1.28 - 5 + 3.6 = -0.12 < 0$

Put $x = 4$ in(i)

=> $f(4) = \ln 4 - 5 + 4 = 1.39 - 5 + 4 = 0.386 > 0$

Thus the root is lying in the interval $[3.6 \quad 4]$.

The mid point of the interval $[3.6 \quad 4]$ is

$$c_1 = \frac{3.6+4}{2} = \frac{7.6}{2} = 3.8$$

Second iteration :

Put $x = 3.8$ in(i)

=> $f(3.8) = \ln 3.8 - 5 + 3.8 = 1.335 - 5 + 3.8 = 0.135 > 0$

Put $x = 3.6$ in (i)

$f(3.6) = \ln(3.6) - 5 + 3.6 = 1.28 - 5 + 3.6 = -0.12 < 0$

Thus the root is lying in the interval $[3.6 \quad 4]$

The mid point of the interval $[3.6 \quad 3.8]$ is

$$c_2 = \frac{3.6+3.8}{2} = \frac{7.4}{2} = 3.7$$

Third iteration :

Put $x = 3.7$ in(i)

=> $f(3.7) = \ln 3.7 - 5 + 3.7 = 1.308 - 5 + 3.7 = 0.008 > 0$

Put $x = 3.6$ in (i)

$f(3.6) = \ln(3.6) - 5 + 3.6 = 1.28 - 5 + 3.6 = -0.12 < 0$

Thus the root is lying in the interval $[3.6 \quad 3.7]$.

The mid point of the interval $[3.6 \quad 3.7]$ is

$$c_4 = \frac{3.6+3.7}{2} = \frac{7.3}{2} = 3.65$$

Fourth iteration :

Put $x = 3.6$ **in (i)**

$$f(3.65) = \ln(3.65) - 5 + 3.65 = 1.29 - 5 + 3.65 = -0.05 < 0$$

Put $x = 3.7$ **in(i)**

$$\Rightarrow f(3.7) = \ln 3.7 - 5 + 3.7 = 1.308 - 5 + 3.7 = 0.008 > 0$$

Thus the root is lying in the interval $[3.6 \quad 3.65]$.

(d) $f(x) = x^2 - 10x + 23 \;,\; [3.2\,,\; 4]$

Solution: $f(x) = x^2 - 10x + 23$ ......*(i)*

Put $x = 3.2$ **in (i)**

$$f(3.2) = (3.2)^2 - 10(3.2) + 23 = 10.24 - 32 + 23$$

$$\Rightarrow f(3.2) = 1.24 > 0$$

Put $x = 4$ **in (i)**

$$\Rightarrow f(4) = 16 - 40 + 23 = -1 < 0$$

Then $f(x)$ **has opposite signs in the interval** $[3.2 \quad 4]$.

Thus the root is lying in the interval $[3.2 \quad 4]$.

The mid point of $[3.2 \quad 4]$ ***is*** $c_o = \frac{3.2+4}{2} = \frac{7.2}{2} = 3.6$

First iteration :

$$f(3.6) = (3.6)^2 - 10(3.6) + 23 = 1.296 - 36 + 23$$

$$\Rightarrow f(3.6) = -11.70 < 0 \quad \textbf{and}$$

$$f(3.2) = (3.2)^2 - 10(3.2) + 23 = 10.24 - 32 + 23$$

$$\Rightarrow f(3.2) = 1.24 > 0$$

Then $f(x)$ ***has opposite signs in the interval*** $[3.2 \quad 3.6]$.

Thus the root is lying in the interval $[3.2 \quad 3.6]$.

The mid point of $[3.2 \quad 3.6]$ *is*

$$c_1 = \frac{3.2+3.6}{2} = \frac{6.8}{2} = 3.4$$

2nd iteration:

$$f(3.4) = (3.4)^2 - 10(3.4) + 23 = 11.15 - 34 + 23 = 0.15 > 0$$

And $f(3.6) = (3.6)^2 - 10(3.6) + 23 = 1.296 - 36 + 23$

$\Rightarrow f(3.6) = -11.70 < 0$

Then $f(x)$ *has opposite signs in the interval* $[3.4 \quad 3.6]$.

Thus the root is lying in the interval $[3.4 \quad 3.6]$.

The mid point of $[3.4 \quad 3.6]$ *is*

$$c_2 = \frac{3.4+3.6}{2} = \frac{7}{2} = 3.5$$

3rd iteration:

$$f(3.5) = (3.5)^2 - 10(3.5) + 23 = 12.25 - 35 + 23 = 0.25 > 0$$

And $f(3.6) = (3.6)^2 - 10(3.6) + 23 = 1.296 - 36 + 23$

$\Rightarrow f(3.6) = -11.70 < 0$

Then $f(x)$ *has opposite signs in the interval* $[3.5 \quad 3.6]$.

Thus the root is lying in the interval $[3.5 \quad 3.6]$.

The mid point of $[3.5 \quad 3.6]$ *is*

$$c_3 = \frac{3.5+3.6}{2} = \frac{7.1}{2} = 3.55$$

4th iteration :

$$f(3.55) = (3.55)^2 - 10(3.55) + 23 = 12.6025 - 35.5 + 23$$
$$\Rightarrow f(3.55) = 0.1025 > 0$$

And $f(3.6) = (3.6)^2 - 10(3.6) + 23 = 1.296 - 36 + 23$

$$\Rightarrow f(3.6) = -11.70 < 0$$

Then $f(x)$ has opposite signs in the interval $[3.55 \quad 3.6]$.

Thus the root is lying in the interval $[3.55 \quad 3.6]$.

Q.3: Compute four iterates of the regular -falsi method for the following functions with indicated interval $[a_o\ ,\ b_o]$.

(a) $f(x) = e^x - 2 - x \quad , \quad [-2.4\ ,\ -1.6]$

Solution: $f(x) = e^x - 2 - x$ .....(i)

Here $a_o = -2.4\ ,\ b_o = -1.6$

$$f(a_o) = f(-2.4) = e^{-2.4} - 2 - (-2.4) = 0.09072 - 2 + 2.4$$
$$= 0.490718 > 0$$
$$f(-1.6) = e^{-1.6} - 2 - (-1.6) = 0.2019 - 2 + 1.6 = -0.1981 < 0$$

Then $f(x)$ has opposite signs in the interval $[-2.4 \quad -1.6]$.

Thus the root is lying in the interval $[-2.4 \quad -1.6]$.

Then c_o is lying in the interval $[-2.4 \quad -1.6]$.

First iteration :

As by regula falsi methoa

$$c_o = a_o - \frac{(a_o - b_o)f(a_o)}{f(a_o) - f(b_o)}\ , \quad \text{Put values}$$

$$c_o = -2.4 - \frac{(-2.4 - (-1.6))(0.490718)}{0.490718 - (-0.1981)}$$

$$= -2.4 - \frac{(-2.4 + 1.6)(0.490718)}{0.490718 + 0.1981}$$

$$=> \ c_o = -2.4 + \frac{0.392574}{0.68882} - 02.4 + 0.569922$$

$$=> \ c_o = -1.830078$$

$$f(c_o) = f(-1.830078) = e^{-1.830078} - 2 - (-1.830078)$$

$$f(c_o) = 0.160401 - 2 + 1.830078 = -0.00952 < 0$$

Since the function values $f(a_o)$ *and* $f(c_o)$ have opposite in signs . Then the root c_1 is lying in the interval $[-2.4 , -1.830078]$ and discard $b_o = -1.6$.

