**UNIT- 2 :**

## Functions and Limits :

**Function :** A function $y = f(x)$ is a rule that assigns for each value of the independent variable x a unique value of the dependent variable y .

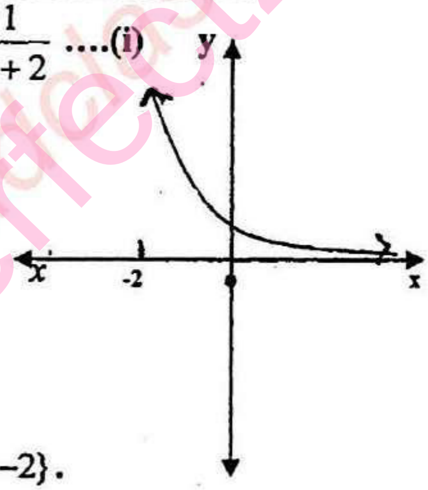
**Domain and range of function :** Let $y = f(x)$ be a function then those values of independent variable (input) x for which dependent variable (out put) y is defined is called domain of the function .

Those values of dependent variable (output) y which is obtained from independent variable (input) x is called range of the function .

**Note:** Every real number is defined but complex numbers and $\infty$ are undefined .

**Domain and range from the graph of function :**

Consider the function $y = \dfrac{1}{x+2}$ ....(i)




From (i) it is clear that $y$ is undefined for $x = -2$ because if we put $x = -2$ in (i) then $y = \dfrac{1}{-2+2} = \dfrac{1}{0} = \infty$

but y is defined for all real numbers except -2 .

Thus domain of f(x) is

$$D = \{x / x \in R \ , \ x \neq -2\}.$$

Now we find range: from (i)

$$y = \frac{1}{x+2}$$

$$\Rightarrow (x+2)y = 1$$

$$\Rightarrow x + 2 = \frac{1}{y}$$

$$\Rightarrow \quad x = \frac{1}{y} - 2$$

$$\Rightarrow \quad x = \frac{1-2y}{y} \quad .......(ii)$$

From (ii) we see that for $y = 0$ there will be no real number of $x$ because if we put $y = 0$ in (ii)

$$x = \frac{1-0}{0} = \infty$$

Thus for out put $y = 0$ there will be no value of input $x$ But for every value of y other than 0 there must be a real number $x$ .

Thus range of f(x) is $R = \{y / y \in R \ , \ y \neq 0\}$

**Absolute or modulus of $x$ :** The absolute or modulus of $x$ is denoted and given by

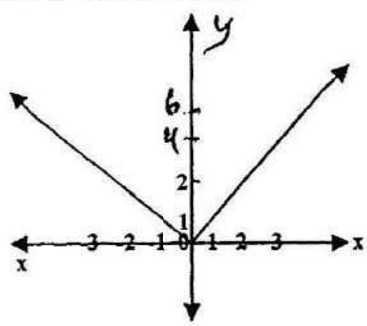
$$|x| = \begin{cases} -x & if \quad x < 0 \\ x & if \quad x \geq 0 \end{cases}$$

**Graph of the following absolute functions and find its Domain and range:**

$$f(x) = |x|$$

**Solution:** $f(x) = |x|$ , then by definition of modulus value function

$$f(x) = \begin{cases} -x & if \quad x < 0 \\ x & if \quad x \geq 0 \end{cases}$$




It is clear from the definition of absolute function the domain of $f(x)$ is set of real numbers . i.e.

Domain of $f(x) = \{x / x \in R\}$.

The range of f(x) is non – negative . i.e.

*Range of* $f(x) = \{f(x) / \ f(x) \geq 0\}$.

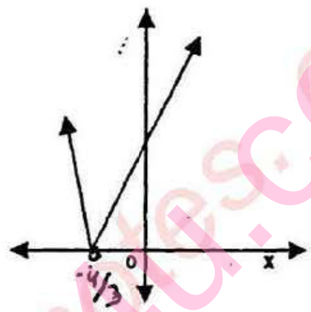
**Example. 2**: Graph of the following absolute functions and find its domain and range: $f(x) = |3x + 4|$

**Solution:**

$$f(x)=\begin{cases}3x+4 & \text{if } 3x+4\geq 0\\ -(3x+4) & \text{if } 3x+4<0\end{cases}$$

**Or**

$$f(x)=\begin{cases}3x+4 & \text{if } x\geq -\frac{4}{3}\\ -(3x+4) & \text{if } x<-\frac{4}{3}\end{cases}$$




**The of $f(x)$ is set of real numbers**
**i.e. Domain of $f(x)=\{x/x\in R\}$.**
**The range of f(x) is non – negative . i.e.**
*Range of* $f(x)=\{f(x)/\ f(x)\geq 0\}$.

**Compound function : A function that defined by more than one equation is called compound function .**

**For example** $f(r)=\begin{cases}2x & \text{if } 0<x<2\\ 3x^2-1 & \text{if } x\geq 2\end{cases}$

**Definition : One – to – one function :**

**A function f(x) is called one – to – one if distinct elements of $x$ mapped to distinct elements of y .**

**Or**

**A function f(x) is called one – to – one if each range value corresponds to exactly one domain value .**

**If each range value corresponds to more than one domain values then the function is called may –to – one function .**

**Composite function : Let $f(x)$ and $g(x)$ are two functions Then the functions $g(f(x))$ is called composite function .**

**Example .3 : Let $f(x)=2x-1$ and $g(x)=\sqrt{3x+5}$ .**

**Find each of the following :**

**(i) (a)** $g[f(4)]$ **(b)** $f[g(4)]$ **(c)** $f[g(-2)]$

**Solution :** $f(x)=2x-1$ **….(i) ,** $g(x)=\sqrt{3x+5}$ **…..(ii)**

**(a)** $g[f(4)]$ **: put** $x=4$ **in (i)**

$$f(4)=2(4)-1=7$$

$g[f(4)] = g(7) = \sqrt{3(7)+5} = \sqrt{26}$ *Ans.*

**(b)** $f[g(4)]$ **: put** $x = 4$ **in (ii)**

$g(4) = \sqrt{3(4)+5} = \sqrt{17}$

$f[g(4)] = f(\sqrt{17}) = 2(\sqrt{17}) - 1 = 2\sqrt{17} - 1$ *Ans.*

**(c)** $f[g(-2)]$ **: put** $x = -2$ **in (ii)**

$g(-2) = \sqrt{3(-2)+5} = \sqrt{-1} = i$

**Which is complex. Since** $-2$ **is not domain of** $g(x)$.

**Hence** $f[g(-2)]$ **does not exist.**

**Example.4:** **Let** $f(x) = 4x+1$ , $g(x) = 2x^2 + 5x$.

**Find each of the following : (a)** $g[f(x)]$ **(b)** $f[g(x)]$

**Solution :** $f(x) = 4x+1$ , $g(x) = 2x^2 + 5x$

**(a)** $g[f(x)] = g(4x+1)$

$\Rightarrow g[f(x)] = 2(4x+1)^2 + 5(4x+1)$

$\Rightarrow g[f(x)] = 2(16x^2 + 8x + 1) + 20x + 5$

$\Rightarrow g[f(x)] = 32x^2 + 16x + 2 + 20x + 5$

$\Rightarrow g[f(x)] = 32x^2 + 36x + 7$ Ans.

**(b)** $f[g(x)] = f(2x^2 + 5x)$

$\Rightarrow f[g(x)] = 4(2x^2 + 5x) + 1$

$\Rightarrow f[g(x)] = 8x^2 + 20x + 1$ *Ans.*

**Inverse of a function :** **Let** $y = f(x)$ **be a one-to-one function then the function** $x = f^{-1}(y)$ **is called inverse function.**

**if** $f(x)$ **is not one-to-one then** $f(x)$ **does not have inverse function.**

**Example .6:** **Find the inverse function of** $f(t) = 3t - 8$.

**Solution :** $f(t) = 3t - 8$ **....(i)**

*Let* $y = f(t) \Rightarrow t = f^{-1}(y)$ ......*(ii)*

$As \quad y = f(t) = 3t - 8$

$\Rightarrow y = 3t - 8$

$\Rightarrow y + 8 = 3t$

$\Rightarrow t = \dfrac{y+8}{3}$ ..........(iii)

**Put (iii) in(ii)**

$f^{-1}(y) = \dfrac{y+8}{3}$ **replace** $y$ **by** $t$

$\Rightarrow f^{-1}(t) = \dfrac{t+8}{3}$ **Ans.**

**Example .7.** **Let** $f(x) = 2x + 3$ **and** $g(x) = 3x$ **and** $h(x) = f[g(x)]$

**(a)** $h(x)$ **(b)** $f^{-1}(x)$ **(c)** $g^{-1}(x)$ **(d)** $h^{-1}(x)$

**Solution :** $f(x) = 2x + 3$ **and** $g(x) = 3x$

**(a)** $h(x)$: **As** $h(x) = f[g(x)]$

$\Rightarrow h(x) = f(3x)$

$= 2(3x) + 3$

$\Rightarrow h(x) = 6x + 3$ *Ans.*

**(b)** $f^{-1}(x)$:

*Let* $y = f(x)$

$\Rightarrow x = f^{-1}(y)$ .......(i)

$As \quad y = f(x) = 2x + 3$

$\Rightarrow y - 3 = 2x$

$\Rightarrow x = \dfrac{y-3}{2}$ ......(ii)

**Put (ii) in (i)**

$f^{-1}(y) = \dfrac{y-3}{2}$ **replace** $y$ *by* $x$

$\Rightarrow f^{-1}(x) = \dfrac{x-3}{2}$ **Ans.**

**(c)** $g^{-1}(x)$:

$$Let \quad y = g(x)$$

$$\Rightarrow \ x = g^{-1}(y) \ .......(i)$$

$$As \quad y = g(x) = 3x$$

$$\Rightarrow \ y = 3x$$

$$\Rightarrow \ x = \frac{y}{3} \ ......(ii)$$

**Put (ii) in (i)**

$$f^{-1}(y) = \frac{y}{3} \quad \textbf{replace } y \ by \ x$$

$$\Rightarrow \ f^{-1}(x) = \frac{x}{3} \quad \textbf{Ans.}$$

**(d)** $h^{-1}(x)$:

$$Let \quad y = h(x)$$

$$\Rightarrow \ x = h^{-1}(y) \ ........(i)$$

$$As \quad y = h(x) = 6x + 3$$

$$\Rightarrow \ y - 3 = 6x$$

$$\Rightarrow \ x = \frac{y-3}{6} \ ......(ii)$$

**Put (ii) in (i)**

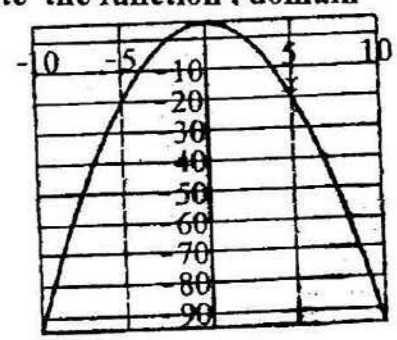
$$h^{-1}(y) = \frac{y-3}{6} \quad replace \ \ y \ \ by \ \ x$$

$$\Rightarrow \ h^{-1}(x) = \frac{x-3}{6} \quad Ans.$$

## Exercise 2.1:

**Q.1: Read the graphs and write the function, domain and range of f:**

**(a)**



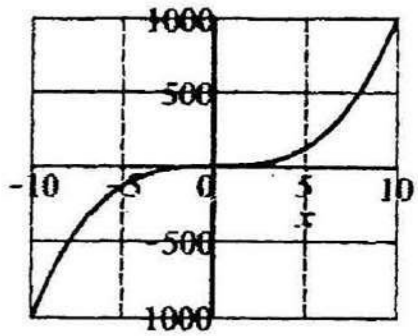

**Solution: The function is**

$$f(x) = -x^2 + 5$$

**It is defined for each real number. Thus**

**Domain = R =** $(-\infty, \ \infty)$

Range $= (-\infty,\ 5)$




**(b)**

**Solution: The function is**

$$f(x) = x^3 + x + 3$$

**Domain = R** $=(-\infty,\quad \infty)$

**Range =** $(-\infty,\ \infty)$

**(c)**

**Solution: The function is**

$$f(x) = \frac{1}{x+2}$$

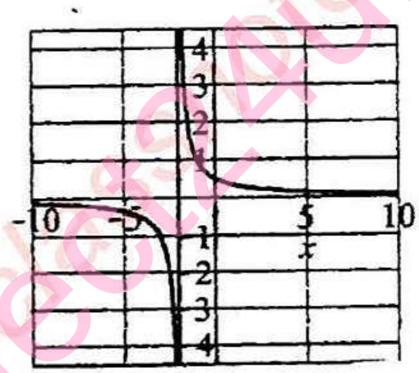
**It is defined for all real numbers except** $x = -2$

**Thus Domain =** $R - \{-2\}$

$= (-\infty,\ -2) U (-2,\ \infty)$

**Range =** $R - \{0\}$

$= (-\infty,\ 0) U (0,\ \infty)$



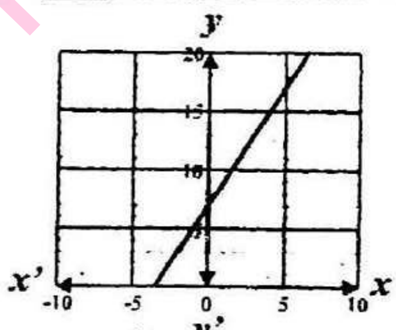

**(d)**

**Sollution: The function is**

$$f(x) = 2x + 7$$

**Domain = R** $=(-5,\quad 7)$

**Range =** $(0,\ 20)$



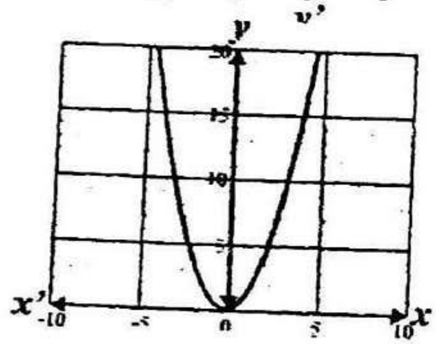

**(e)**

**Sollution: The function is**

$$f(x) = x^2$$

**Domain = R** $=(-5,\quad 5)$

**Range = (0 , 20)**



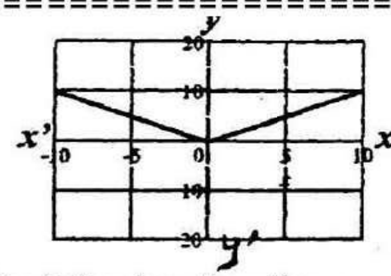
(f)

**Sollution: The function is**

$f(x) = |x|$

**Domain = ( - 10 , 10)**

**Range = (0 , 10)**



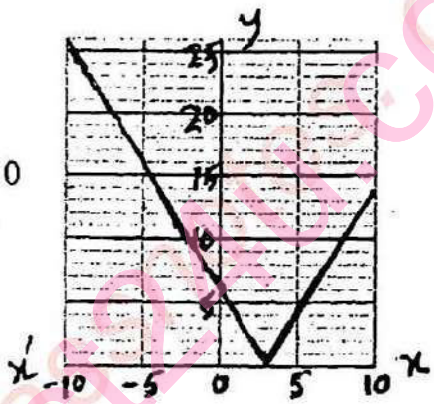

**Q.2: Draw Graphs each of the following functions .**

(a) $y = 2|x-3|$

**Solution:** $y = 2|x-3|$

$$\Rightarrow y = \begin{cases} 2(x-3) & if \quad x-3 \geq 0 \\ -2(x-3) & if \quad x-3 < 0 \end{cases}$$

$$\Rightarrow y = \begin{cases} 2x-6 & if \quad x \geq 3 \\ -2x+6 & if \quad x < 3 \end{cases}$$