The initial interval is reset to obtain the first interval

$$[a_1, \ b_1] = [-2.4, \ -1.830078]$$

2nd iteration :

$$c_1 = a_1 - \frac{(a_1 - b_1) f(a_1)}{f(a_1) - f(b_1)} \quad \text{put values}$$

$$c_1 = -2.4 - \frac{(-2.4 - (-1.830078))(0.490718)}{0.490718 - (-0.00952)}$$

$$= -2.4 - \frac{(-2.4 + 1.830078)(0.490718)}{0.490718 + 0.00952}$$

$$=> \ c_1 = -2.4 + \frac{0.27967}{0.50024} - 2.4 + 0.55907$$

$$=> \ c_1 = -1.8409$$

$$f(c_1) = f(-1.8409) = e^{-1.8409} - 2 - (-1.8409)$$

$$f(c_1) = 0.15867 - 2 + 1.8409 = -0.00043 < 0$$

Put $a_1 = -2.4$ in(i)

$$f(a_1) = f(-2.4) = e^{-2.4} - 2 - (-2.4) = 0.09072 - 2 + 2.4$$

$$= 0.490718 > 0$$

Since the function values $f(a_1)$ *and* $f(c_1)$ have opposite in signs . **Then the root** c_2 lying in the interval $[-2.4 , -1.8409]$

The initial interval is reset to obtain the second interval

$$[a_2, \ b_2] = [-2.4, \ -1.8409]$$

3rd iteration :

$$c_2 = a_2 - \frac{(a_2 - b_2)f(a_2)}{f(a_2) - f(b_2)} \quad \textbf{put values}$$

$$c_2 = -2.4 - \frac{(-2.4-(-1.8409))(0.490718)}{0.490718-(-0.00043)}$$

$$= -2.4 - \frac{(-2.4+1.8409)(0.490718)}{0.490718+0.00043}$$

$$\Rightarrow \ c_2 = -2.4 + \frac{0.2743604}{0.491148} - 2.4 + 0.55861$$

$$\Rightarrow \ c_2 = -1.84138$$

$$f(c_2) = f(-1.84138) = e^{-1.84138} - 2 - (-1.84138)$$

$$f(c_2) = 0.158598 - 2 + 1.84138 = -0.000022 < 0$$

Put $a_2 = -2.4$ **in(i)**

$$f(a_2) = f(-2.4) = e^{-2.4} - 2 - (-2.4) = 0.09072 - 2 + 2.4$$

$$= 0.490718 > 0$$

Since the function values $f(a_2)$ *and* $f(c_2)$ have opposite in **signs. Then the root** c_3 **lying in the interval** $[-2.4, -1.84138]$.

The initial interval is reset to obtain the 3rd interval

$$[a_3, b_3] = [-2.4, -1.830078]$$

4th iteration :

$$c_3 = a_3 - \frac{(a_3 - b_3)f(a_3)}{f(a_3) - f(b_3)} \quad \textbf{put values}$$

$$c_3 = -2.4 - \frac{(-2.4-(-1.84138))(0.490718)}{0.490718-(-0.000022)}$$

$$= -2.4 - \frac{(-2.4+1.84138)(0.490718)}{0.490718+0.000022}$$

$$\Rightarrow \ c_3 = -2.4 + \frac{0.2741248}{0.49074} - 2.4 + 0.558594$$

$$\Rightarrow \ c_3 = -1.841405$$

(b) $f(x)=\cos x+1-x$, $[0.8\ ,\ 1.6]$ x is in radian .

Solution: $f(x)=\cos x+1-x$ ......(i)

Here $a_o=0.8$ *and* $b_o=1.6$ in (i) ,

$f(0.8)=\cos 0.8+1-0.8=0.6967+1-0.8=0.897\ >0$

Put $x=b_o=1.6$ in (i) => $f(1.6)=\cos 1.6+1-1.6$

$f(1.6)=-0.0292+1-1.6=-0.629<0$

As $f(x)$ has opposite signs in the interval $[0.8\quad 1.6]$.

Then the root c is lying in the interval $[0.8\quad 1.6]$.

First iteration:

As by regula falsi method

$$c_o=a_o-\frac{(a_o-b_o)f(a_o)}{f(a_o)-f(b_o)}$$ put values

$$c_o=0.8-\frac{(0.8-1.6)(0.897)}{0.897-(-0.629)}=0.8+\frac{0.7176}{1.526}$$

=> $c_o=0.8+0.47=1.27$

$f(c_o)=f(1.27)=\cos 1.27+1-1.27=0.29628+1-1.27$

=> $f(c_o)=0.026>0$

Since the function values $f(b_o)$ *and* $f(c_o)$ have opposite

in signs .Then the root c_1 is lying in the interval $[1.27\ ,1.6]$

The initial interval is reset to obtain the first interval

$[a_1,\ b_1]=[1.27,\ 1.6]$

2nd iteration:

$$c_1=a_1-\frac{(a_1-b_1)f(a_1)}{f(a_1)-f(b_1)}$$ *put values*

$$c_1=1.27-\frac{(1.27-1.6)(0.026)}{0.026-(-0.629)}=1.27+\frac{0.00858}{0.655}$$

=> $c_1=1.27+0.013=1.283$

$f(c_1) = f(1.283) = \cos 1.283 + 1 - 1.283 = 0.28383 + 1 - 1.283$

$\Rightarrow\ f(c_1) = 0.00083 > 0$

Put $b_1 = 1.6$ **in (i)** $\Rightarrow\ f(1.6) = \cos 1.6 + 1 - 1.6$

$f(1.6) = -0.0292 + 1 - 1.6 = -0.629 < 0$

Since the function values $f(b_o)$ *and* $f(c_1)$ **have opposite in signs.** Then the root c_2 is lying in the interval $[1.283\ \ ,1.6]$

The initial interval is reset to obtain the second interval

$[a_2,\ b_2] = [1.283,\ 1.6]$

3rd iteration:

$$c_2 = a_2 - \frac{(a_2 - b_2)f(a_2)}{f(a_2) - f(b_2)} \quad \textit{put values}$$

$$c_2 = 1.283 - \frac{(1.283 - 1.6)(0.00083)}{0.00083 - (-0.629)} = 1.283 + \frac{0.00026}{0.62983}$$

$\Rightarrow\ c_2 = 1.283 + 0.000412 = 1.2834$

$f(c_2) = f(1.2834) = \cos 1.2834 + 1 - 1.2834 = 0.28345 + 1 - 1.2834$

$\Rightarrow\ f(c_2) = 0.000056 > 0$

Put $b_2 = 1.6$ **in (i)** $\Rightarrow\ f(1.6) = \cos 1.6 + 1 - 1.6$

$f(1.6) = -0.0292 + 1 - 1.6 = -0.629 < 0$

Since the function values $f(b_2)$ *and* $f(c_2)$ **have opposite in signs. Then the root** c_3 **is lying in the interval**

$[1.2834\ \ ,1.6]$

4th iteration:

$$c_{23} = c_2 - \frac{(c_2 - b_o)f(c_2)}{f(c_2) - f(b_o)} \quad \textit{put values}$$

$$c_3 = 1.2834 - \frac{(1.2834-1.6)(0.000056)}{0.000056-(-0.629)} = 1.2834 + \frac{0.0000018}{0.629056}$$

$\Rightarrow\ c_3 = 1.2834 + 0.0000028 = 1.283402$

(c) $f(x) = \ln x - 5 + x\ ,\ \ [-2.4\ ,\ -1.6]$

Solution: $f(x) = \ln x - 5 + x$ ......(i)

Put $a_o = -2.4\ ,\ b_o = -1.6$ **in(i)**

$\Rightarrow f(-2.4) = \ln(-2.4) - 5 - 2.4 = \textit{does not defined.}$

$\Rightarrow f(-1.6) = \ln(-1.6) - 5 - 1.6 = \textit{does not defined.}$

Thus the regula – falsi method is not valid .

(d) $f(x) = x^2 - 10x + 23\ ,\ \ [-2.4\ ,\ -1.6]$

Solution: $f(x) = x^2 - 10x + 23$ .....(i)

Here $a_o = -2.4\ ,\ b_o = -1.6$

$f(-2.4) = (-2.4)^2 - 10(-2.4) + 23 = 5.76 + 24 + 23$

$\Rightarrow f(-2.4) = 52.76 > 0$

$f(-1.6) = (-1.6)^2 - 10(-1.6) + 23 = 2.56 + 16 + 23$

$\Rightarrow f(-1.6) = 41.56 > 0$

Since $f(a_o)$ **and** $f(b_o)$ **have same signs in the interval** $[-2.4\ ,\ -1.6]$.

Thus $f(x)$ **has no root in the interval** $[-2.4\ ,\ -1.6]$

Q.4: What will happen if the bisection method is used with the function $f(x) = \dfrac{1}{x-2}$ **for the following intervals (a)** $[3,\ 7]$ **(b)** $[1,\ 7]$

Solution: $f(x) = \dfrac{1}{x-2}$ .....(i)

(a) $[a_o\ , b_o] = [3,\ 7]$

Put $x = a_o = 3$ in(i) => $f(3) = \frac{1}{3-2} = \frac{1}{1} = 1 > 0$

put $x = b_o = 7$ in(i) => $f(7) = \frac{1}{7-2} = \frac{1}{5} = 0.2 > 0$

Since $f(a_o)$ *and* $f(b_o)$ have same signs then bisection Method is not valid.

(b) $[1, 7]$

Put $x = a_o = 1$ in(i) => $f(1) = \frac{1}{1-2} = \frac{1}{-1} = -1 < 0$

put $x = b_o = 7$ in(i) => $f(7) = \frac{1}{7-2} = \frac{1}{5} = 0.2 > 0$

Since $f(a_o)$ *and* $f(b_o)$ have opposite signs then bisection Method is valid.

Q.5: Find iterate x_3 of Newton – Raphson iterative method for the following functions with initial start x_o :

(a) $f(x) = x^3 - 3$; $x_o = 1$

Solution: $f(x) = x^3 - 3$ ......(i)

=> $f'(x) = 3x^2$ ....(ii)

As we know by Newton – Rapshon method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad ......(A)$$

Put $i = 0$ in (A) => $x_1 = x_o - \frac{f(x_o)}{f'(x_o)}$

=> $x_1 = x_o - \frac{x_o^3 - 3}{3x_o^2}$, Put $x_o = 1$

$$\Rightarrow x_1 = 1 - \frac{(1)^3 - 3}{3(1)^2} = 1 - \frac{1-3}{3} = 1 + \frac{2}{3}$$

$$\Rightarrow x_1 = 1 + 0.667 = 1.667$$

Put $i = 1$ **in (A)** $\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$\Rightarrow x_2 = x_1 - \frac{x_1^3 - 3}{3x_1^2} \quad \text{, Put } x_1 = 1.667$$

$$\Rightarrow x_2 = 1.667 - \frac{(1.667)^3 - 3}{3(1.667)^2} = 1.667 - \frac{4.6324079 - 3}{8.336667}$$

$$\Rightarrow x_2 = 1.667 - 0.195811 = 1.471$$

Put $i = 2$ **in (A)** $\Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$\Rightarrow x_3 = x_2 - \frac{x_2^3 - 3}{3x_2^2} \quad \text{, Put } x_2 = 1.471$$

$$\Rightarrow x_3 = 1.471 - \frac{(1.471)^3 - 3}{3(1.471)^2} = 1.471 - \frac{3.183010 - 3}{6.491523}$$

$$\Rightarrow x_3 = 1.471 - 0.02819 = 1.443$$

(b) $f(x) = \sin x \quad , \quad x_o = 1$

Solution: $f(x) = \sin x$ **……(i)**

$\Rightarrow f'(x) = \cos x$ **……(ii)**

As by Newton –Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{……(A)}$$

Put $i = 0$ **in (A)** $\Rightarrow x_1 = x_o - \frac{f(x_o)}{f'(x_o)}$

$$x_1 = x_o - \frac{\sin x_o}{\cos x_o} \quad \textbf{put} \quad x_o = 1$$

$$\Rightarrow x_1 = 1 - \frac{\sin 1}{\cos 1} = 1 - \frac{0.84147}{0.540302}$$

$$\Rightarrow x_1 = 1 - 1.557 = -0.557$$

Put $i = 1$ **in (A)** $\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$x_2 = x_1 - \frac{\sin x_1}{\cos x_1} \quad \textbf{put} \quad x_o = -0.557$$

$$\Rightarrow x_2 = -0.557 - \frac{\sin(-0.557)}{\cos(-0.557)} = -0.557 - \frac{-0.5286}{0.8488}$$

$$\Rightarrow x_2 = -0.577 + 0.6228 = 0.0458$$

Put $i = 2$ **in (A)** $\Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$x_3 = x_2 - \frac{\sin x_2}{\cos x_2} \quad \textbf{put} \quad x_o = 0.0458$$

$$\Rightarrow x_3 = 0.0458 - \frac{\sin(0.0458)}{\cos(0.0458)} = 0.0458 - \frac{0.0458}{0.9989}$$

$$\Rightarrow x_3 = 0.0458 - 0.0.0458 = 0.0000$$

(c) $f(x) = x^3 + 2x - 1$, $x_o = 0$

Solution: $f(x) = x^3 + 2x - 1$ ....*(i)*

$\Rightarrow f'(x) = 3x^2 + 2$ ....(ii)

As by Newton –Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad ......(A)$$

Put $i = 0$ **in (A)** $\Rightarrow x_1 = x_o - \frac{f(x_o)}{f'(x_o)}$

$$\Rightarrow x_1 = x_o - \frac{x_o^3 + 2x_o - 1}{3x_o^2 + 2} \qquad \textbf{put } x_o = 0$$

$$\Rightarrow x_1 = 0 - \frac{0+0-1}{0+2} = \frac{1}{2} = 0.5$$

Put $i = 1$ **in (A)** $\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$\Rightarrow x_2 = x_1 - \frac{x_1^3 + 2x_1 - 1}{3x_1^2 + 2} \qquad \textbf{put } x_1 = 0.5$$

$$\Rightarrow x_2 = 0.5 - \frac{(0.5)^3 + 2(0.5) - 1}{3(0.5)^2 + 2} = 0.5 - \frac{0.125 + 1 - 1}{0.75 + 2}$$

$$\Rightarrow x_2 = 0.5 - 0.0909 = 0.4041$$

Put $i = 2$ **in (A)** $\Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$\Rightarrow x_3 = x_2 - \frac{x_2^3 + 2x_2 - 1}{3x_2^2 + 2} \qquad \textbf{put } x_2 = 0.4041$$

$$\Rightarrow x_3 = 0.4041 - \frac{(0.4041)^3 + 2(0.4041) - 1}{3(0.4041)^2 + 2}$$

$$= 0.4041 - \frac{0.065988 + 0.8082 - 1}{0.1633 + 2}$$

$$\Rightarrow x_3 = 0.4041 - 0.0581 = 0.3459$$

(d) $f(x) = \sin x \quad , \quad x_o = -2$

<u>Solution</u>: $f(x) = \sin x$ **......(i)**

$\Rightarrow f'(x) = \cos x$ **......(ii)**

As by Newton –Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \textbf{......(A)}$$