**The table is given by**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | - 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| $y$ | 8 | 6 | 4 | 2 | 0 | 2 | 4 |

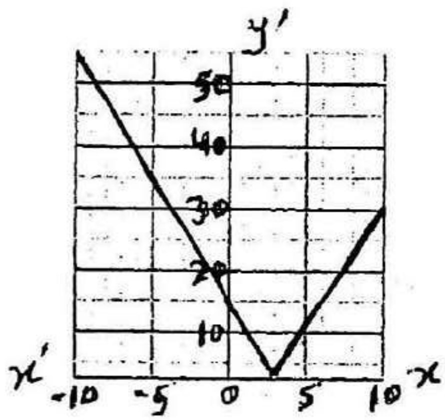
(b) $y = 4|x-3| + 3$

**Solution:**

$$y = \begin{cases} 4(x-3)+3 & if \quad x-3 \geq 0 \\ -4(x-3)+3 & if \quad x-3 < 0 \end{cases}$$

$$y = \begin{cases} 4x-12+3 & if \quad x \geq 3 \\ -4x+12+3 & if \quad x < 3 \end{cases}$$

$$y = \begin{cases} 4x-9 & if \quad x \geq 3 \\ -4x+15 & if \quad x < 3 \end{cases}$$




**The table is given by**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | - 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| $y$ | 19 | 15 | 11 | 7 | 3 | 7 | 11 |

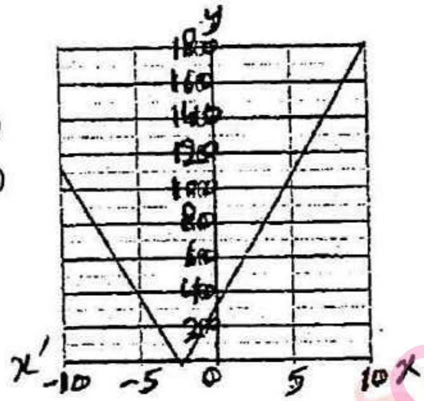
(c) $y = 5|3t+7|-2$

**Solution:** $y = 5|3t+7|-2$

$$y = \begin{cases} 5(3t+7)-2 & if\ 3t+7 \geq 0 \\ -5(3t+7)-2 & if\ 3t+7 < 0 \end{cases}$$

$$y = \begin{cases} 15t+35-2 & if\ 3t \geq -7 \\ -15t-35-2 & if\ 3t < -7 \end{cases}$$

$$y = \begin{cases} 15t+33 & if\ t \geq -\dfrac{7}{3} \\ -15t-37 & if\ t < -\dfrac{7}{3} \end{cases}$$




| t | -5 | -4 | -3 | -2 | -1 | 0 | |
|---|---|---|---|---|---|---|---|
| $y$ | 38 | 23 | 8 | 3 | 18 | 33 | |

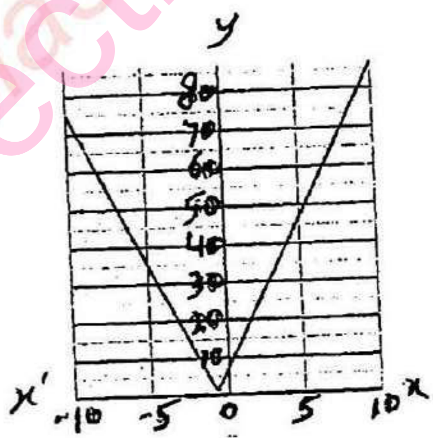
(d) $y = 2|4x+3|+1$

**Solution :** $y = 2|4x+3|+1$

$$y = \begin{cases} 2(4x+3)+1 & if\ 4x+3 \geq 0 \\ -2(4x+3)+1 & if\ 4x+3 < 0 \end{cases}$$

$$y = \begin{cases} 8x+6+1 & if\ 4x \geq -3 \\ -8x-6+1 & if\ 4x < -3 \end{cases}$$

$$y = \begin{cases} 8x+7 & if\ x \geq -\dfrac{3}{4} \\ -8x-5 & if\ x < -\dfrac{3}{4} \end{cases}$$




The table is given by

| $x$ | -4 | -3 | -2 | -1 | 0 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 27 | 19 | 11 | 3 | 7 | 15 | 31 |

**Q.3: Find the composite functions $f[g(x)]$ and $g[f(x)]$ of the following functions.**

(a) $f(x) = x^2 + 1$ , $g(x) = 2x$

**Solution :** $f(x) = x^2 + 1$ ....(i) **and** $g(x) = 2x$ .....(ii)

$$f[g(x)] = f(2x) \quad put \quad g(x) = 2x$$

$$\Rightarrow f[g(x)] = (2x^2) + 1 = 4x^2 + 1 \quad Ans.$$

$$Now \quad g[f(x)] = g(x^2 + 1) \quad put \quad f(x) = x^2 + 1$$

$$\Rightarrow g[f(x)] = 2(x^2 + 1) = 2x^2 + 2 \quad Ans.$$

(b) $f(x) = \sin x$ **and** $g(x) = 1 - x^2$

**Solution:** $f(x) = \sin x$ ...(i) **and** $g(x) = 1 - x^2$ ...(ii)

$$f[g(x)] = f(1 - x^2) \quad put \quad g(x) = 1 - x^2$$

$$\Rightarrow f[g(x)] = \sin(1 - x^2) \quad Ans$$

$$Now \quad g[f(x)] = g(\sin x) \quad put \quad f(x) = \sin x$$

$$\Rightarrow g[f(x)] = 1 - \sin^2 x = \cos^2 x \quad Ans.$$

(c) $f(x) = \dfrac{x-1}{x+1}$ , $g(u) = \dfrac{x+1}{1-x}$

**Solution:** $f(x) = \dfrac{x-1}{x+1}$ ....(i) **and** $g(x) = \dfrac{x+1}{1-x}$ .....(ii)

$$f[g(x)] = f\left(\frac{x+1}{1-x}\right) \quad put \quad g(x) = \frac{x+1}{1-x}$$

$$\Rightarrow f[g(x)] = \frac{\left(\frac{x+1}{1-x}\right) - 1}{\left(\frac{x+1}{1-x}\right) + 1} = \frac{\frac{x+1-1+x}{1-x}}{\frac{x+1+1-x}{1-x}}$$

$$\Rightarrow f[g(x)] = \frac{2x}{2} = x \quad Ans.$$

$$\textbf{Now} \quad g[f(x)] = g\left(\frac{x-1}{x+1}\right) \quad put \quad f(x) = \frac{x-1}{x+1}$$

$$\Rightarrow \quad g[f(x)] = \frac{\frac{x-1}{x+1}+1}{1-\frac{x-1}{x+1}} = \frac{\frac{x-1+x+1}{x+1}}{\frac{x+1-(x-1)}{x+1}}$$

$$\Rightarrow \quad g[f(x)] = \frac{2x}{2} = x \quad Ans.$$

**(d)** $f(x) = \sin x$ **and** $g(x) = 2x+3$

**Solution :** $f(x) = \sin x$ ....(i) **and** $g(x) = 2x+3$ .....(ii)

$f[g(x)] = f(2x+3)$ *put* $g(x) = 2x+3$

$\Rightarrow f[g(x)] = \sin(2x+3)$ *Ans.*

*Now* $g[f(x)] = g(\sin x)$ *put* $f(x) = \sin x$

$\Rightarrow g[f(x)] = 2\sin x + 3$ *Ans.*

**Q.4: Determine the inverse function of $f(g(x))$ and $g(f(x))$ for the following functions.**

**(a)** $f(x) = x+5$ , $g(x) = x-4$

**Solution : Let** $h(x) = f(g(x))$ **, put** $g(x) = x-4$

$\Rightarrow h(x) = f(x-4) \quad \Rightarrow h(x) = x-4+5$

$\Rightarrow h(x) = x+1$

**Let** $y = h(x)$

$\Rightarrow x = h^{-1}(y)$ ......(i)

*As* $y = x+1 \Rightarrow x = y-1$ .......(ii)

**Put (ii) in (i)**

$h^{-1}(y) = y-1$ *replace* $y$ *by* $x$

$\Rightarrow h^{-1}(x) = x-1 \Rightarrow h^{-1}(x) = f^{-1}(g(x)) = x-1$ *Ans.*

**Let** $k(x) = g(f(x))$ **, put** $f(x) = x+5$

$\Rightarrow k(x) = g(x+5) \quad \Rightarrow s(x) = x+5-4$

$\Rightarrow k(x) = x+1$

**Let** $y = k(x)$

$\Rightarrow x = k^{-1}(y)$ ......(iii)

*As* $y = x+1 \Rightarrow x = y-1$ .......(iv)

**Put (ii) in (i)**

$$k^{-1}(y) = y - 1 \qquad replace \quad y \quad by \quad x$$

$$\Rightarrow \quad k^{-1}(x) = x - 1 \Rightarrow \quad k^{-1}(x) = g^{-1}(f(x)) = x - 1 \quad Ans.$$

**(b)** $f(x) = 2x + 7$ , $g(x) = 2x$

**<u>Solution</u>:** **Let** $h(x) = f(g(x))$ **, put** $g(x) = 2x$

$$\Rightarrow h(x) = f(2x) \Rightarrow h(x) = 2(2x) + 7$$

$$\Rightarrow \quad h(x) = 4x + 7$$

**Let** $y = h(x)$

$$\Rightarrow \quad x = h^{-1}(y) \quad ......\text{(i)}$$

$As \quad y = 4x + 7 \quad \Rightarrow \quad y - 7 = 4x$

$$\Rightarrow x = \frac{y-7}{4} \quad ......(ii)$$

**Put (ii) in (i)**

$$f^{h-1}(y) = \frac{y-7}{4} \qquad replace \quad y \quad by \quad x$$

$$\Rightarrow h^{-1}(x) = \frac{x-7}{4} \quad \Rightarrow h^{-1}(x) = f^{-1}(g(x)) = \frac{x-7}{4} \qquad Ans.$$

**Let** $k(x) = g(f(x))$ **, put** $f(x) = 2x + 7$

$$\Rightarrow \quad k(x) = g(2x+7) \quad \Rightarrow \quad k(x) = 2(2x+7)$$

$$\Rightarrow \quad k(x) = 4x + 14$$

**Let** $y = k(x)$

$$\Rightarrow \quad x = k^{-1}(y) \quad ......\text{(iii)}$$

$As \quad y = 4x + 14 \quad \Rightarrow \quad y - 14 = 4x$

$$or \quad x = \frac{y-14}{4} \quad .......(iv) \qquad , put \quad (iv) \; in \; (iii)$$

$$\Rightarrow \quad k^{-1}(y) = \frac{y-14}{4} \quad , \quad replace \quad y \quad by \quad x$$

$$\Rightarrow \quad k^{-1}(x) = \frac{x-14}{4} \quad or \quad k^{-1}(x) = g^{-1}(f(x)) = \frac{x-14}{4} \quad \textbf{ANS.}$$

**(c)** $f(x) = 2(x-4)$ , $g(x) = \dfrac{x+5}{2}$

**Solution :**

Let $h(x)=f(g(x))$ , put $g(x)=\dfrac{x+5}{2}$

$\Rightarrow\ h(x)=f\left(\dfrac{x+5}{2}\right)\quad \Rightarrow\ h(x)=2\left(\dfrac{x+5}{2}-4\right)$

$\Rightarrow\ h(x)=2\left(\dfrac{x+5-8}{2}\right)\quad \Rightarrow\ h(x)=x-3$

Let $y=h(x)$

$\Rightarrow\ x=h^{-1}(y)$ ......(i)

*As* $y=x-3\quad \Rightarrow\ x=y+3$ .....(ii)

Put (ii) in (i)

$h^{-1}(y)=y+3$ *replace y by x*

$\Rightarrow\ h^{-1}(x)=x+3\ \Rightarrow\ h^{-1}(x)=f^{-1}(g(x))=x+3$ *Ans.*

Let $k(x)=g(f(x))$ , put $f(x)=2(x-4)$

$\Rightarrow\ k(x)=g(2x-8)\quad \Rightarrow\ k(x)=\dfrac{2x-8+5}{2}$

$\Rightarrow\ k(x)=\dfrac{2x-3}{2}$

Let $y=k(x)\ \Rightarrow\ x=k^{-1}(y)$ .....(iii)

*Then* $y=\dfrac{2x-3}{2}\quad \Rightarrow\ 2y=2x-3$

$\Rightarrow\ 2y+3=2x$ *or* $x=\dfrac{2y+3}{2}$ ......(iv)

Put (iv) in (iii)

$\Rightarrow\ k^{-1}(y)=\dfrac{2y+3}{2}$ , replace y by x

$\Rightarrow\ k^{-1}(x)=\dfrac{2x+3}{2}\ \Rightarrow\ k^{-1}(x)=g^{-1}(f(x))=\dfrac{2x+3}{2}$ Ans.

(d) $y=f(x)=\dfrac{x+4}{2}$, $g(x)=2x-4$

**Solution :** Let $h(x)=f(g(x))$ , put $g(x)=2x-4$

$$\Rightarrow \quad h(x) = f(2x-4)$$

$$\Rightarrow \quad h(x) = \frac{2x-4+4}{2} \qquad \Rightarrow \quad h(x) = \frac{2x}{2}$$

$$\Rightarrow \quad h(x) = x$$

**Let** $y = h(x) \quad \Rightarrow \quad x = h^{-1}(y)$ **......(i)**

*As* $\quad y = x \quad ......(ii)$ **Put (ii) in (i)**

$h^{-1}(y) = y \qquad$ *replace* $y$ *by* $x$

$$\Rightarrow \quad h^{-1}(x) = x \quad \Rightarrow \quad \Rightarrow \quad h^{-1}(x) = f^{-1}(g(x)) = x \qquad Ans.$$

**Let** $k(x) = g(f(x))$ **, put** $y = f(x) = \dfrac{x+4}{2}$

$$\Rightarrow \quad k(x) = g\left(\frac{x+4}{2}\right) \quad \Rightarrow \quad k(x) = 2\left(\frac{x+4}{2}\right) - 4$$

$$\Rightarrow \quad k(x) = x+4-4 \quad \Rightarrow \quad k(x) = x$$

**Let** $y = k(x) \quad \Rightarrow \quad x = k^{-1}(y)$ **......(ii)**

*As* $\quad y = x \quad ......(iv)$ **Put (ii) in (i)**

$k^{-1}(y) = y \qquad$ *replace* $y$ *by* $x$

$$\Rightarrow \quad k^{-1}(x) = x \quad \Rightarrow \quad \Rightarrow \quad k^{-1}(x) = g^{-1}(f(x)) = x \qquad Ans.$$

**<u>Transcendental function</u>: A function which is not algebraic is called transcendental function. Trigonometric , hyperbolic exponential and logarithm functions are called transcendental function.**

**<u>Polynomial</u>: A polynomial** $P_n(x)$ **is a function of the form**

$f(x) = P_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ....... + a_1 x + a_o$ **is called Polynomial of degree n.**

**The constants** $a_n \,, a_{n-1} \,, a_{n-2} \,, ......, a_1$ *and* $a_o$ **are called coefficients.** $a_n$ **is called leading coefficient and** $a_o$ **is called constant term.**

**If** $n = 0$ **then** $f(x) = a_o$ **is called constant function.**

**If** $n = 1$ **then** $f(x) = a_1 x + a_o$ **is called linear function.**

**If** $n = 2$ **then** $f(x) = a_2 x^2 + a_1 x + a_o$ **is called quadratic**

function .