Put $i=0$ in (A) $\Rightarrow x_1 = x_o - \dfrac{f(x_o)}{f'(x_o)}$

$$x_1 = x_o - \frac{\sin x_o}{\cos x_o} \quad \text{put } x_o = -2$$

$$\Rightarrow x_1 = 1 - \frac{\sin(-2)}{\cos(-2)} = -2 - \frac{-0.909}{-0.42}$$

$$\Rightarrow x_1 = -2 - 2.16 = -4.16$$

Put $i=1$ in (A) $\Rightarrow x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$$x_2 = x_1 - \frac{\sin x_1}{\cos x_1} \quad \text{put } x_o = -4.16$$

$$\Rightarrow x_2 = -4.16 - \frac{\sin(-4.16)}{\cos(-4.16)} = -4.16 - \frac{-0.8513}{-0.5247}$$

$$\Rightarrow x_2 = -4.16 - 1.6225 = -5.7825$$

Put $i=2$ in (A) $\Rightarrow x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$$x_3 = x_2 - \frac{\sin x_2}{\cos x_2} \quad \text{put } x_o = -5.7825$$

$$\Rightarrow x_3 = -5.7825 - \frac{\sin(-5.7825)}{\cos(-5.7825)} = -5.7825 - \frac{0.48}{0.88}$$

$$\Rightarrow x_3 = -5.7825 - 0.5454 = -6.-3279$$

Q.6: Use Newton – Raphson iterative method to approximate the actual r of the following non –linear equations with indicated interval .

(a) $f(x) = x^3 + 3x - 1 = 0$ in $(0\,,\,1)$

Solution: $f(x) = x^3 + 3x - 1 = 0$ …….(i)

$\Rightarrow f'(x) = 3x^2 + 3$ ……(ii)

As by Newton –Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad ......(A)$$

Put $i=0$ *in (A)* $\Rightarrow x_1 = x_o - \dfrac{f(x_o)}{f'(x_o)}$

$$\Rightarrow x_1 = x_o - \frac{x_o^3 + 3x_o - 1}{3x_o^2 + 3}$$

Here in interval $(0\ ,\ 1)$ **we start any where.**

Let $x_o = 0.5$ **, then put** $x_o = 0.5$

$$x_1 = 0.5 - \frac{(0.5)^3 + 3(0.5) - 1}{3(0.5)^2 + 3} = 0.5 - \frac{0.125 + 1.5 - 1}{0.75 + 3}$$

$$\Rightarrow x_1 = 0.5 - 0.16666 = 0.333$$

Put $i = 1$ **in (A)** $\Rightarrow x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$$\Rightarrow x_2 = x_1 - \frac{x_1^3 + 3x_1 - 1}{3x_1^2 + 3}$$

$$x_2 = 0.333 - \frac{(0.333)^3 + 3(0.333) - 1}{3(0.333)^2 + 3}$$

$$\Rightarrow x_2 = 0.333 - \frac{0.0369 + 0.999 - 1}{0.333}$$

$$\Rightarrow x_2 = 0.333 - 0.109 = 0.225$$

Put $i = 2$ **in (A)** $\Rightarrow x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$$\Rightarrow x_3 = x_2 - \frac{x_2^3 + 3x_2 - 1}{3x_2^2 + 3}$$

$$x_3 = 0.225 - \frac{(0.225)^3 + 3(0.225) - 1}{3(0.225)^2 + 3}$$

$$\Rightarrow x_3 = 0.225 - \frac{0.01139 + 0.675 - 1}{3.1518}$$

$$\Rightarrow x_3 = 0.225 + 0.099 = 0.324$$

(b) $f(x) = x^3 + 2x^2 - x + 1$ *on* $(-3, -2)$

Solution: $f(x) = x^3 + 2x^2 - x + 1$ **……(i)**

$$\Rightarrow f'(x) = 3x^2 + 4x - 1 \quad \ldots\ldots\text{(ii)}$$

Let here we start $x_o = 2.5$

As by Newton –Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \ldots\ldots\textbf{(A)}$$

Put $i = 0$ **in (A)** $\Rightarrow x_1 = x_o - \dfrac{f(x_o)}{f'(x_o)}$

$$\Rightarrow x_1 = x_o - \frac{x_o^3 + 2x_o^2 - x_o + 1}{3x_o^2 + 4x_o - 1} \quad \textbf{put } x_o = -2.5$$

$$\Rightarrow x_1 = -2.5 - \frac{(-2.5)^3 + 2(-2.5)^2 - (-2.5) + 1}{3(-2.5)^2 + 4(-2.5) - 1}$$

$$\Rightarrow x_1 = -2.5 - \frac{-15.625 + 12.5 + 2.5 + 1}{18.75 - 10 - 1}$$

$$\Rightarrow x_1 = -2.5 - 0.04838 = -2.5484$$

Put $i = 1$ **in (A)** $\Rightarrow x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$$\Rightarrow x_2 = x_1 - \frac{x_1^3 + 2x_1^2 - x_1 + 1}{3x_1^2 + 4x_1 - 1} \quad \textbf{put } x_1 = -2.5484$$

$$\Rightarrow x_2 = -2.5484 - \frac{(-2.5484)^3 + 2(-2.5484)^2 - (-2.5484) + 1}{3(-2.5484)^2 + 4(-2.5484) - 1}$$

$$=> x_2 = -2.5484 - \frac{-16.5502 + 6.4943 + 2.5484 + 1}{19.4830 - 10.1936 - 1}$$

$$=> x_2 = -2.5474$$

Put $i = 2$ **in (A)** $\quad => x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$=> x_3 = x_2 - \frac{x_2^3 + 2x_2^2 - x_2 + 1}{3x_2^2 + 4x_2 - 1}$$

put $x_2 = -2.54874$

$$=> x_3 = -2.5474 - \frac{(-2.5474)^3 + 2(-2.5474)^2 - (-2.5474) + 1}{3(-2.5474)^2 + 4(-2.5474) - 1}$$

$$=> x_3 = -2.5474 - \frac{-16.5307 + 12.9785 + 2.5474 + 1}{19.4677 - 10.1896 - 1}$$

$$=> x_3 = -2.5468$$

(c) $\sqrt[3]{x-3} = x + 1$ **on** $[-3, -2]$

<u>Solution</u>: Let $f(x) = (x-3)^{\frac{1}{3}} - x - 1$ **....(i)**

$$f'(x) = \frac{1}{3}(x-3)^{-\frac{2}{3}} - 1 = \frac{1}{3(x-3)^{\frac{2}{3}}} - 1 \text{ .....(ii)}$$

Since here -2 is also include in the interval.