If $n=3$ then $f(x)=a_3x^3+a_2x^2+a_1x+a_o$ is called cubic function .

**Algebraic function** : A function $f(x)$ is called algebraic if it can be constructed using algebraic operations such as (adding , subtracting , multiplying , dividing or taking Square roots) starting with polynomial . For example

$$f(x)=2x^2+4x+5 \quad , \quad f(x)=\frac{2x-4}{x^2+1}$$

**Exponential function** : If $a>0$ *and* $a\neq 1$ then the function of the form $f(x)=a^x$ is called common exponential function . And

$f(x)=e^x$ , where $2<e<3$ is called natural exponential function .

**Exponents Law** : If $a>0$ *and* $b>0$ and $a\neq 1, b\neq 1$

Then (i) $a^x.a^y=a^{x+y}$ (ii) $(a^x)^y=a^{xy}$

(iii) $(ab)^x=a^x.b^x$ (iv) $\left(\frac{a}{b}\right)^x=\frac{a^x}{b^x}$

(v) $a^x=a^y$ if and only if $x=y$

(vi) $a^x=b^x$ for $x\neq 0$ if and only if $a=b$

**Example .8**: Cholera an intestinal disease , is caused by a cholera bacterium that multiples exponentially by cell division as given by $N=N_oe^{1.386t}$, with N is the number of bacteria present at the start $t=0$ . If we start with 25 bacteria , how many bacteria (to nearest unit) will be present in

(a) 1 hour ? (b) 3 hours ? (c) 4 hours ? (d) interpret .

**Solution:** $N=N_oe^{1.386t}$ .....(i)

(a) Put $N=25$ and $t=1$ in (i)

$N=25e^{1.386}=(25)(3.998)$ *use calculater.*

$=99.97$ *bacteria*

(b) put $t=3$ $in(i)$

$$N = 25e^{(1.386)(3)} = 25(63.94)$$

$$= 1599 \ \ bacteria$$

**(c) Put $t = 4$ in (i)**

$$N = 25e^{(1.386)(4)} = (25)(255.698)$$

$$= 6392 \ \ bacteria$$

**(d) From above calculation we see that the population of bacteria is increases when time t increases.**

**Logarithmic function: If**

$$a^x = y \iff \log_a y = x$$

**is called logarithm function. Where $a$ called base.**

**If base is 10 then**

$10^x = y \iff \log_{10} y = x$ **it is called common logarithm.**

**If base is $e$ then it is called natural logarithm. i.e.**

$$e^x = y \iff \log_e y = x$$

$$or \quad \ln y = x$$

**Example .9: Evaluate the following logarithmic functions:**

**(a)** $\log 10000$ **(b)** $\log 0.01$ **(c)** $\log\sqrt{10} = \frac{1}{2}$

**Solution: (a)**

$$\log 10000 = \log 10^4$$

$$= 4\log 10 \qquad As \ \ \log 10 = 1$$

$$= (4)(1) = 4 \ \ Ans.$$

**(b)** $\log 0.01$:

$$\log 0.01 = \log 10^{-2}$$

$$= (-2)\log 10$$

$$= (-2)(1) = -2 \quad Ans.$$

**(c)** $\log\sqrt{10} = \frac{1}{2}$

$$\Rightarrow \log\sqrt{10} = \log(10)^{\frac{1}{2}}$$
$$= \frac{1}{2}\log 10 = \frac{1}{2}(1)$$
$$= \frac{1}{2} \quad \textbf{Ans.}$$

**<u>Laws of logarithm</u> : If b , M , p , x and N are any positive real numbers with $b \neq 1$ , then**

(i) $\log_b 1 = 0$ (ii) $\log_b b = 1$

(iii) $\log_b b^x = x$ (iv) $b^{\log_b x} = x$

(v) $\log_b MN = \log_b M + \log_b N$

(vi) $\log_b \frac{M}{N} = \log_b M - \log_b N$

(vii) $\log_b M^p = p\log_b M$

(viii) $\log_b M = \log_b N$ *then* $M = N$

**<u>Example .10</u>: Find $x$ to four decimal places for the following indicated exponential functions .**

(a) $10^x = 2$ (b) $e^x = 3$ (c) $3^x = 4$

**<u>Solution</u> : (a)** $10^x = 2$ **,Taking log of both sides .**

$\Rightarrow \log 10^x = \log 2$ *use calculater*

$\Rightarrow x\log 10 = 0.3010$ *As* $\log 10 = 1$

$\Rightarrow x = 0.3010$ *Ans.*

**(b)** $e^x = 3$ **: Taking ln of both sides .**

$\ln e^x = \ln 3$

$\Rightarrow x\ln e = 1.0986$ *use calculater*

$\Rightarrow x = 1.0986$ *As* $\ln e = 1$

**(c )** $3^x = 4$ **,Taking log of both sides .**

$\log 3^x = \log 4 \Rightarrow x\log 3 = \log 4$

$\Rightarrow x = \frac{\log 4}{\log 3}$ *Use calculater.*

$\Rightarrow \quad x = 1.2619 \quad Ans$

**Example.11;** Two people with the covid – 19 positive visited the campus of Peshawer university. The number of days T that took for the corona virus to infect n people is given by $T(n) = -1.43\ln\left(\frac{10.000-n}{4998n}\right)$

How many days will it take for the virus to infect (a) 500 people (b) 5000 people ?

**Solution:** $T(n) = -1.43\ln\left(\frac{10,000-n}{4998n}\right)$ .....(i)

(a) put $n = 500$ in (i)

$$T(500) = -1.43\ln\left(\frac{10,000-500}{4998\times 500}\right)$$

$$= -1.43\ln\left(\frac{950}{2499000}\right) = -1.43\ln(0.00380)$$

$$= -1.43(-5.57275) = 7.96903 = 8 \ days$$

(b) put $n = 5000$ in (i)

$$T(5000) = -1.43\ln\left(\frac{10,000-5000}{4998\times 5000}\right)$$

$$= -1.43\ln\left(\frac{5000}{24990000}\right) = -1.43\ln\left(\frac{1}{4998}\right)$$

$$= 12.17 = 12 \ days$$

**Hyperbolic function :**

(i) $\sinh x = \frac{e^x - e^{-x}}{2}$ (ii) $\cosh x = \frac{e^x + e^{-x}}{2}$

(iii) $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ (iv) $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, $x \neq 0$

(v) $\sec hx = \frac{2}{e^x + e^{-x}}$ (vi) $\cos echx = \frac{2}{e^x - e^{-x}}$, $x \neq 0$

are called hyperbolic functions.

**Note :** $\cosh^2 x - \sinh^2 x = 1$

**Explicit function :** A function in which dependent variable $y$

is completely expressed in terms of $x$ is called Explicit function. For example $y = 3x^2 + 4x$

**Implicit function**: A function in which dependent variable $y$ is cannot be expressed in terms of $x$ is called implicit function. For example $yx^3 = 3y^2x^2 + 4xy^3$.

**Parametric equations**: Those equations in which two or more than two variables can be express in terms of a single variable is called parametric equation. If $x = f(t)$ and $y = g(t)$ these equations is called parametric equation and t is called parameter.

**Properties of the graph of the exponential function** $f(x) = b^x$ :

With $b > 0$ and $b \neq 1$.

(i) All graphs will passes through the point $(0, 1)$.

(ii) All graphs will continues, with no jumps.

(iii) If $b > 1$, then the graph of $f(x)$ will increases if $x$ increases.

(iv) $0 < b < 1$, then, then the graph of $f(x)$ will decreases if $x$ decreases.

**Example 12**: Sketch the graph of the parametric functions $(x(t), y(t)) = (3-t, 2t)$ for all t.

**Solution**: As $(x(t), y(t)) = (3-t, 2t)$

=> $x = 3 - t$ *and* $y = 2t$

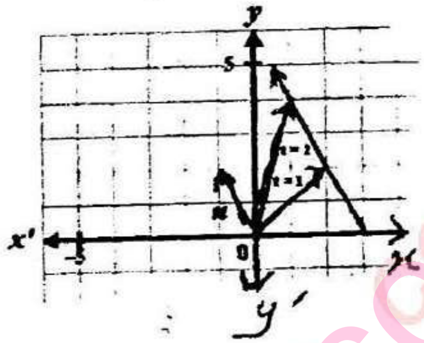
Note: $\frac{x-3}{-1} = t$ *and* $\frac{y}{2} = t$ Thus in symmetric form

=> $\frac{x-3}{-1} = \frac{y}{2} = t$, where -1 and 2 direction ratios of the vector parallel to the given curve.

Now the table is given

| $t$ | -2 | -1 | 0 | 1 |
|---|---|---|---|---|
| $x = 3 - t$ | 5 | 4 | 3 | 2 |
| $y = 2t$ | -4 | -2 | 0 | 2 |

Thus the graph of above function is obtained by plotting the points $(x , y)$.
it is a straight line passing through the point P(3 ,0) and parallel to the vector u =(-1 , 2).



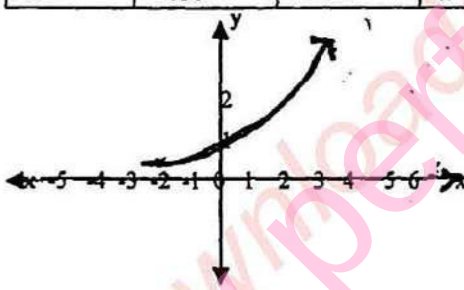

**Example.13:**Graph of $y = e^x$ and $y = e^{-x}$:

The table of $y = e^x$ is obtained following by using scientific calculator.

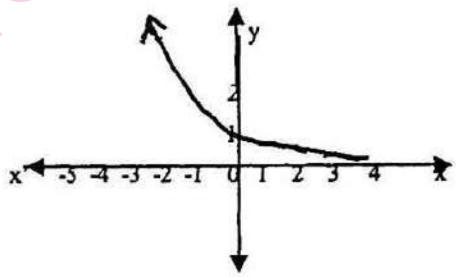
| | | | | | |
|---|---|---|---|---|---|
| $x$ | -2 | -1 | 0 | 1 | 2 |
| $y$ | 0.14 | 0.37 | 1 | 2.72 | 7.39 |

The table of $y = e^{-x}$ is obtained following by using scientific calculator.

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | -2 | -1 | 0 | 1 | 2 | 3 |
| $y$ | 7.39 | 2.72 | 1 | 0.37 | 0.14 | 0.05 |




Graph of $y = e^x$




Graph of $y = e^{-x}$

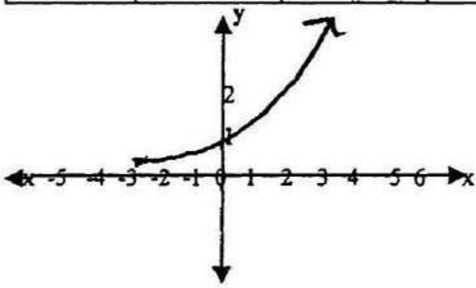
**Example.14:** Graph of the function $y = 2^x$ and $y = 2^{-x}$ :

The table of $y = 2^x$ is obtained following by using scientific calculator.

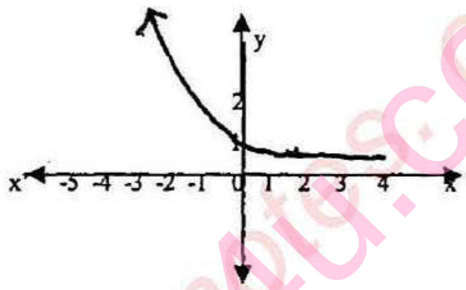
| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | -2 | -1 | 0 | 1 | 2 | 3 |
| $y$ | 0.25 | 0.5 | 1 | 2 | 4 | 8 |

The table of $y=2^{-x}$ is obtained following by using scientific calculator .

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | -2 | -1 | 0 | 1 | 2 | 3 |
| $y$ | 4 | 2 | 1 | 0.5 | 0.25 | 0.125 |




Graph of $y=2^x$




Graph of $y=2^{-x}$

**<u>Example.15:</u>Graph of** $y=\log_2 x$ **:**

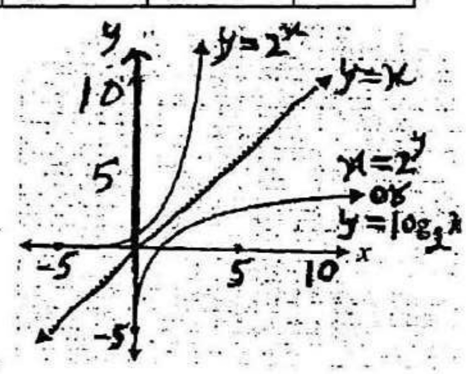
**<u>Solution</u>: As** $\log_2 x = y$

$\Rightarrow$ $2^y = x$ **As** $y=\log_2 x$ **then domain** $x>0$ .

**Thus the graph of** $\log_2 x = y$ **is same as the graph of** $x=2^y$

**The table of** $x=2^y$ **is below .**

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | 1 | 2 | 4 | 8 | 16 | |
| $y$ | 0 | 1 | 2 | 3 | 4 | |

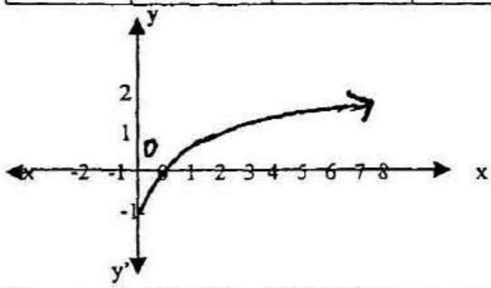



**Graph of** $y=\ln x$ **: Here** $x>0$ .

**The table of** $y=\ln x$ **is given below using scientific**

Calculator .

| $x$ | 0.5 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | -0.69 | 0 | 0.69 | 1.09 | 1.39 | 1.6 |




**Example.22**: Use MAPLE command to draw the graphs of the given function.

(a) Function $f(x)=-(x+2)^2$ with domain $[0\ ,-4]$

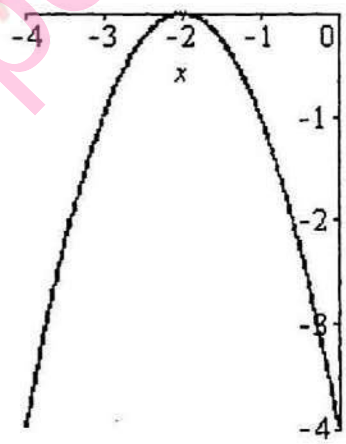
(b) Parametric function $(x(t),\ y(t))=(\cos(t),\ \sin(t))$ for $t=-3.5$ to $3.5$ , x from -1.5 to 1.5 and y from - 1.5 to 1.5 .

(c ) An implicit function $x^2-y^2=1$ , x from $-5$ to 5 and y from $-5$ to 5 .

**Solution**: (a)Open the MAPLE programme .Then open the new document . Use the keyboard keys and write $plot(-(x+2)\text{^}2,\ x=0..-4)$ ; . Press enter key . you get your result .

**Command.**

> $plot(-(x+2)^2, x=0..-4);$



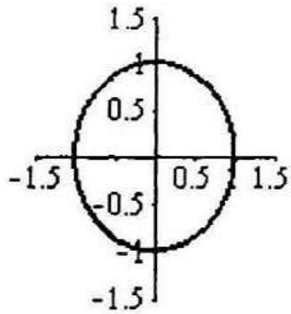
**(b)Open the MAPLE programme .Then open the new document . Use the keyboard keys and write**

$plot([\cos(t),\sin(t),t=-3.5..3.5],-1.5..1.5,-1.5..1.5)$ ;

**Press enter key . you get your result .**

**Command.**

> $plot([\cos(t),\sin(t),t=-3.5..3.5],-1.5..1.5,-1.5..1.5);$



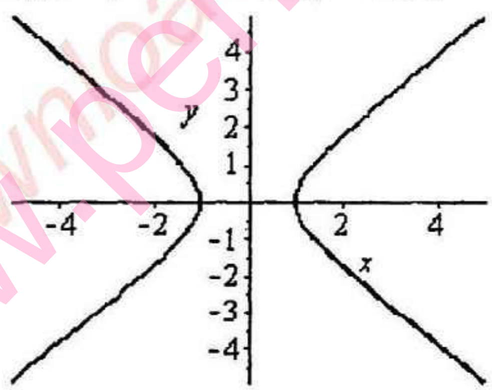

**(c)Open the MAPLE programme .Then open the new document . Use the keyboard keys and write**

$plots[implicitplot](x\text{^}2-y\text{^}2-1,x=-5..5,y=-5..5)$;

**Press enter key . you get your result .**

**Command.**

> $plots[implicitplot](x^2-y^2-1,x=-5..5,y=-5..5);$




**<u>Example 23</u>: Use MAPLE commands to draw the following functions .**

**(a)** $f(x)=x^2-b,\ x\ from -1\ to\ 10\ \ and\ \ y\ from\ -5\ to\ 5$

(b) $(x(t),\ y(t)) = \cos(tx).\sin(ty)$ , t from $-1$ to 2 , x from $-\pi$ to $\pi$ and y from $-\pi$ to $\pi$ .

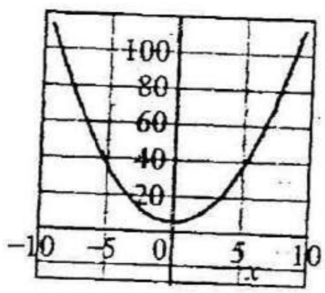
**Solution**: (a) Open the MAPLE programme .Then open the new document . Use the keyboard keys and write

$plots[animate](plot,[x\verb|^|2-b, x=10..10, b=-5..5);$

Press enter key . you get your result .

**Command.**

$plots[animate](plot,[x^2-b, x=-10..10], b=-5..5);$




(b) Open the MAPLE programme .Then open the new document . Use the keyboard keys and write

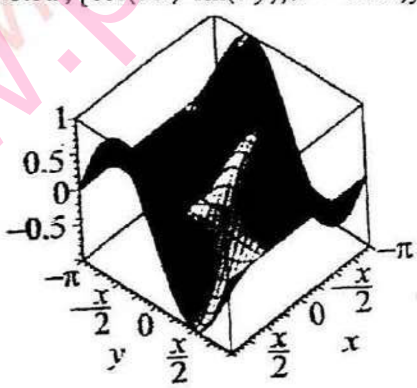
$plots[animate]('plot3d',[\cos(t*x)*\sin(t*y), x=-\pi..\pi,$

$y=-\pi...\pi], t=1..2);$

Press enter key . you get your result .

**Command:**

$plots[animate]('plot3d',[\cos(t\cdot x)\cdot\sin(t\cdot y), x=-\pi..\pi, y=-\pi..\pi], t=1..2);$



## Exercise 2.2:

**Q.1**: **Recognize and write the type for each of the following functions.**

(a) $y = a\sin^2 x + x$ (b) $y = 7x^4 + 3x^2 + 4x + 5$

(c) $y = \arctan x - 7$ (d) $y = \log_2 16 + 7\log_2(x)$

(e) $x^2 + y^2 = 36$ (f) $y = \dfrac{e^{3x} + e^{-3x}}{e^{3x} - e^{-3x}}$

(g) $y = \sqrt{x-8}$

**Solution**: (a) triagonometric (b) algebric
(c) Inverse triagonometric (d) logarithm
(e) Implicit (f) Hyperbolic (g) explicit

**Q.2**: **A sealed box cantains radium. The number of grams present at a time t is given by** $Q(t) = 100e^{-0.00043t}$.
**Where t is measured in years. Find the amount of radium in the box at the following times.**
(a) t = 0 (b) t = 800 (c) 1600 (d) 5000
(e) How did you guess from the above result.

**Solution**: $Q(t) = 100e^{-0.00043t}$ ......(i)

(a) Put $t = 0$ in (i) => $Q(0) = 100e^0 = 100 \times 1 = 100\ g$

(b) Put $t = 800$ in (i)
=> $Q(800) = 100e^{-(0.00043)(800)} = 100 \times 0.709 = 70.9g$

(c) Put $t = 1600$ in (i)
=> $Q(1600) = 100e^{-(0.00043)(1600)} = 100 \times 0.503 = 50.3g$

(d) Put $t = 5000$ in (i)
=> $Q(5000) = 100 \times e^{-(0.00043)(5000)} = 100 \times 0.116 = 11.6g$

(e) decline the number of radium grams is seen.

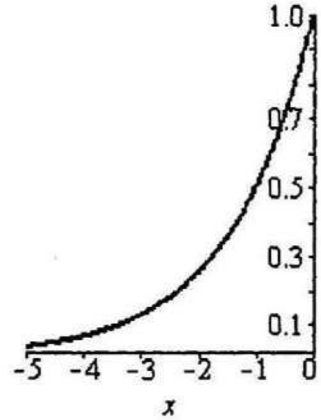
**Q.3**: **Using a calculator and point by point to plot the following exponential functions.**

(a) $h(x) = 2^x$ ; $[-5\ ,0]$

**Solution**:

The table of $h(x) = 2^x$ is below.

| $x$ | -5 | -4 | -3 | -2 | -1 | 0 |
|---|---|---|---|---|---|---|
| $h$ | **0.03125** | **0.0625** | **0.125** | **0.25** | **0.5** | **1** |

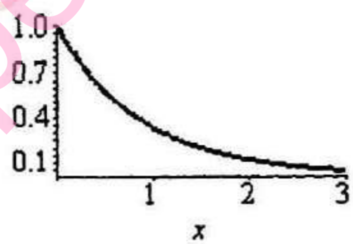



(b) $m(x) = 3^{-x}$ ; $[0\ ,3]$

**Solution:**

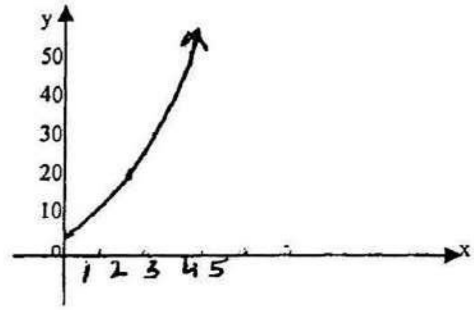
**The table of** $m(x) = 3^{-x}$ **is below .**

| $x$ | 0 | 1 | 2 | 3 | | |
|---|---|---|---|---|---|---|
| $m$ | **1** | **0.333** | **0.111** | **0.037** | | |



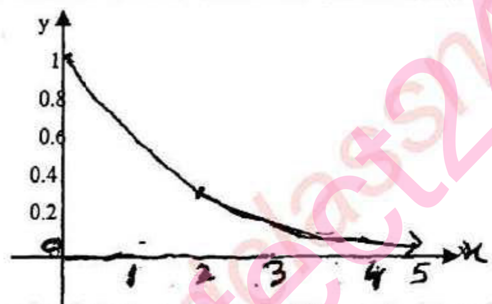

(c) $N = e^t$ , $[0\ ,5]$

| t | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $N = e^t$ | 1 | 2.718 | 7.389 | 20.08 | 54.6 | 148.4 |




(d) $N = e^{-t}$ , $[0\ ,5]$

| t | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $N = e^t$ | 1 | 0.368 | 0.135 | 0.05 | 0.018 | 0.0067 |



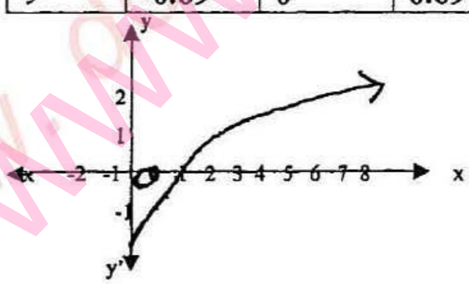

**Q.4: Using a calculator and point by point to plot the Following logarithm functions .**

**(a)** $y = \ln x$

**Solution:** As $x > 0$

**The table of $y = \ln x$ is given below using scientific Calculator .**

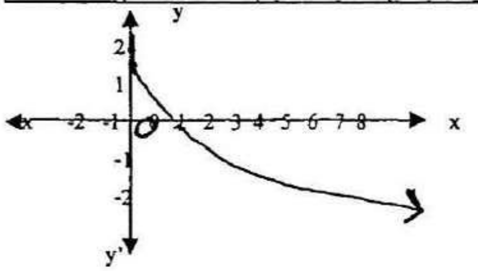
| $x$ | **0.5** | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|---|
| $y$ | **-0.69** | **0** | **0.69** | **1.09** | **1.39** | **1.6** |



(b) $u = -\ln x$

**Solution** :The table of $u = -\ln x$ is given below using scientific calculator .

| $x$ | 0.5 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | 0.69 | 0 | -0.69 | -1.09 | -1.39 | -1.6 |



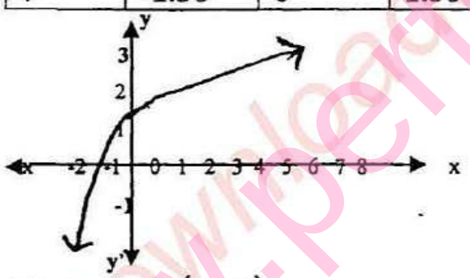

(c) $y = 2\ln(x+2)$

**Solution**: Here we take $x > -2$ because if $x \leq -2$

$\Rightarrow$ $x+2 \leq 0$ and $\ln(x+2)$ does not defined for $x+2 \leq 0$ .

The table of $y = 2\ln(x+2)$ is given below using scientific Calculator .

| $x$ | -1.5 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $y$ | -1.38 | 0 | 1.38 | 2.19 | 2.78 | 3.12 |

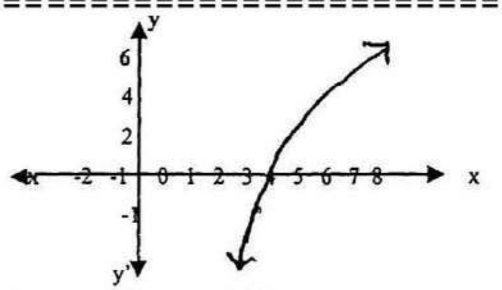



(d) $y = 4\ln(x-3)$

**Solution**: Here we take $x > 3$ because if $x \leq 3$ then $x-3 \leq 0$ and $\ln(x-3)$ does not defined for $x-3 \leq 0$ .

The table of $y = 4\ln(x-3)$ is given below using scientific Calculator .

| $x$ | 3.5 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| $y$ | -2.77 | 0 | 2.77 | 5.49 | 6,93 | 8.05 |

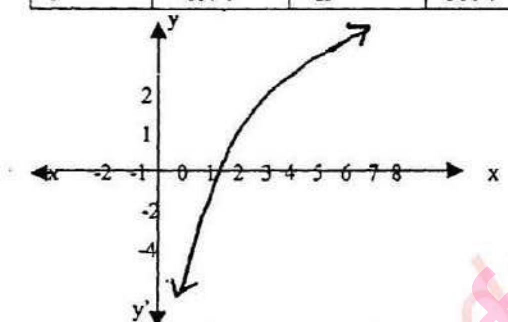



(e) $y = 4\ln x - 2$

**Solution:** $y = 4\ln x - 3$ **, Here** $x > 0$

**The table of** $y = 4\ln x - 2$ **is given below using scientific Calculator .**

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | 0.5 | 1 | 2 | 3 | 4 | 5 |
| $y$ | -4.77 | -2 | 0.77 | 2.39 | 3.54 | 4.43 |




**Q.5: Sketch the following parametric curves.**

**(a)** $(x(t)\ , y(t)) = (3 - t\ ,\ 2t)$ **, t is real number .**

**Solution: As** $(x(t)\ ,\ y(t)) = (3 - t\ ,\ 2t)$

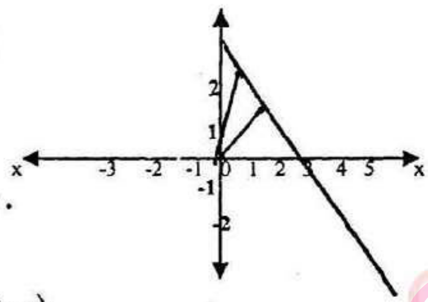
$\Rightarrow\ x = 3 - t \quad and \quad y = 2t$

> **Note:** $\dfrac{x-3}{-1} = t \quad and \quad \dfrac{y}{2} = t$ **Thus in symmetric form**
>
> $\Rightarrow \dfrac{x-3}{-1} = \dfrac{y}{2} = t$ **, where - 1 and 2 direction ratios**
>
> **of the vector parallel to the given curve .**

**Now the table is given**

| | | | | |
|---|---|---|---|---|
| $t$ | -2 | -1 | 0 | 1 |
| $x = 3 - t$ | 5 | 4 | 3 | 2 |
| $y = 2t$ | -4 | -2 | 0 | 2 |

Thus the graph of above function is obtained by plotting the points $(x\ ,\ y)$.
it is a straight line passing through the point P(3 ,0) and parallel to the vector u =(-1 , 2).




(b) $(x(t)\ ,\ y(t)) = (4\cos t,\ -3\sin t)$

**Solution:** As $x = 4\cos t$ and $y = -3\sin t$

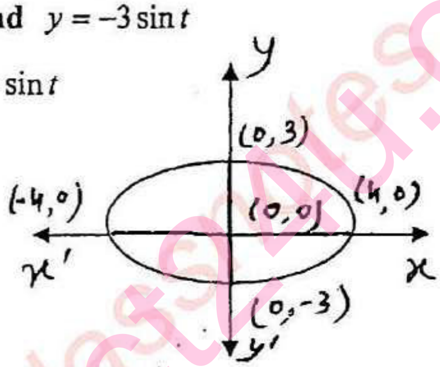
=> $\dfrac{x}{4} = \cos t$ *and* $\dfrac{y}{-3} = \sin t$

Squaring and adding

$$\frac{x^2}{16} + \frac{y^2}{9} = \cos^2 t + \sin^2 t$$

=> $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

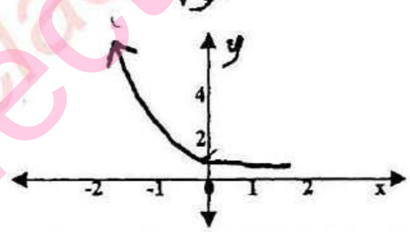
It is ellipse.