So we start at $x_o = -2$

As by Newton –Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \text{ ......(A)}$$

Put $i = 0$ **in (A)** $\quad => x_1 = x_o - \frac{f(x_o)}{f'(x_o)}$

$$\Rightarrow \quad x_1 = x_o - \frac{(x_o - 3)^{\frac{1}{3}} - x_o - 1}{\frac{1}{3(x_o - 3)^{\frac{2}{3}}} - 1} \qquad \textbf{put} \quad x_o = -2$$

$$\Rightarrow \quad x_1 = -2 - \frac{(-2-3)^{\frac{1}{3}} - (-2) - 1}{\frac{1}{3(-2-3)^{\frac{2}{3}}} - 1} = -2 - \frac{(-5)^{\frac{1}{3}} + 2 - 1}{\frac{1}{3(-5)^{\frac{2}{3}}} - 1}$$

$$\Rightarrow \quad x_1 = -2 - \frac{-1.70997 + 1}{\frac{1}{3(2.9240)} - 1} = -2 + \frac{0.70997}{-0.8860016}$$

$$\Rightarrow \quad x_1 = -2.8013$$

Put $i = 1$ **in (A)** $\quad \Rightarrow \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$\Rightarrow \quad x_2 = x_1 - \frac{(x_1 - 3)^{\frac{1}{3}} - x_1 - 1}{\frac{1}{3(x_1 - 3)^{\frac{2}{3}}} - 1} \qquad \textbf{put} \quad x_1 = -2.8013$$

$$\Rightarrow \quad x_2 = -2.8013 - \frac{(-2.8013 - 3)^{\frac{1}{3}} - (-2.8013) - 1}{\frac{1}{3(-2.8013 - 3)^{\frac{2}{3}}} - 1}$$

$$= -2.8013 - \frac{(-5.8013)^{\frac{1}{3}} + 2.8013 - 1}{\frac{1}{3(-5.8013)^{\frac{2}{3}}} - 1}$$

$$\Rightarrow \quad x_2 = -2.8013 + \frac{0.004464}{-0.896757}$$

$$\Rightarrow \quad x_2 = -2.796322$$

Put $i = 2$ **in (A)** $\quad \Rightarrow \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$\Rightarrow x_3 = x_2 - \frac{(x_2-3)^{\frac{1}{3}} - x_2 - 1}{\frac{1}{3(x_2-3)^{\frac{2}{3}}} - 1} \quad \textbf{put} \quad x_2 = -2.796322$$

$$\Rightarrow x_2 = -2.796322 - \frac{(-2.796322-3)^{\frac{1}{3}} - (-2.796322) - 1}{\frac{1}{3(-2.796322-3)^{\frac{2}{3}}} - 1}$$

$$= -2.796322 - \frac{(-5.796322)^{\frac{1}{3}} + 2.796322 - 1}{\frac{1}{3(-5.796322)^{\frac{2}{3}}} - 1}$$

$$\Rightarrow x_3 = -2.796322 + \frac{0.000000086}{0.8966976}$$

$$\Rightarrow x_3 = -2.796322$$

Q.7:Use MAPLE command "fsolve" to solve $3x^2 + 4x - 3 = 0$
With initial start $x_o = 2$

Solution: Command:

$fsolve(3 \cdot x^2 + 4 \cdot x - 3)$

$-1.868517092, 0.5351837585$

Trapezoidal Rule: If $f(x)$ **is continuous on** $[a\ ,\ b]$**, then the trapezoidal rule is**

$$\int_a^b f(x)dx = \frac{\Delta x}{2}\left[f_o + 2(f_1 + f_2 + \dots\dots + f_{n-1}) + f_n\right] \quad \textbf{is called}$$

Trapezoidal rule .

Here $\Delta x = \frac{b-a}{n} \quad \Rightarrow n\Delta x = b - a$

$\Rightarrow b = a + n\Delta x$

The function values $f_o, f_1, f_2, \ldots\ldots, f_n$ can be obtained
by $f_k = f(a+k\Delta x)$ where $k = 0, 1, 2, \ldots\ldots, n$ because

if $k=0$ then $f_o = f(a) = f(x_o)$

if $k=1$ then $f_1 = f(a+\Delta x) = f(x_1)$

if $k=2$ then $f_2 = f(a+2\Delta x) = f(x_2)$

and so on

Example .5: Approximate the definite integral

$I = \int_{-1}^{2} x^2 dx$ in $n=4$ **subintervals and then compare your approximate answer with the actual value of the definite integral.**

Solution: Here $a=-1$, $b=2$, $n=4$

Then $\Delta x = \dfrac{b-a}{n} = \dfrac{2-(-1)}{4} = \dfrac{2+1}{4}$

$\Rightarrow \Delta x = \dfrac{3}{4}$

As $f_k = f(a+k\Delta x) = f\left(-1+\dfrac{3k}{4}\right)$

$\Rightarrow f_k = f\left(-1+\dfrac{3k}{4}\right) = \left(-1+\dfrac{3k}{4}\right)^2$ ....(A)

where $k = 0, 1, 2, 3, 4$

Put $k=0$ in (A) $\Rightarrow f_o = (-1+0)^2 = 1$

Put $k=1$ in (A) $\Rightarrow f_1 = \left(-1+\dfrac{3}{4}\right)^2 = \left(-\dfrac{1}{4}\right)^2 = \dfrac{1}{16} = 0.0625$

Put $k=2$ in (A)

$$\Rightarrow f_2=\left(-1+\frac{3\times 2}{4}\right)^2=\left(-1+\frac{3}{2}\right)^2=\frac{1}{4}=0.25$$

Put $k=3$ in (A) $\Rightarrow f_3=\left(-1+\frac{9}{4}\right)^2=\frac{25}{16}=1.5625$

Put $k=4$ in (A) $\Rightarrow f_4=\left(-1+\frac{3\times 4}{4}\right)^2=(2)^2=4$

As by Trapezoidal rule

$$\int_{-1}^{2}x^2dx=\frac{\Delta x}{2}\left[f_o+2(f_1+f_2+f_3)+f_4\right] \text{ put values}$$

$$\Rightarrow \int_{-1}^{2}x^2dx=\frac{3}{2\times 4}\left[1+2(0.0625+0.25+1.5625)+4\right]$$

$$=3.28125$$

Now actual value is

$$\int_{-1}^{2}x^2dx=\left.\frac{x^3}{3}\right|_{-1}^{2}=\frac{1}{3}\left((2)^3-(-1)^3\right)$$

$$=\frac{1}{3}[8-(-1)]=\frac{9}{3}=3$$

Thus error = actual value – approximation

$$=3-3.28125=-0.28125$$

<u>Simpson ,s rule</u>: If $f(x)$ is continuous on closed interval $[a\,,\ b]$ then

$$S_{2n}=\int_a^b f(x)dx$$

$$=\frac{\Delta x}{3}\left[f_o+4(f_1+f_3+\ldots\ldots+f_{2n-1})+2(f_2+f_4+\ldots\ldots+f_{2n-2})+f_{2n}\right]$$

Here $\Delta x = \frac{b-a}{2n}$

Exercise 12.2:

Q.1: **Use the trapezoidal rule to approximate the value of each definite integral . Round the answer to the nearest hundredth and compare your result with the exact value of the definite integral . (a)** $I = \int_1^3 x^2 dx$ and $n = 4$

Solution: **Here** $a = 1$, $b = 3$, $n = 4$

Then $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4}$ => $\Delta x = \frac{1}{2}$