**Q.6**: Sketch the graph of :

(a) $f(x) = e^{-2x}$

**Solution:** $f(x) = e^{-2x}$

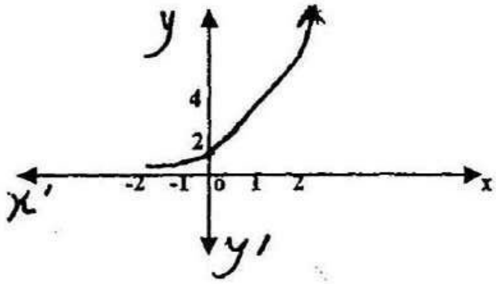



| x | -2 | - 1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 54.6 | 7.39 | 1 | 0.135 | 0.0183 |

**(b)**: $f(x) = \dfrac{2}{3}e^{3x}$

**Solution:** $f(x) = \dfrac{2}{3}e^{3x}$

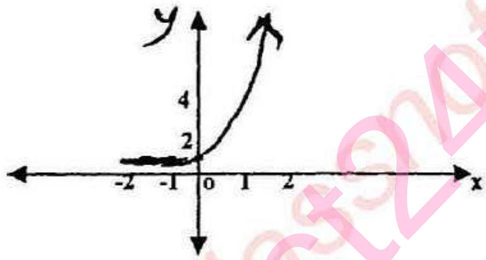
| x | -2 | - 1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 0.0017 | 0.03 | 0.67 | 13.46 | 270.3 |



**(c)** : $f(x) = 4^x$

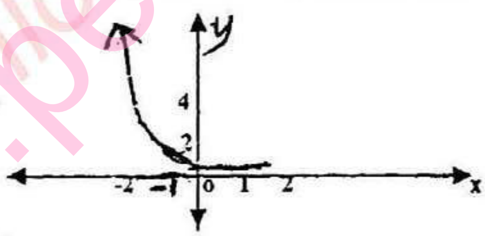
**Solution:** : $f(x) = 4^x$

| x | -2 | - 1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 0.0625 | 0.25 | 1 | 4 | 16 |



**(d):** $f(x) = 4^{-x}$

| x | -2 | - 1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 16 | 4 | 1 | 0.25 | 0.0625 |



**(e):** $f(x) = \log_2(x+5)$

**Solution: Let** $\log_2(x+5) = y$ **…(i)**

$\Rightarrow 2^y = x + 5$ **….(ii)**

**Put** $x = -4$ **in (ii)**

$\Rightarrow 2^y = -4 + 5 \Rightarrow 2^y = 1$

$\Rightarrow 2^y = 2^0 \Rightarrow y = 0$

**Put** $x = -3$ **in (ii)**

$\Rightarrow\ 2^y = -3+5 \ \Rightarrow\ 2^y = 2$

$\Rightarrow\ y = 1$

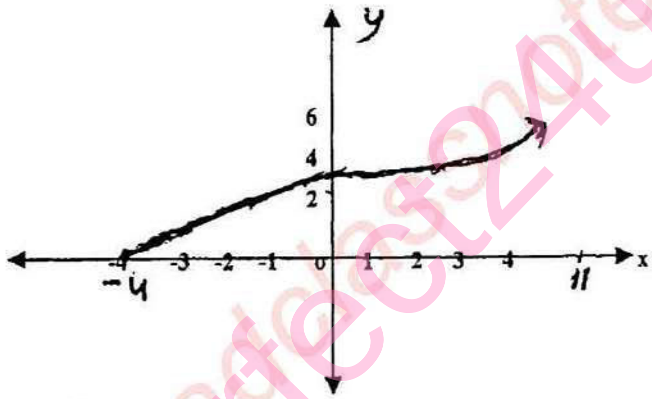
**Put** $x=-1$ **in (ii)** $\Rightarrow\ 2^y = -1+5$

$\Rightarrow\ 2^y = 4 \ \Rightarrow\ 2^y = 2^2 \ \Rightarrow\ y = 2$

**Put** $x=3$ **in (ii)** $\Rightarrow\ 2^y = 3+5$

$\Rightarrow\ 2^y = 8 \ \Rightarrow\ 2^y = 2^3 \ \Rightarrow\ y = 3$ **and son .**

| x | -4 | -3 | -1 | 3 | 11 |
|---|---|---|---|---|---|
| $f(x)$ | 0 | 1 | 2 | 3 | 4 |




**(f):** $f(x) = \log_2 x^2$

**Solution: Let** $\log_2 x^2 = y$ **...(i)** $\Rightarrow\ 2\log_2 x = y$

$\Rightarrow\ \log_2 x = \dfrac{y}{2} \Rightarrow\ 2^{\frac{y}{2}} = x$ **....(ii)**

**Put** $x=1$ **in (ii)**

$\Rightarrow\ 2^{\frac{y}{2}} = 1 \Rightarrow 2^{\frac{y}{2}} = 2^0 \Rightarrow\ \dfrac{y}{2} = 0 \ \Rightarrow\ y = 0$

**Put** $x=2$ **in (ii)**

$\Rightarrow\ 2^{\frac{y}{2}} = 2 \ \Rightarrow\ \dfrac{y}{2} = 1 \ \Rightarrow\ y = 2$

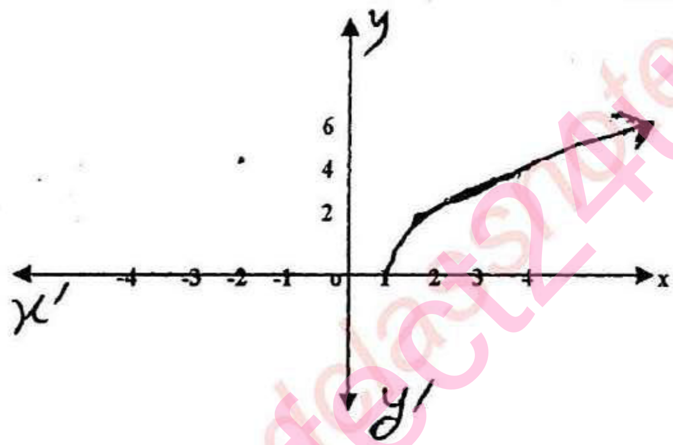
**Put** $x=4$ **in (ii)** $\Rightarrow\ 2^{\frac{y}{2}} = 4 \ \Rightarrow\ 2^{\frac{y}{2}} = 2^2$

$$\Rightarrow \frac{y}{2} = 2 \Rightarrow y = 4$$

**Put** $x = 8$ **in (ii)** $\Rightarrow 2^{\frac{y}{2}} = 8$

$\Rightarrow 2^{\frac{y}{2}} = 2^3 \Rightarrow \frac{y}{2} = 3 \Rightarrow y = 6$ **and son .**

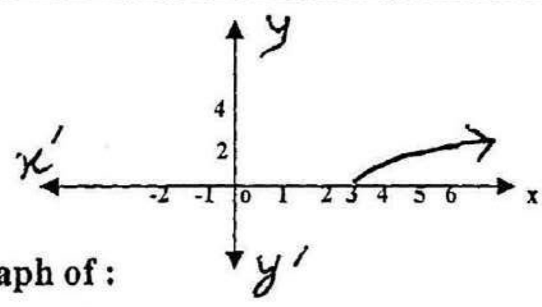
| x | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| $f(x)$ | 0 | 2 | 4 | 6 | 8 |




(g) $f(x) = \log_e(2x-5)$

**Solution:** $f(x) = \log_e(2x-5) = \ln(2x-5)$

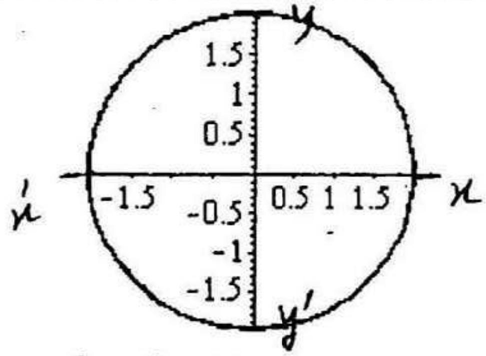
| x | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| $f(x)$ | 0 | 1.1 | 1.61 | 1.95 | 2.2 |




**Q.7: Sketch the graph of :**

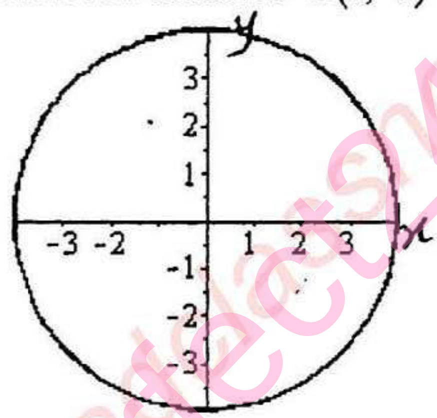
(a) $x^2 + y^2 = 4$

**Solution: It is a circle . Its center is** $C(0, 0)$ **and raidu is 2 .**



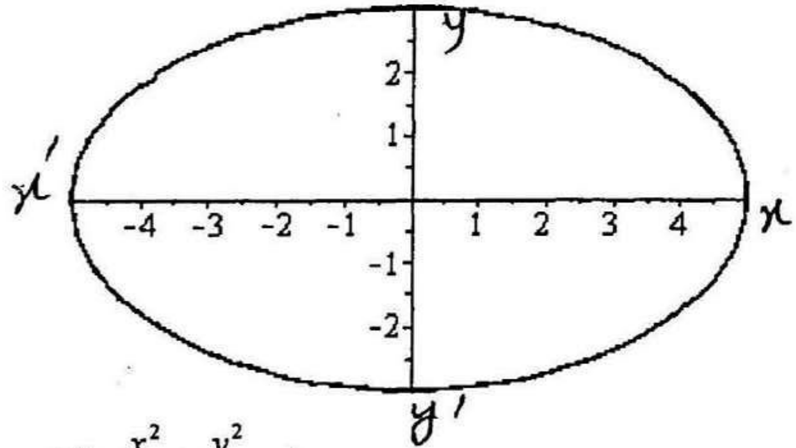

**(b):** $x^2 + y^2 = 16$

**Solution:** It is a circle . Its center is $C(0,\ 0)$ and raidu is 4 .



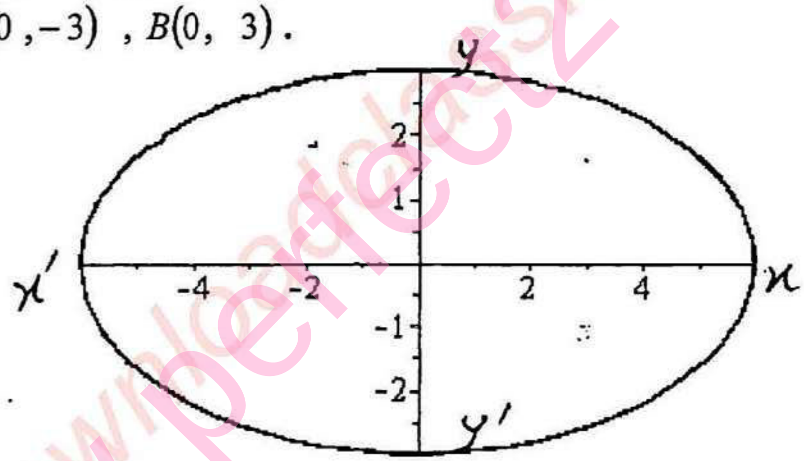

**(c) :** $\frac{x^2}{25} + \frac{y^2}{} = 1$

**Solution :** : is an ellipse with center C ( 0 , 0) , Vertices are $A'(\ 5\ ,0)$ , $A(5,\ 0)$ .Extrimities of minor axis are $B'(0\ ,-3)$ , $B(0,\ 3)$ .




**(d):** $\dfrac{x^2}{36}+\dfrac{y^2}{9}=1$

**Solution :** It is an ellipse with center C ( 0 , 0) , Vertices are $A'(-6\ ,0)$ , $A(6,\ 0)$ .Extrimities of minor axis are $B'(0\ ,-3)$ , $B(0,\ 3)$ .




**Q.8:** **Use maple command to plot the graph of the functions given in Q. 1:**

**(a)** $y = a\sin^2 x + x$ **(b)** $y = 7x^4 + 3x^2 + 4x + 5$

**(c )** $y = \arctan x - 7$ **(d)** $y = lo_2 16 + 7\log_2(x)$

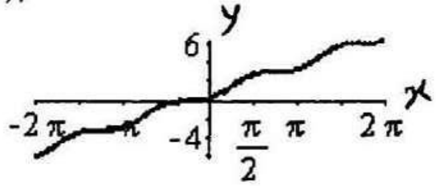
**(e)** $x^2 + y^2 = 36$ **(f)** $y = \dfrac{e^{3x}+e^{-3x}}{e^{3x}-e^{-3x}}$

**(g)** $y = \sqrt{x-8}$

**Solution:** (a) Open the Maple programme and open the new document and use keybord keys and write $plots(\sin\text{^}2(x), x)$; and use enter key you get the following result and graph.

**Command :**

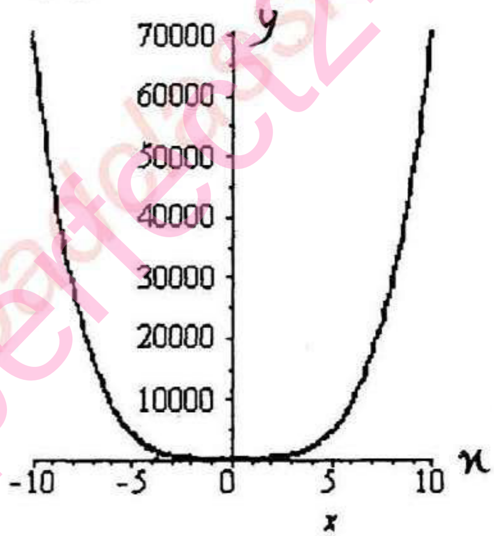
> $plot(\sin^2(x) + x, x);$




**(b): Open the Maple programme and open the new document and use keybord keys and write** $plots(7*x^4+3*x^2-4*x+5,x);$ **and use enter key you get the following result and graph.**

**Command :**

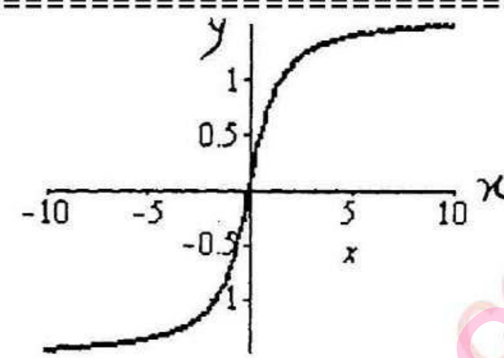
> $plot(7\cdot x^4 + 3\cdot x^2 - 4\cdot x + 5, x);$




**(c ): Open the Maple programme and open the new document and use keybord keys and write** $plots(7*x^4+3*x^2-4*x+5,x);$ **and use enter key you get the following result and graph.**

**Command :**

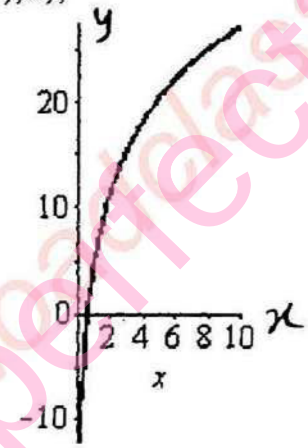
$plot(\arctan(x), x);$




**(d): Open the Maple programme and open the new document and use keybord keys and write** $plot(\log 2(16)+7*\log 2(x),x);$ **and use enter key you get the following result and graph .**

**Command :**

> $plot(log2(16)+7\cdot log2(x),x);$

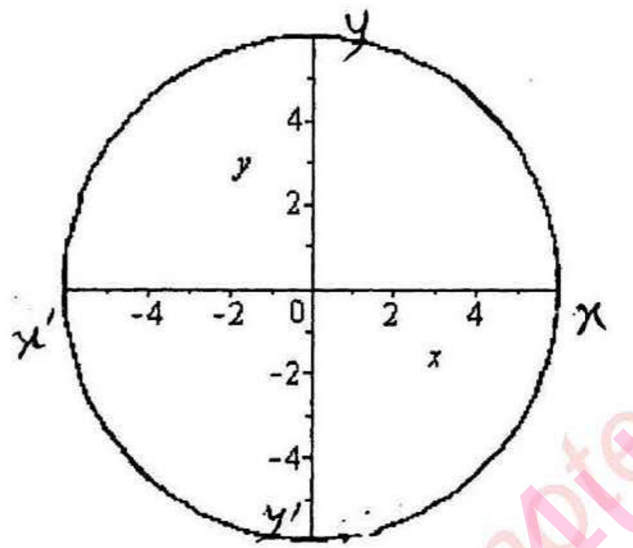



**(e): Open the Maple programme and open the new document and use keybord keys and write**

$plots[implicitplot(x\char`^2+y\char`^2=36,x=-6..6,\quad y=-6..6);$ **and use enter key you get the following result and graph .**

**Command :**

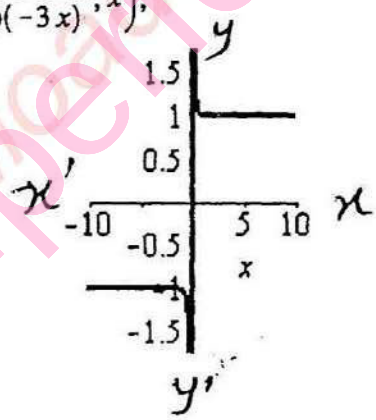
> $plots[implicitplot](x^2+y^2=36,x=-6..6,y=-6..6);$




**(f) Open the Maple programme and open the new document and use keybord keys and write** $plot(\exp(3x)+\exp(-3x)/\exp(3x)-\exp(-3x),x)$; **and use enter key you get the following result and graph .**

**Command :**

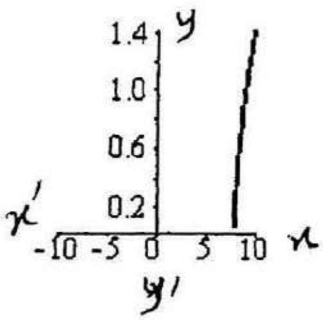
> $plot\left(\dfrac{\exp(3x)+\exp(-3x)}{\exp(3x)-\exp(-3x)},x\right);$




**(g) Open the Maple programme and open the new document and use keybord keys and write** $plot(sqrt(x-8),x)$; **and use enter key you get the following result and graph .**

**Command :**

```
> plot(sqrt(x − 8), x);
```




**Inequality :** (i) $\leq$ = less than or equal .
(ii) $\geq$ = greater than or equal .
(iii) $<$ = less than
(iv) $>$ = greater than .

**Intervals :** (i) $[a \quad b]$ is called closed interval and we write $a \leq x \leq b$ . it contains both $a$ and $b$ .

(ii) $(a \quad b]$ is called half open and half closed interval and we write $a < x \leq b$ . it contains b but does not contain $a$ .

(iii) $(a \quad b)$ is called open interval and we write $a < x < b$ . it does not contain $a$ and $b$ .

**Definition : Limit of a function:** Let $f(x)$ be a function and $a$ is a fixed point (constant number) if $\lim\limits_{x \to a} f(x) = l$ , then $l$ is called limit of $f(x)$ when $x$ approaches to $a$ .

**Some important results:**

(i) constant rule: $\lim\limits_{x \to c} k = k$ , where $k$ is any constant .

(ii) Limit rule: $\lim\limits_{x \to c} x = c$

(iii) Multiple rule: $\lim\limits_{x \to c} kf(x) = k \lim\limits_{x \to c} f(x)$

(iv) Sum or difference rule :

$$\lim_{x \to c}[f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

(v) Polynomial rule : If $p(x)$ is any polynomial then

$$\lim_{x \to c} p(x) = p(c) \text{ .}$$

**(vi) Product rule :**

$$\lim_{x \to c}[f(x).g(x)] = \lim_{x \to c} f(x).\lim_{x \to c} g(x)$$

**(vii) Quotient rule:** $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ , $\lim_{x \to c} g(x) \neq 0$

**(viii) Power rule:** $\lim_{x \to c}[f(x)]^n = [\lim_{x \to c} f(x)]^n$

**(ix) Equal function rule : If f(x) and g(x) are equal for all x , then** $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$ .

**Definition : Sequence:** A sequence $\{S_n\}$ is a function whose domain is the set of natural numbers and its range is the subset of real numbers . The functional notation of the sequence is $S(n)$: Where $n = 1, 2, 3, \ldots\ldots\ldots$

**Limit of sequence :** A sequence $\{S_n\}$ is converging to a real number $L$ if $\lim_{n \to \infty} S_n = L$ .

**Example .27:** Find the limit of each of these convergent / divergent sequence .

(a) $\left\{\dfrac{8n}{n+3}\right\}$ (b) $\left\{\dfrac{n^5 + n^3 + 2}{7n^4 + n^2 + 3}\right\}$ (c) $\{(-1)^n\}$

**Solution:** Note: if the limit of the form $\lim_{x \to \infty} \dfrac{f(x)}{g(x)}$ then divided the denominator and numerator by the highest power variable of the denominator .

(a) $\left\{\dfrac{8n}{n+3}\right\}$:

$$\lim_{n \to \infty}\left\{\frac{8n}{n+3}\right\} = \lim_{n \to \infty} \frac{\frac{8n}{n}}{\frac{n+3}{n}}$$ Divided denominator and Numerator by n .

$$\Rightarrow \lim_{n\to\infty}\left\{\frac{8n}{n+3}\right\} = \lim_{n\to\infty}\frac{\frac{8}{1}}{\frac{n}{n}+\frac{3}{n}} = \lim_{n\to\infty}\frac{8}{1+\frac{3}{n}} \quad \textit{Applying limits}$$

$$\Rightarrow \lim_{n\to\infty}\left\{\frac{8n}{n+3}\right\} = \frac{8}{1+\frac{3}{\infty}} = \frac{8}{1+0} = 8 \quad \textit{Ans.}$$

**It is convergent to 8 .**

**(b)** $\lim_{n\to\infty}\left\{\dfrac{n^5+n^3+2}{7n^4+n^2+3}\right\}$ **divided denominator and numerator by** $n^4$.

$$\lim_{n\to\infty}\left\{\frac{n^5+n^3+2}{7n^4+n^2+3}\right\} = \lim_{n\to\infty}\frac{\frac{n^5+n^3+2}{n^4}}{\frac{7n^4+n^2+3}{n^4}}$$

$$= \lim_{n\to\infty}\frac{\frac{n^5}{n^4}+\frac{n^3}{n^4}+\frac{2}{n^4}}{\frac{7n^4}{n^4}+\frac{n^2}{n^4}+\frac{3}{n^4}}$$

$$= \lim_{n\to\infty}\frac{n+\frac{1}{n}+\frac{2}{n^4}}{7+\frac{1}{n^2}+\frac{3}{n^4}} \quad \textbf{applying limits}$$

$$= \frac{\infty+\frac{1}{\infty}+\frac{2}{\infty^4}}{7+\frac{1}{\infty^2}+\frac{3}{\infty^4}} = \frac{\infty+0+0}{7+0+0} = \infty$$

**It is divergent .**

**(c) The sequence** $\{(-1)^n\}$ **is - 1 , 1 , - 1 , 1 , -1 , ........**

**Thus** $\lim_{n\to\infty}\{(-1)^n\} = \pm 1$ **since the limit is not unique**

**Thus the sequence is divergent .**

**Example. 28**: **If** $f(x)=x^2+2x+3$ **and** $g(x)=x-4$ **then calculate** $\lim_{x\to 2}[f(x)+g(x)]$

**Solution**: $\lim_{x\to 2}[f(x)+g(x)]=\lim_{x\to 2}[x^2+2x+3+x-4]$

$$=\lim_{x\to 2}[x^2+3x-1]$$

$$=(2)^2+3(2)-1=4+6-1=9 \quad \textbf{Ans.}$$

**Example.29**: **If** $f(x)=x-7$ **and** $g(x)=x^2+3x+2$, **then calculate** $\lim_{x\to 1}[f(x)-g(x)]$

**Solution**: $\lim_{x\to 1}[f(x)-g(x)]=\lim_{x\to 1}[x-7-(x^2+3x+2)]$

$$=\lim_{x\to 1}(x-7-x^2-3x-2)$$

$$=\lim_{x\to 1}(-x^2-2x-9)$$

$$=-(1)^2-2(1)-9$$

$$=-1-2-9=-12 \quad \textit{Ans.}$$

**Example. 30**: **If** $f(x)=x+5$ **and** $g(x)=2x-4$, **then calculate** $\lim_{x\to 3}[f(x).g(x)]$

**Solution**: $\lim_{x\to 3}[f(x).g(x)]=\lim_{x\to 3}[(x+5)(2x-4)]$

$$=\lim_{x\to 3}(2x^2-4x+10x-20)$$

$$=\lim_{x\to 3}(2x^2+6x-20)$$

$$=2(3)^2+6(3)-20$$

$$=18+18-20=16 \quad \textit{Ans.}$$

**Example. 31**: **If** $f(x)=x-4$ **and** $g(x)=x^2+3$ **then calculate** $\lim_{x\to -2}\left[\frac{f(x)}{g(x)}\right]$

**Solution**: $\lim_{x\to -2}\left[\frac{f(x)}{g(x)}\right]=\lim_{x\to -2}\left[\frac{x-4}{x^2+3}\right]$

$$= \frac{-2-4}{(-2)^2+3} = -\frac{6}{4+3}$$

$$= -\frac{6}{7} \quad \textbf{Ans.}$$

**Important limits:** (i) $\lim_{x \to a}\left(\frac{x^n - a^n}{x-a}\right) = na^{n-1}$

> **Note:** $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
> $a^4 - b^4 = (a-b)(a^3+a^2b+ab^2+b^3)$
> $a^n - b^n = (a-b)(a^{n-1}+a^{n-2}b+.........+ab^{n-2}+b^{n-1})$

**Proof**: if we apply direct the limit we get of the form $\frac{0}{0}$.

$$\lim_{x \to a}\left(\frac{x^n - a^n}{x-a}\right) = \lim_{x \to a}\left(\frac{(x-a)(x^{n-1}+ax^{n-2}+.......+a^{n-2}x+a^{n-1})}{x-a}\right)$$

$$= \lim_{x \to a}(x^{n-1}+ax^{n-2}+.......+a^{n-2}x+a^{n-1})$$

$$= a^{n-1}+a.a^{n-2}+......+a^{n-2}.a+a^{n-1}$$

$$= a^{n-1}+a^{n-1}+a^{n-1}+......+a^{n-1} \ (n-times)$$

$$= na^{n-1} \quad Ans.$$

(ii) $\lim_{x \to a}\left(\frac{x-a}{\sqrt{x}-\sqrt{a}}\right) = 2\sqrt{a}$

**Proof** : if we apply direct the limit we get of the form $\frac{0}{0}$.

Then multiply and divided by $\sqrt{x}+\sqrt{a}$

$$\lim_{x \to a}\left(\frac{x-a}{\sqrt{x}-\sqrt{a}}\right) = \lim_{x \to a}\frac{x-a}{\sqrt{x}-\sqrt{a}} \times \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$$

$$= \lim_{x \to a} \frac{(x-a)(\sqrt{x}+\sqrt{a})}{(\sqrt{x})^2 - (\sqrt{a})^2}$$

$$= \lim_{x \to a} \frac{(x-a)(\sqrt{x}+\sqrt{a})}{x-a}$$

$$= \lim_{x \to a} (\sqrt{x}+\sqrt{a})$$

$$= \sqrt{a}+\sqrt{a}$$

$$= 2\sqrt{a} \quad Ans.$$

**(iii):** $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e$

**(iv)** $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

**Proof:** As we know that $\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = e$ ......(i)

**Let** $\frac{1}{n} = x$ *then* $n = \frac{1}{x}$

*As* $n \to \infty$ *Then* $x \to 0$ **put in the L . H . S. of (i)**

$\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$ **Proved .**

**(v)** $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a \quad , a > 0$

**Proof:** **Let** $a^x - 1 = y \quad => a^x = 1 + y$

**Take ln of both sides**

$$\log a^x = \log(1+y)$$

$$=> x \log a = \log(1+y)$$

$$=> \quad x = \frac{\log(1+y)}{\log a}$$

**As** $x \to 0$ *and* $y = a^x - 1$ *then* $y \to 0$

*Now* $\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{x\to 0}\frac{y}{\frac{\log(1+y)}{\log a}}$

$$= \lim_{x\to 0}\frac{\log a}{\frac{1}{y}\log(1+y)}$$

$$= (\log a)\lim_{x\to 0}\frac{1}{\log(1+y)^{\frac{1}{y}}}$$

$$= \frac{\log a}{\log \lim_{x\to 0}(1+y)^{\frac{1}{y}}} \qquad As \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

$$\Rightarrow \lim_{x\to 0}\frac{a^x-1}{x} = \frac{\log a}{\log e}$$

$$= \log_e a$$

$$= \ln a$$

*As by logaraithm rule*

$$\frac{\log_b m}{\log_b n} = \log_n m$$

**(vi):** $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

**Proof : As** $\lim_{x\to 0}\frac{a^x-1}{x} = \ln a \quad , a>0$ **put** $a = e$

$$\lim_{x\to 0}\frac{e^x-1}{x} = \ln e = 1$$

**(vii) Prove that** $\lim_{x\to a}\frac{\sqrt{x+a}-\sqrt{a}}{x} = \frac{1}{2\sqrt{a}}$

**Proof : if we apply direct the limit we get of the form** $\frac{0}{0}$ **.**

**Then multiply and divided by** $\sqrt{x+a}+\sqrt{a}$

$$\lim_{x\to 0}\left(\frac{\sqrt{x+a}-\sqrt{a}}{x}\right)=\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}\times\frac{\sqrt{x+a}+\sqrt{a}}{\sqrt{x+a}+\sqrt{a}}$$

$$=\lim_{x\to 0}\frac{\left(\sqrt{x+a}\right)^2-\left(\sqrt{a}\right)^2}{x\left(\sqrt{x+a}+\sqrt{a}\right)}$$

$$=\lim_{x\to 0}\frac{x+a-a}{x\left(\sqrt{x+a}+\sqrt{a}\right)}$$

$$=\lim_{x\to 0}\frac{x}{x\left(\sqrt{x+a}+\sqrt{a}\right)}=\lim_{x\to 0}\frac{1}{\left(\sqrt{x+a}+\sqrt{a}\right)}$$

$$=\frac{1}{\sqrt{0+a}+\sqrt{a}}=\frac{1}{\sqrt{a}+\sqrt{a}}$$

$$=\frac{1}{2\sqrt{a}}\quad = R.H.S.$$

**(viii)** $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=n$

**Proof:** Let $(1+x)^n-1=z$ ....(i)

$\Rightarrow\ (1+x)^n=1+z$

$\log(1+x)^n=\log(1+z)$

$\Rightarrow n\log(1+x)=\log(1+z)$ ......(ii)