As $f_k = f(a + k\Delta x) = f\left(1 + \frac{k}{2}\right)$

=> $f_k = f\left(1+\frac{k}{2}\right) = \left(1+\frac{k}{2}\right)^2$ ....(A) **where** $k = 0, 1, 2, 3, 4$

Put $k = 0$ **in (A)** => $f_o = (1+0)^2 = 1$

Put $k = 1$ **in (A)** => $f_1 = \left(1+\frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2.25$

Put $k = 2$ **in (A)** => $f_2 = \left(1+\frac{2}{2}\right)^2 = (1+1)^2 = 4$

Put $k = 3$ **in (A)** => $f_3 = \left(1+\frac{3}{2}\right)^2 = \frac{25}{4} = 6.25$

Put $k = 4$ **in (A)** => $f_4 = \left(1+\frac{4}{2}\right)^2 = (3)^2 = 9$

As by Trapezoidal rule

$$\int_{-1}^{2} x^2 dx = \frac{\Delta x}{2}\left[f_o + 2(f_1 + f_2 + f_3) + f_4\right] \quad \textbf{put values}$$

$$\Rightarrow \int_{-1}^{2} x^2 dx = \frac{1}{2\times 2}\left[1 + 2(2.25 + 4 + 6.25) + 9\right] = 8.75$$

Now actual value is

$$\int_{1}^{3} x^2 dx = \left.\frac{x^3}{3}\right|_1^3 = \frac{1}{3}\left((3)^3 - (1)^3\right) = \frac{1}{3}[27 - 1] = \frac{26}{3} = 8.67$$

Thus error = actual value − approximation

$$= 8.67 - 8.75 = -0.08$$

(b) $\int_0^1 \left(\frac{x^2}{2} + 1\right) dx \quad , \quad n = 4$

<u>Solution</u>: Here $a = 0$ and $b = 1$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$$

As $f_k = f(a + k\Delta x) = f\left(0 + \frac{k}{4}\right)$

$$\Rightarrow f_k = f\left(\frac{k}{4}\right) = \frac{1}{2}\left(\frac{k}{4}\right)^2 + 1$$

$$\Rightarrow f_k = \frac{k^2}{32} + 1 \quad \text{......(A) where } k = 0, 1, 2, 3, 4$$

Put $k = 0$ in (A) $\Rightarrow f_o = 1$

Put $k = 1$ *in (A)* $\Rightarrow f_1 = \frac{1}{32} + 1 = 1.03125$

Put $k=2$ *in (A)* $\Rightarrow f_2 = \frac{2^2}{32} + 1 = \frac{1}{8} + 1 = 1.125$

Put $k=3$ **in (A)** $\Rightarrow f_3 = \frac{3^2}{32} + 1 = \frac{9}{32} + 1 = 1.28125$

Put $k=4$ **in (A)** $\Rightarrow f_4 = \frac{4^2}{32} + 1 = \frac{1}{2} + 1 = 1.5$

As by Trapezoidal rule

$$\int_0^1 \left(\frac{x^2}{2}+1\right)dx = \frac{\Delta x}{2}\left[f_o + 2(f_1+f_2+f_3) + f_4\right] \quad \textbf{put values}$$

$$= \frac{1}{2\times 4}\left[1 + 2(1.03125 + 1.125 + 1.28125) + 1.5\right]$$

$$= 1.171875$$

Now actual value is $\int_0^1 \left(\frac{x^2}{2}+1\right)dx = \left.\frac{x^3}{6} + x\right|_0^1$

$$= \frac{1}{6}\left[(1)^3 - 0\right] + (1-0)$$

$$= \frac{1}{6} + 1 = 0.16667 + 1 = 1.16667$$

(c) $I = \int_1^3 \frac{dx}{x}$, $n = 6$

Solution: **Here** $a=1$, $b=3$, $n=6$

$\Delta x = \frac{b-a}{n} = \frac{3-1}{6} = \frac{2}{6} = \frac{1}{3}$ **and** $f(x) = \frac{1}{x}$

Now $f_k = f(a + k\Delta x) = f\left(1 + \frac{k}{3}\right)$

$$\Rightarrow \quad f_k = f\left(\frac{3+k}{3}\right)$$

$$f_k = \frac{1}{\frac{3+k}{3}} = \frac{3}{3+k} \quad \text{.........(A) Where } k = 0, 1, 2, 3, 4, 5, 6$$

Put $k = 0$ in (A) $\quad f_o = \frac{3}{3+0} = \frac{3}{3} = 1$

Put $k = 1$ ***in (A)*** $\Rightarrow f_1 = \frac{3}{3+1} = \frac{3}{4} = 0.75$

Put $k = 2$ **in (A)** $\Rightarrow f_2 = \frac{3}{3+2} = \frac{3}{5} = 0.6$

Put $k = 3$ ***in (A)*** $\Rightarrow f_3 = \frac{3}{3+3} = \frac{3}{6} = 0.5$

Put $k = 4$ ***in (A)*** $\Rightarrow f_4 = \frac{3}{3+4} = \frac{3}{7} = 0.42857$

Put $k = 5$ ***in (A)*** $\Rightarrow f_5 = \frac{3}{3+5} = \frac{3}{8} = 0.375$

Put $k = 6$ ***in (A)*** $\Rightarrow f_6 = \frac{3}{3+6} = \frac{3}{9} = 0.333$

As by Trapezoidal rule

$$\int_1^3 \frac{dx}{x} = \frac{\Delta x}{2}\left[f_o + 2(f_1 + f_2 + f_3 + f_4 + f_5) + f_6\right] \quad \textbf{put values}$$

$$\Rightarrow \int_1^3 \frac{dx}{x} = \frac{1}{2\times 3}\left[1 + 2(0.75 + 0.6 + 0.5 + 0.42857 + 0.375) + 0.333\right]$$

$$= \frac{1}{6}(6.6404) = 1.11$$

Now actual value : $\int_1^3 \frac{dx}{x} = \ln x\Big|_1^3$

$= \ln(3) - \ln 1 = 1.099$

(d) $\int_0^2 \sqrt{1+x^2}\,dx$, $n = 6$

Solution: Here $a = 0$, $b = 2$ and $n = 6$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$$

As $f(x) = \sqrt{1+x^2}$

Now $f_k = f(a + k\Delta x) = f\left(0 + \frac{k}{3}\right)$

$$f_k = f\left(\frac{k}{3}\right) = \sqrt{1 + \left(\frac{k}{3}\right)^2}$$

$$\Rightarrow f_k = \sqrt{\frac{9+k^2}{9}} = \frac{\sqrt{9+k^2}}{3} \quad \text{.........(A)}$$

Where $k = 0, 1, 2, 3, 4, 5, 6$

Put $k = 0$ in (A) $f_o = \frac{\sqrt{9+0}}{3} = \frac{3}{3} = 1$

Put $k = 1$ in (A) $\Rightarrow f_1 = \frac{\sqrt{9+1}}{3} = \frac{\sqrt{10}}{3} = 1.054$

Put $k = 2$ in (A) $\Rightarrow f_2 = \frac{\sqrt{9+4}}{3} = \frac{\sqrt{13}}{3} = 1.202$

Put $k = 3$ *in (A)* $\Rightarrow f_3 = \frac{\sqrt{9+9}}{3} = \frac{\sqrt{18}}{3} = 1.414$