**As** $x\to 0$ *then* $z\to 0$ **because** $(1+x)^n-1=z$

*Now* $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=\lim_{z\to 0}\frac{z}{x}$, $\times$ *and* $\div$ *by* $\log(1+z)$

$$=\lim_{z\to 0}\frac{z}{x}\times\frac{\log(1+z)}{\log(1+z)}$$

$$=\lim_{z\to 0}\frac{z}{\log(1+z)}\lim_{z\to 0}\frac{\log(1+z)}{x}\ ....(iii)$$

**Put (ii) in (iii)**

$$\lim_{x\to0}\frac{(1+x)^n-1}{x}=\lim_{z\to0}\frac{z}{\log(1+z)}.\lim_{x\to0}\frac{n\log(1+x)}{x}$$

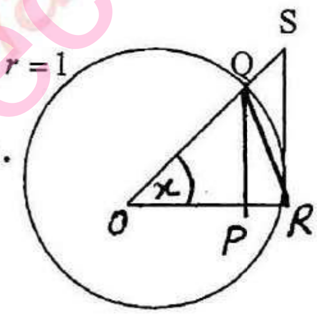
$$=\lim_{z\to0}\frac{1}{\frac{1}{z}\log(1+z)}.n\lim_{x\to0}\frac{1}{x}\log(1+x)$$

$$=n\lim_{z\to0}\frac{1}{\log(1+z)^{\frac{1}{z}}}\lim_{x\to0}\log(1+x)^{\frac{1}{x}}$$

$$=n\frac{1}{\log\lim_{z\to0}(1+z)^{\frac{1}{z}}}\log\lim_{x\to0}(1+x)^{\frac{1}{x}}$$

$$=n\frac{1}{\log e}.\log e=n \qquad \text{Proved.}$$

**(ix) Prove that** $\lim_{x\to0}\frac{\sin x}{x}=1$ **, x is an angle measured in radian .**




**Proof:Consider a circle of radius** $r=1$

**Let** $x$ **be the positive centeral angle measured in radiain .**

**From** $\Delta OPQ$

$$\sin x=\frac{|QP|}{|QO|}\quad \text{....(i)}$$

**But** $|QO|=r=1$ **= radius of the circle .....(ii)**

**Put (ii) in (i)**

$\sin x=|QP|$ **.....(iii)**

**From** $\Delta ORS$ , $\tan x=\frac{|RS|}{|OR|}$**....(iv)**

**But** $|OR|=r=1$ **= radius of the circle .....(v)**

**Put (v) in (iv) =>** $\tan x=|RS|$**....(vi) .**

$\overline{OQ}$ produce to S, so that $SR$ perpendicular to $RO$.
Join Q and R.

Area of $\Delta ORQ = \frac{1}{2}|OR||PQ|$ , put (iii) and (v)

Area of $\Delta ORQ = \frac{1}{2}.1.\sin x = \frac{1}{2}\sin x$ ......(a)

Area of sector $ORQ = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2.x$ , put $r = 1$, $\theta = x$

Area of sector $ORQ = \frac{1}{2}x$ ......(b)

Area of $\Delta ORS = \frac{1}{2}|OR||RS|$ , Put (v) and (vi)

Area of $\Delta ORS = \frac{1}{2}.1.\tan x$ ......(c)

From the figure it is clear that

Area of $\Delta ORQ$ < Area of sector $ORQ$ < Area of $\Delta ORS$

From (a), (b) and (c)

$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x$ multiplied by 2

$\Rightarrow \sin x < x < \frac{\sin x}{\cos x}$ ......(d)

Since sinx is positive then ineq(d) divided by $\sin x$

$\Rightarrow \frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\sin x}{\sin x.\cos x}$

$\Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$

$\Rightarrow \cos x < \frac{\sin x}{x} < 1$

| |
|---|
| use result if $a, b, c > 0$ and $a < b < c$ then $\frac{1}{a} > \frac{1}{b} > \frac{1}{c}$ |

$\lim_{x\to 0} \cos x < \lim_{x\to 0} \frac{\sin x}{x} < \lim_{x\to 0} 1$

=> $\cos 0 < \lim_{x\to 0}\frac{\sin x}{x} < 1$

=> $1 < \lim_{x\to 0}\frac{\sin x}{x} < 1$

=> $\lim_{x\to 0}\frac{\sin x}{x} = 1$ **proved .**

**Example. 32 .: Evaluate** $\lim_{x\to 1}\frac{x^3-3x+7}{5x^2+9x+6}$

**Solution:** $\lim_{x\to 1}\frac{x^3-3x+7}{5x^2+9x+6} = \frac{\lim_{x\to 1}(x^3-3x+7)}{\lim_{x\to 1}(5x^2+9x+6)}$

$$= \frac{(1)^3-3(1)+7}{5(1)^2+9(1)+6}$$

$$= \frac{1-3+7}{5+9+6}$$

$$= \frac{5}{20} = \frac{1}{4} \quad Ans.$$

**Example .33: Evaluate** $\lim_{x\to -2}\sqrt[3]{(x^2-3x-2)}$

**Solution:** $\lim_{x\to -2}\sqrt[3]{(x^2-3x-2)} = \lim_{x\to -2}(x^2-3x-2)^{\frac{1}{3}}$

$$= \left(\lim_{x\to -2}(x^2-3x-2)\right)^{\frac{1}{3}}$$

$$= \left((-2)^2-3(-2)-2\right)^{\frac{1}{3}}$$

$$= (4+6-2)^{\frac{1}{3}}$$

$$= (8)^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2 \quad Ans.$$

**Example .34: Evaluate the limits.**

**(a)** $\lim_{x\to 0}\sin^2 x$ **(b)** $\lim_{x\to 0}(1-\cos x)$

**Solution : (a)** $\lim_{x\to 0}\sin^2 x = \left[\lim_{x\to 0}\sin x\right]^2 = (\sin 0)^2 = 0$

(b) $\lim_{x\to 0}(1-\cos x)=1-\cos 0=1-1=0$ *Ans.*

**MAPLE command to evaluate limnit of a function:**

**Example. 35: (a)** $f(x)=x^2+2x+2$ **, when x tends to 2 .**

**(b)** $f(x)=(2x^3)(x-4)$ **, when x tends to 3 .**

**(c)** $f(x)=\dfrac{x^3-a^3}{x-a}$ **, when x tends to** $a$ **.**

**Solution; (a) Open the MAPLE programme and the open new document . Use keyboard and write**

**limit**$((x\^2)+(2*x+2),x=2)$; **then press enter key and get**

**Command:**

> $limit((x^2)+(2\cdot x+2),x=2);$

$$10$$

**(b) Open the MAPLE programme and the open new document . Use keyboard and write**

**limit**$((2*x\^3)*(x-4),x=3)$ **then press enter key and get**

**Command:**

> $limit((2\cdot x^3)\cdot(x-4),x=3);$

$$-54$$

**(c ) Open the MAPLE programme and the open new document . Use keyboard and write**

**limit**$((x\^3-a\^3)/(x-a),x=a)$; **then press enter key and get**

**Command:**

> $limit\left(\dfrac{(x^3-a^3)}{(x-a)},x=a\right);$

$$3a^2$$

**Exercise 2.3:**

**Q.1: Evaluate the following limits :**

**(a)** $\lim_{x\to 4}\left(\dfrac{1}{x}+\dfrac{3}{x-5}\right)$

**Solution:** $\lim_{x\to 4}\left(\dfrac{1}{x}+\dfrac{3}{x-5}\right)=\lim_{x\to 4}\left(\dfrac{1}{x}\right)+\lim_{x\to 4}\left(\dfrac{3}{x-5}\right)$

$$= \frac{1}{4} + \frac{3}{4-5} = \frac{1}{4} - \frac{3}{1}$$
$$= \frac{1-12}{4} = -\frac{11}{4} \quad Ans.$$

**(b)** $\lim\limits_{x \to 1}\left(\frac{x^2+3x+2}{x^2+x+2}\right)^2$

**Solution:** $\lim\limits_{x \to 1}\left(\frac{x^2+3x+2}{x^2+x+2}\right)^2 = \left[\lim\limits_{x \to 1}\left(\frac{x^2+3x+2}{x^2+x+2}\right)\right]^2$

$$= \left(\frac{\lim\limits_{x \to 1}(x^2+3x+2)}{\lim\limits_{x \to 1}(x^2+x+2)}\right)^2$$
$$= \left(\frac{(2)^2+3(2)+2}{(2)^2+2+2}\right)^2$$
$$= \left(\frac{4+6+2}{4+4}\right)^2 = \left(\frac{12}{8}\right)^2$$
$$= \left(\frac{3}{2}\right)^2 = \frac{9}{4} \quad Ans.$$

**(c)** $\lim\limits_{x \to 1}\frac{\sqrt{x}-1}{x-1}$

**Solution:** **multiplying and divided by** $\sqrt{x}+1$

$$\lim_{x \to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x \to 1}\frac{\sqrt{x}-1}{x-1} \times \frac{\sqrt{x}+1}{\sqrt{x}+1}$$
$$= \lim_{x \to 1}\frac{\left(\sqrt{x}\right)^2-(1)^2}{(x-1)\left(\sqrt{x}+1\right)}$$
$$= \lim_{x \to 1}\frac{x-1}{(x-1)\left(\sqrt{x}+1\right)} = \lim_{x \to 1}\frac{1}{\sqrt{x}+1}$$

$$= \frac{1}{\sqrt{1}+1} = \frac{1}{1+1} = \frac{1}{2} \quad Ans.$$

**(d)** $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

**Solution:** $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1} = \lim\limits_{x\to 1} \dfrac{1}{x-1}\left(\dfrac{1}{x}-1\right)$

$$= \lim_{x\to 1} \frac{1}{(x-1)}\left(\frac{1-x}{x}\right)$$

$$= \lim_{x\to 1} \frac{-1}{(x-1)}\left(\frac{x-1}{x}\right)$$

$$= - = \lim_{x\to 1} \frac{1}{x} = -\frac{1}{1} = -1 \quad Ans.$$

**(e)** $\lim\limits_{x\to 0} \dfrac{1-\sin x}{\cos^2 x}$

**Solution:** $\lim\limits_{x\to 0} \dfrac{1-\sin x}{\cos^2 x} = \lim\limits_{x\to 0} \dfrac{1-\sin x}{1-\sin^2 x}$

$$= \lim_{x\to 0} \frac{1-\sin x}{(1-\sin x)(1+\sin x)} = \lim_{x\to 0} \frac{1}{1+\sin x}$$

$$= \frac{1}{1+\sin 0} = \frac{1}{1+0} = 1 \quad Ans.$$

**(f)** $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$

**Solution:** $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = \lim\limits_{x\to 0} \dfrac{\sin x}{x\cos x}$

$$= \lim_{x\to 0} \frac{\sin x}{x} . \lim_{x\to 0} \frac{1}{\cos x}$$

$$\boxed{As \ \lim_{x\to 0} \frac{\sin x}{x} = 1}$$

$$=(1)\left(\frac{1}{\cos 0}\right)$$

$$=(1)\left(\frac{1}{1}\right)=1 \quad Ans$$

**(g)** $\lim_{x\to 0}\frac{\sec x-1}{x\sec x}$

**Solution:** $$\lim_{x\to 0}\frac{\sec x-1}{x\sec x}=\lim_{x\to 0}\frac{\frac{1}{\cos x}-1}{\frac{x}{\cos x}}$$

$$=\lim_{x\to 0}\frac{\frac{1-\cos x}{\cos x}}{\frac{x}{\cos x}}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{\cos x}\times\frac{\cos x}{x}$$

$$\Rightarrow \lim_{x\to 0}\frac{\sec x-1}{x\sec x}=\lim_{x\to 0}\frac{1-\cos x}{x} \quad ,\times \text{ and } \div \text{ by } 1+\cos x$$

$$=\lim_{x\to 0}\frac{1-\cos x}{x}\times\frac{1+\cos x}{1+\cos x}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}.\frac{\sin x}{1+\cos x}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}.\lim_{x\to 0}\frac{\sin x}{1+\cos x}$$

$$=(1)\left(\frac{\sin 0}{1+\cos 0}\right)=(1)(0)$$

$$=0 \quad Ans.$$

(h) $\lim\limits_{x\to 0}\dfrac{\sin^2(7x)}{7x}$

**Solution :** $\lim\limits_{x\to 0}\dfrac{\sin^2(7x)}{7x}=\lim\limits_{x\to 0}\dfrac{\sin^2(7x)}{7x}\times\dfrac{7x}{7x}$

$$=\lim_{x\to 0}\frac{\sin^2(7x)}{(7x)^2}\times\lim_{x\to 0}(7x)$$

$$=\left(\lim_{x\to 0}\frac{\sin(7x)}{7x}\right)^2.(7(0))$$

$$=(1)^2.0=0 \quad \textbf{Ans.}$$

**Q.2: Use algebra and the rules of limits to evaluate the following limits.**

(a) $\lim\limits_{x\to 4}\dfrac{-6}{(x-4)^2}$

**Solution :** $\lim\limits_{x\to 4}\dfrac{-6}{(x-4)^2}=\dfrac{-6}{(4-4)^2}$

$$=\frac{-6}{0}=-\infty$$

**Thus limit does not exist.**

(b) $\lim\limits_{x\to 0}\dfrac{\left[\dfrac{1}{x+3}-\dfrac{1}{3}\right]}{x}$

**Solution:** $\lim\limits_{x\to 0}\dfrac{\left[\dfrac{1}{x+3}-\dfrac{1}{3}\right]}{x}=\lim\limits_{x\to 0}\dfrac{1}{x}\left[\dfrac{1}{x+3}-\dfrac{1}{3}\right]$

$$=\lim_{x\to 0}\frac{1}{x}\left[\frac{3-x-3}{3(x+3)}\right]$$

$$=\lim_{x\to 0}\frac{-x}{3x(x+3)}$$

$$=\lim_{x\to 0}\frac{-1}{3(x+3)}=-\frac{1}{3(0+3)}=-\frac{1}{9} \quad \textit{Ans.}$$