$Put \ \ k=4 \ \ in \ (A) \ \ => f_4 = \dfrac{\sqrt{9+16}}{3} = \dfrac{5}{3} = 1.667$

$Put \ \ k=5 \ \ in \ (A) \ \ => f_5 = \dfrac{\sqrt{9+25}}{3} = \dfrac{\sqrt{34}}{3} = 1.944$

$Put \ \ k=6 \ \ in \ (A) \ \ => f_6 = \dfrac{\sqrt{9+36}}{3} = \dfrac{\sqrt{45}}{3} = 2.236$

As by Trapezoidal rule

$$\int_0^2 \sqrt{1+x^2}\,dx = \frac{\Delta x}{2}\left[f_o + 2(f_1+f_2+f_3+f_4+f_5)+f_6\right] \ \ \textbf{put values}$$

$$= \frac{1}{2\times 3}\left[1+2(1.054+1.202+1.414+1.667+1.944)+2.236\right]$$

$$=> \int_0^2 \sqrt{1+x^2}\,dx = \frac{1}{6}(17.798) = 2.97$$

Actual value : As we know the integration of

$$\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|$$

See chapter no 5 integration by parts . Then

$$\int_0^2 \sqrt{1+x^2}\,dx = \left.\frac{x\sqrt{1+x^2}}{2}\right|_0^2 + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)\Big|_0^2$$

$$= \frac{2\sqrt{1+4}}{2} - 0 + \frac{1}{2}\left[\ln\left(2+\sqrt{1+2}\right) - \ln\left(0+\sqrt{1+0}\right)\right]$$

$$= \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right) - \frac{1}{2}\ln(1) = 2.236 + \frac{1}{2}\ln(4.236) + 0$$

$$= 2.236 + 0.7218 = 2.96$$

<u>Q.2</u>: Use Simpson's rule to to approximate the value of each definite integral . Round the answer to the nearest hundredth and compare your result with the exact value of the definite integral . (a) $I = \int_2^4 x^2 dx$, $n = 3$

<u>Solution</u>: Here $a = 2$, $b = 4$, $n = 3$ => $2n = 6$

$$\Delta x = \frac{b-a}{2n} = \frac{4-2}{6} = \frac{1}{3} \quad \textbf{Let} \quad f(x) = x^2$$

Now $f_k = f(a + k\Delta x) = f\left(2 + \frac{k}{3}\right)$

=> $f_k = f\left(2 + \frac{k}{3}\right) = \left(2 + \frac{k}{3}\right)^2$ ....(A)

Where $k = 0, 1, 2, 3, 4, 5, 6$

Put $k = 0$ **in (A)** => $f_o = (2)^2 = 4$

Put $k = 1$ **in (A)** => $f_1 = \left(2 + \frac{1}{3}\right)^2 = \frac{7}{9} = 5.44$

Put $k = 2$ **in (A)** => $f_2 = \left(2 + \frac{2}{3}\right)^2 = \frac{64}{9} = 7.11$

Put $k = 3$ **in (A)** => $f_3 = \left(2 + \frac{3}{3}\right)^2 = (3)^2 = 9$

Put $k=4$ in (A)

$$\Rightarrow f_4=\left(2+\frac{4}{3}\right)^2=\frac{100}{9}=11.11$$

Put $k=5$ in (A) $\Rightarrow f_5=\left(2+\frac{5}{3}\right)^2=\frac{121}{9}=13.44$

Put $k=6$ in (A) $\Rightarrow f_6=\left(2+\frac{6}{3}\right)^2=(4)^2=16$

As by Simpson's rule

$$\int_2^4 x^2dx=\frac{\Delta x}{3}\left[f_o+4\left(f_1+f_3+_5\right)+2\left(f_2+f_4\right)+f_6\right]$$ **put values**

$$\int_2^4 x^2dx=\frac{1}{3\times 3}\left[1+4(5.44+9+13.44)+2(7.11+11.11)+16\right]$$
$$=18.3288$$

Actual value : $\int_2^4 x^2dx=\left.\frac{x^3}{3}\right|_2^4$

$$=\frac{1}{3}\left[(4)^3-(2)^3\right]=\frac{1}{3}(64-8)=18.67$$

(b) $\int_2^3\left(\frac{x^2}{3}-1\right)dx$, $n=4$

Solution: Here $a=2$, $b=3$, $n=4$

$\Delta x=\frac{b-a}{2n}=\frac{3-2}{8}=\frac{1}{8}$ **and** $f(x)=\frac{x^2}{3}-1$

Now $f_k=f(a+k\Delta x)=f\left(2+\frac{k}{8}\right)$

$$\Rightarrow f_k = \frac{1}{3}\left(2+\frac{k}{8}\right)^2 - 1 \qquad \text{.....(A)}$$

Put $k=0$ **in (A)** $\Rightarrow f_o = \frac{1}{3}(2)^2 - 1 = 0.33$

Put $k=1$ **in (A)** $\Rightarrow f_1 = \frac{1}{3}\left(2+\frac{1}{8}\right)^2 - 1 = 0.505$

Put $k=2$ **in (A)** $\Rightarrow f_2 = \frac{1}{3}\left(2+\frac{2}{8}\right)^2 - 1 = 0.6875$

Put $k=3$ **in (A)** $\Rightarrow f_3 = \frac{1}{3}\left(2+\frac{3}{8}\right)^2 - 1 = 0.8802$

Put $k=4$ **in (A)** $\Rightarrow f_4 = \frac{1}{3}\left(2+\frac{4}{8}\right)^2 - 1 = 1.083$

Put $k=5$ **in (A)** $\Rightarrow f_5 = \frac{1}{3}\left(2+\frac{5}{8}\right)^2 - 1 = 1.297$

Put $k=6$ **in (A)** $\Rightarrow f_6 = \frac{1}{3}\left(2+\frac{6}{8}\right)^2 - 1 = 1.5208$

Put $k=7$ **in (A)** $\Rightarrow f_7 = \frac{1}{3}\left(2+\frac{7}{8}\right)^2 - 1 = 1.755$

Put $k=8$ **in (A)** $\Rightarrow f_8 = \frac{1}{3}\left(2+\frac{8}{8}\right)^2 - 1 = 2$

As by Simpson's rule

$$\int_2^4 x^2 dx = \frac{\Delta x}{3}\left[f_o + 4(f_1+f_3+f_5+f_7) + 2(f_2+f_4+f_6) + f_8\right]$$

$$= \frac{1}{3\times 8}\begin{bmatrix} 0.33 + 4(0.505+0.8802+1.297+1.755) \\ +2(0.6875+1.083+1.52)+2 \end{bmatrix}$$

$$= 1.111$$

Actual value : $\int_2^3 \left(\frac{x^2}{3}-1\right)dx = \frac{x^3}{9}-x\Big|_2^3$

$$=\frac{1}{9}\left[(3)^3-(2)^3\right]-(3-2)$$

$$=\frac{1}{9}(27-8)-1=1.11$$

(c) $I=\int_1^3 \frac{dx}{x}$, $n=3$

<u>Solution</u>: Here $a=1$, $b=3$, $2n=6$

$$\Delta x=\frac{b-a}{2n}=\frac{3-1}{6}=\frac{1}{3}$$

$f(x)=\frac{1}{x}$ **then** $f_k=f(a+k\Delta x)=f\left(1+\frac{k}{3}\right)$

$\Rightarrow f_k=\frac{1}{1+\frac{k}{3}}=\frac{3}{3+k}$ **.......(A) Now table is under**

k	0	1	2	3	4	5	6
$f_k=\frac{3}{k+3}$	1	0.75	0.6	0.5	0.43	0.375	0.33

As by Simpson's rule

$\int_1^3 \frac{1}{x}dx=\frac{\Delta x}{3}\left[f_o+4(f_1+f_3+f_5)+2(f_2+f_4)+f_6\right]$ **put values**

$$=\frac{1}{3\times 3}\left[1+4(0.75+0.5+0.375)+2(0.6+0.43)+0.33\right]$$

$$=1.10$$

Actual value :

$$\int_1^3 \frac{1}{x}dx = \ln x\Big|_1^3 = \ln 3 - \ln 1 = 1.098 \;\; or \;\; 1.10$$

(d) $I = \int_0^1 e^{2x}dx$, $n = 4$

Solution: Here $a = 0$, $b = 1$, $2n = 8$

$$\Delta x = \frac{b-a}{2n} = \frac{1-0}{8} = \frac{1}{8} \quad \text{and} \quad f(x) = e^{2x} \quad \text{Then}$$

$$f_k = f(a + k\Delta x) = f\left(0 + \frac{k}{8}\right) = f\left(\frac{k}{8}\right)$$

$\Rightarrow f_k = e^{2\left(\frac{k}{8}\right)} = e^{\frac{k}{4}}$ ....(A) **Now table is under**

k	0	1	2	3	4	5	6	7	8
f_k	1	1.28	1.65	2.11	2.72	3.49	4.48	5.75	7.39
	f_o	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8

As by Simpson's rule

$$\int_0^1 e^{2x}dx = \frac{\Delta x}{3}[f_o + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6) + f_8]$$

$$= \frac{1}{3\times 8}[1 + 4(1.28 + 2.11 + 3.49 + 5.75) + 2(1.65 + 2.72 + 4.48) + 7.39]$$

$$= 3.19$$

Now exact value is $\int_0^1 e^{2x}dx = \frac{e^{2x}}{2}\Big|_0^1 = \frac{1}{2}(e^2 - e^0)$

$$= \frac{1}{2}(7.39 - 1) = 3.195$$

<u>Q. 3</u>: A quarter circle of radius 1 has the equation

$y=\sqrt{1-x^2}$ **for** $0 \le x \le 1$, **which means that**

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4} \quad , \quad n=4$$

Approximate the definite integral on the left by trapezoidal rule that equal the right side when $\pi = 3.1$

Solution: **Here** $f(x)=\sqrt{1-x^2}$ **and**

$a=0$, $b=1$ **and**

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$$

Now $f_k = f(a+k\Delta x) = f\left(0+\frac{k}{4}\right) = f\left(\frac{k}{4}\right)$

$$\Rightarrow f_k = \sqrt{1-\left(\frac{k}{4}\right)^2}$$

Now the table is under

k	0	1	2	3	4
f_k	1	0.9682	0.87	0.66	0
	f_o	f_1	f_2	f_3	f_4

Now by Trapezoidal rule

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\Delta x}{2}\left[f_o + 2(f_1+f_2+f_3)+f_4\right] \text{ put values}$$

$$= \frac{1}{2\times 4}\left[1+2(0.9682+0.87+0.66)+0\right]$$

$$= 0.7495 \quad \text{.....(i)}$$

But $\frac{\pi}{4} = \frac{3.1}{4} = 0.775$ **.....(ii)**

From (i) and (ii) it is proved approximately that

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}.$$

Q.4: A quarter circle of radius 1 has the equation $y=\sqrt{1-x^2}$ for $0 \le x \le 1$, which means that

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4} \quad , \quad n=4$$

Approximate the definite integral on the left by simpson,s rule that equal the right side when $\pi = 3.1$

Solution: Here $f(x)=\sqrt{1-x^2}$ and

$a=0$, $b=1$ and $2n=8$

$$\Delta x = \frac{b-a}{2n} = \frac{1-0}{8} = \frac{1}{8}$$

Now $f_k = f(a+k\Delta x) = f\left(0+\frac{k}{8}\right) = f\left(\frac{k}{8}\right)$

$$\Rightarrow f_k = \sqrt{1-\left(\frac{k}{8}\right)^2}$$

Now the table is under

k	0	1	2	3	4	5	6	7	8
f_k	1	0.99	0.96	0.92	0.8	0.78	0.66	0.48	0
	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8

Now by Simpson's rule

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\Delta x}{3}\left[f_0 + 4(f_1+f_3+f_5+f_7) + 2(f_2+f_4+f_6) + f_8\right]$$

Put values

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{3\times 8}\begin{bmatrix}1+4(0.99+0.92+0.78+0.48)\\ +2(0.96+0.8+0.66)+0\end{bmatrix}$$

$$= 0.7802 \quad \ldots\ldots(i)$$

But $\frac{\pi}{4} = \frac{3.1}{4} = 0.775 \quad \ldots\ldots(ii)$

From (i) and (ii) it is proved approximately that

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}.$$

Review Exercise 12.

Q.1: Choose the correct options.

(i) When $f(x)=0$, the roots of the equation $f(x)=6x^2+5x-6$ are :

(a) $\left(\frac{2}{3},-\frac{2}{3}\right)$ (b) $\left(\frac{2}{3},-\frac{3}{2}\right)$ (c) $\left(\frac{3}{2},-\frac{2}{3}\right)$ (d) $\left(-\frac{3}{2},-\frac{2}{3}\right)$

(ii) When $f(x)=0$, then the value of x in the equation $f(x)=x^{-2}+6x-5$ is :

(a) $x=-3.7187$ (b) $x=0.37187....$ (c) -0.37187 (d) $x=4.7847$

(iii) The Newton Raphson method is also known as :

(a) tangent method (b) second method
(c) diameter method (d) chord method

(iv) The iterative formula for Newton Raphson method

$x_3=......$ (a) $x_1-\frac{f(x_2)}{f'(x_2)}$ (b) $x_2-\frac{f(x_3)}{f'(x_3)}$
(c) $x_2-\frac{f(x_2)}{f'(x_2)}$ (d) $x_o+\frac{f(x_2)}{f'(x_2)}$

(v) The function values $f(a)$ and $f(c)$ at $x=a$ and $x=c$ have the opposite signs, then the roots lies in the interval : (a) $[a,\ b]$ (b) $[a,\ c]$ (c) (a,b) (d) $(a,\ c)$

(vi) The Newton Raphson method fails at :

(a) stationary point (b) floating point
(c) continuous point (d) non of these

(vii) The Newton Raphson method fails at :

(a) stationary point (b) floating point
(c) continues point (d) non of these

(viii) The MAPLE command to solve equation $x^2-5x+3=0$

(a) $solve(x^2-5x+3=0)$ (b) $fsolve(x^2-5x+3=0)$
(c) $simplon(x^2-5x+3=0)$ (d) $psolve(x^2-5x+3=0)$

(ix) The roots of the equation $x - \sin x - \frac{1}{2} = 0$ by using the bisection method between $1 < x < 2$.

(a) 1.88 (b) 1.48 (c) 1.49 (d) 1.897

(x) The bisection method also known as :

(a) Newton method (b) Binary Chopping method

(c) Quaternary method (d) non of these

Answers: (i) b (ii) c (iii) a (iv) c (v) b
(vi) a (vii) d (viii) b (ix) c (x) b