(c) $\lim\limits_{x\to 5}\dfrac{\sqrt{x}-\sqrt{5}}{x-5}$

**Solution:** Divided and multiplied by $\sqrt{x}+\sqrt{5}$

$$\lim_{x\to 5}\frac{\sqrt{x}-\sqrt{5}}{x-5}=\lim_{x\to 5}\frac{\sqrt{x}-\sqrt{5}}{x-5}\times\frac{\sqrt{x}+\sqrt{5}}{\sqrt{x}+\sqrt{5}}$$

$$=\lim_{x\to 5}\frac{\left(\sqrt{x}\right)^2-\left(\sqrt{5}\right)^2}{(x-5)\left(\sqrt{x}+\sqrt{5}\right)}$$

$$=\lim_{x\to 5}\frac{x-5}{(x-5)\left(\sqrt{x}+\sqrt{5}\right)}$$

$$=\lim_{x\to 5}\frac{1}{\sqrt{x}+\sqrt{5}}=\frac{1}{\sqrt{5}+\sqrt{5}}$$

$$=\frac{1}{2\sqrt{5}} \qquad Ans.$$

**Q.3: Find the limit of the convergent of the following Sequences .**

(a) $\left\{\dfrac{5n}{n+7}\right\}$

**Solution:** $\lim\limits_{n\to\infty}\left\{\dfrac{5n}{n+7}\right\}=\lim\limits_{n\to\infty}\dfrac{5n}{n\left(1+\dfrac{7}{n}\right)}$

$$=\lim_{n\to\infty}\frac{5}{1+\dfrac{7}{n}}$$

$$\Rightarrow \lim_{n\to\infty}\left\{\frac{5n}{n+7}\right\}=\frac{5}{1+\dfrac{7}{\infty}}=\frac{5}{1+0}$$

$$=5 \qquad Ans.$$

(b) $\left\{\dfrac{4-7n}{8+n}\right\}$

**Solution:** $\lim_{n\to\infty}\left\{\dfrac{4-7n}{8+n}\right\} = \lim_{n\to\infty}\dfrac{n\left(\dfrac{4}{n}-7\right)}{n\left(\dfrac{8}{n}+1\right)}$

$$= \lim_{n\to\infty}\frac{\dfrac{4}{n}-7}{\dfrac{8}{n}+1}$$

$$\Rightarrow \lim_{n\to\infty}\left\{\frac{4-7n}{8+n}\right\} = \frac{\dfrac{4}{\infty}-7}{\dfrac{8}{\infty}+1} = \frac{0-7}{0+1}$$

$$= -7 \qquad Ans.$$

(c) $\dfrac{(-1)^n}{n}$

**Solution:** $\lim_{n\to\infty}\left\{\dfrac{(-1)^n}{n^2}\right\} = \left[\lim_{n\to\infty}(-1)^n\right]\lim_{n\to\infty}\dfrac{1}{n^2}$ .....(i)

**But** $\lim_{n\to\infty}(-1)^n$ **does not approaches to a single value .**

**Then** $\lim_{n\to\infty}(-1)^n$ **does not exist . But** $\lim_{n\to\infty}\dfrac{1}{n^2} - = \dfrac{1}{\infty} = 0$

**Thus from (i)** $\lim_{n\to\infty}\left\{\dfrac{(-1)^n}{n^2}\right\} = \textit{does not exist} \times 0$

**= does not exist .**

**Q.4: Weekly sales (in rupees) at big store $x$ weeks after the end of an advertising campaign are given by :**

$S(x) = 5000 + \dfrac{3600}{x+2}$ **.Find the sale for the indicated weeks limits :**

**(a)** $S(2)$ **(b)** $\lim_{x\to 5} S(x)$ **(c)** $\lim_{x\to 16} S(x)$

**Solution :** $S(x) = 5000 + \frac{3600}{x+2}$ ........(i)

(a) *Put* $x = 2$ in (i)

$$S(2) = 5000 + \frac{3600}{2+2} = 5000 + \frac{3600}{4}$$

$$= 5000 + 9000 = 14000 \quad Ans.$$

(b) $\lim\limits_{x \to 5} S(x) = \lim\limits_{x \to 5}\left(5000 + \frac{3600}{x+2}\right) = 5000 + \frac{3600}{5+2}$

$$= 5000 + 514.28 = 5514.28 \quad Ans.$$

(c) $\lim\limits_{x \to 16} S(x) = \lim\limits_{x \to 5}\left(5000 + \frac{3600}{x+2}\right) = 5000 + \frac{3600}{16+2}$

$$= 5000 + 200 = 50200 \quad Ans.$$

**Q.5: Use MAPLE command "limit" to evaluate the limit of All parts of Q.1:**

(a) $\lim\limits_{x \to 4}\left(\frac{1}{x} + \frac{3}{x-5}\right)$

**Solution: Open the MAPLE programme and the open new document . Use keyboard and write**

limit $((3/(x) + 1/(x-5), x = 4)$; **then press enter key and get**

**Command:**

> $limit\left(\frac{3}{(x)} + \frac{1}{(x-5)}, x = 4\right);$

$$-\frac{1}{4}$$

(b) $\lim\limits_{x \to 1}\left(\frac{x^2 + 3x + 2}{x^2 + x + 2}\right)^2$

**Solution: Open the MAPLE programme and the open new document . Use keyboard and write**

$\lim it(((x\char`^2) + (3 * x + 2)/(x\char`^2) + (x + 2), x = 1));$

**then press enter key and get**

**Command:**

$$> limit\left(\left(\frac{(x^2)+(3\cdot x+2)}{(x^2)+(x+2)}\right)^2, x=1\right);$$

$$\frac{9}{4}$$

**(c)** $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}$

**Solution:** **Open the MAPLE programme and the open new document. Use keyboard and write**

$\lim it(((sqrt(x))-1)/(x-1), x=1);$

**then press enter key and get**

**Command:**

$$> limit\left(\frac{((\text{sqrt}(x))-1)}{(x-1)}, x=1\right);$$

$$\frac{1}{2}$$

**(d)** $\lim\limits_{x\to 1}\dfrac{\frac{1}{x}-1}{x-1}$

**Solution:** **Open the MAPLE programme and the open new document. Use keyboard and write**

$\lim it(((1/x)-1)/(x-1)), x=1);$

**then press enter key and get**

**Command:**

$$> limit\left(\frac{\left(\left(\frac{1}{x}\right)-1\right)}{(x-1)}, x=1\right);$$

$$-1$$

**(e)** $\lim\limits_{x\to 0}\dfrac{1-\sin x}{\cos^2 x}$

**Solution:** **Open the MAPLE programme and the open new document. Use keyboard and write**

$\lim it((1-\sin(x))/(\cos\verb|^|2(x)), x=0);$

**then press enter key and get**

**Command:**

$$> limit\left(\frac{(1-\sin(x))}{(\cos^2(x))}, x=0\right);$$

$$1$$

**(f)** $\lim\limits_{x\to 0}\frac{\tan x}{x}$

**Solution:** Open the MAPLE programme and the open new document . Use keyboard and write $\lim it((\tan(x)/(x), x=0)$; then press enter key and get

**Command:**

$$> limit\left(\frac{(\tan(x))}{(x)}, x=0\right);$$

$$1$$

**(g)** $\lim\limits_{x\to 0}\frac{\sec x-1}{x\sec x}$

**Solution:** Open the MAPLE programme and the open new document . Use keyboard and write $\lim it((\sec(x)-1)/((x)*(\sec(x))), x=0)$;
then press enter key and get

**Command:**

$$> limit\left(\frac{(\sec(x)-1)}{((x)\cdot(\sec(x)))}, x=0\right);$$

$$0$$

**(h)** $\lim\limits_{x\to 0}\frac{\sin^2(7x)}{7x}$

**Solution:** Open the MAPLE programme and the open new document . Use keyboard and write
$\lim it((\sin\text{^}2(7*x))/(7*x), x=0)$; then press enter key and get

**Command:**

$$> limit\left(\frac{(\sin^2(7\cdot x))}{(7\cdot x)}, x=0\right);$$

$$0$$

**Q.6:** Use algebric techniques to evaluate the following ,

(a) $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

**Solution:** $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$$=\lim_{h\to 0}\frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}\ \textbf{Ans.}$$

(b) $\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}$

**Solution:** $\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}=\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}\times\frac{x-a}{x-a}$

$$=\lim_{x\to a}\frac{x^n-a^n}{x-a}\times\frac{x-a}{x^m-a^m}$$

$$=\lim_{x\to a}\frac{x^n-a^n}{x-a}\times\lim_{x\to a}\frac{x-a}{x^m-a^m}$$

$$\Rightarrow \lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}=\lim_{x\to a}\frac{x^n-a^n}{x-a}\times\lim_{x\to a}\frac{1}{\frac{x^m-a^m}{x-a}}\quad \text{.....(i)}$$

**Since we have results** $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$ **....(ii) and**

$\lim_{x\to a}\frac{x^m-a^m}{x-a}=ma^{m-1}$ **.....(iii). put (ii) and (iii) in (i)**

$$\Rightarrow \quad \lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}=\frac{na^{n-1}}{ma^{m-1}} \quad or \quad \lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}=\frac{n}{m}a^{n-m} \quad \textbf{Ans}$$

**(c)** $\lim\limits_{\theta\to 0}\dfrac{1-\cos p\theta}{1-\cos q\theta}$

**Note: use result** $1-\cos x = 2\sin^2\dfrac{x}{2}$

**Solution:** $$\lim_{\theta\to 0}\frac{1-\cos p\theta}{1-\cos q\theta}=\lim_{\theta\to 0}\frac{2\sin^2\frac{p\theta}{2}}{2\sin^2\frac{q\theta}{2}}$$

$$\Rightarrow \quad \lim_{\theta\to 0}\frac{1-\cos p\theta}{1-\cos q\theta}=\lim_{\theta\to 0}\frac{\dfrac{\left(\frac{p\theta}{2}\right)^2}{\left(\frac{p\theta}{2}\right)^2}\sin^2\dfrac{p\theta}{2}}{\dfrac{\left(\frac{q\theta}{2}\right)^2}{\left(\frac{q\theta}{2}\right)^2}\sin^2\dfrac{q\theta}{2}}$$

$$=\lim_{\theta\to 0}\frac{\dfrac{p^2\theta^2}{4}\times\dfrac{\sin^2\frac{p\theta}{2}}{\left(\frac{p\theta}{2}\right)^2}}{\dfrac{q^2\theta^2}{4}\times\dfrac{\sin^2\frac{q\theta}{2}}{\left(\frac{q\theta}{2}\right)^2}}$$

$$= \frac{p^2}{q^2}\lim_{\theta\to 0}\frac{\dfrac{\sin^2\frac{p\theta}{2}}{\left(\frac{p\theta}{2}\right)^2}}{\dfrac{\sin^2\frac{q\theta}{2}}{\left(\frac{q\theta}{2}\right)^2}} = \frac{p^2}{q^2}\cdot\frac{\left(\lim\limits_{\theta\to 0}\dfrac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\right)^2}{\left(\lim\limits_{\theta\to 0}\dfrac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right)^2}$$

$$\Rightarrow \lim_{\theta\to 0}\frac{1-\cos p\theta}{1-\cos q\theta} = \frac{p^2}{q^2}\cdot\frac{1}{1} = \frac{p^2}{q^2} \quad \textbf{Ans.}$$

**(d)** $\lim\limits_{\theta\to 0}\dfrac{\tan\theta-\sin\theta}{\sin^3\theta}$

**Solution:** $\lim\limits_{\theta\to 0}\dfrac{\tan\theta-\sin\theta}{\sin^3\theta} = \lim\limits_{\theta\to 0}\dfrac{1}{\sin^3\theta}\left(\dfrac{\sin\theta}{\cos\theta}-\sin\theta\right)$

$$= \lim_{\theta\to 0}\frac{1}{\sin^3\theta}\left(\frac{\sin\theta-\sin\theta\cos\theta}{\cos\theta}\right)$$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{\sin^3\theta}\left(\frac{1-\cos\theta}{\cos\theta}\right)$$

$$= \lim_{\theta\to 0}\frac{1}{\sin^2\theta}\left(\frac{1-\cos\theta}{\cos\theta}\right)$$

$$= \lim_{\theta\to 0}\frac{1}{(1-\cos^2\theta)}\left(\frac{1-\cos\theta}{\cos\theta}\right)$$

$$= \lim_{\theta\to 0}\frac{1}{(1-\cos\theta)(1+\cos\theta)}\left(\frac{1-\cos\theta}{\cos\theta}\right)$$

$$= \lim_{\theta\to 0}\frac{1}{(1+\cos\theta)}\left(\frac{1}{\cos\theta}\right)$$

$$= \frac{1}{(1+\cos 0)(\cos 0)} = \frac{1}{(1+1)(1)} = \frac{1}{2} \quad \textbf{Ans.}$$

(e) $\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{n}$

**Solution:** $\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{n}=\lim_{n\to+\infty}\left(\left(1+\left(-\frac{1}{n}\right)\right)^{-n}\right)^{-1}$

$$=\left(\lim_{n\to+\infty}\left(1+\left(-\frac{1}{n}\right)\right)^{-n}\right)^{-1}$$

$$=(e)^{-1}=\frac{1}{e}\quad \textbf{Ans.}$$

(f) $\lim_{x\to0}\left(1+2x^{2}\right)^{\frac{1}{x^{2}}}$

**Solution:** $\lim_{x\to0}\left(1+2x^{2}\right)^{\frac{1}{x^{2}}}=\lim_{x\to0}\left[\left(1+2x^{2}\right)^{\frac{1}{2x^{2}}}\right]^{2}$

$$=\left[\lim_{x\to0}\left(1+2x^{2}\right)^{\frac{1}{2x^{2}}}\right]^{2}$$

$$=(e)^{2}=e^{2}\quad \textbf{Ans}$$

(g) $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x}$

**Solution:** $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x}=\lim_{x\to\infty}\left(\frac{1+x}{x}\right)^{-x}$

$$=\lim_{x\to\infty}\left(\frac{1}{x}+\frac{x}{x}\right)^{-x}$$

$$=\lim_{x\to\infty}\left(\frac{1}{x}+1\right)^{-x}=\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right)^{-1}$$

$$=(e)^{-1}=\frac{1}{e}\quad \textbf{ANS.}$$

**Left hand side limit(L.H.L) :** If $x < a$, then $\lim_{x \to a^-} f(x) = l_1$ is called left hand side limit of $f(x)$

As $x < a$ then we put $x = a - h$ where $h > 0$

Also $x \to a^-$ then $h \to 0$.

**Right hand side limit(R . H .L) :** If $x > a$, then $\lim_{x \to a^+} f(x) = l_2$ is called right hand side limit of $f(x)$

As $x > a$ then we put $x = a + h$ where $h > 0$

Also $x \to a^+$ then $h \to 0$.

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x)$ then limit of $f(x)$ is exist. i.e. if left and right hand sides limit are equal then the limit of the function is exist.

**Example. 36:** Determine whether $\lim_{x \to 3} f(x)$ and $\lim_{x \to 5} f(x)$ exist, if $f(x) = \begin{cases} 3x + 4 & if\ 0 \le x < 3 \\ 16 - x & if\ 3 \le x < 5 \\ x & if\ 5 \le x < 14 \end{cases}$

**Solution:** (a) $\lim_{x \to 3^-} f(x) = \lim_{x \to 3}(3x + 4) = 3(3) + 4 = 13$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3}(16 - x) = 16 - 3 = 13$

Since $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 13$

Thus $\lim_{x \to 3} f(x) = 13$ **Ans**

(b) $\lim_{x \to 5^-} f(x) = \lim_{x \to 5}(16 - x) = 16 - 5 = 11$

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5}(x) = 5$

Since $\lim_{x \to 5^-} f(x) \ne \lim_{x \to 5^+} f(x)$

Thus $\lim_{x \to 5} f(x)$ does not exist.

**Definition : Continuity:** A function $f(x)$ is continues at $x = a$ if the following three conditions are satisfied.

(a) $f(a)$ is defined.

(b) $\lim_{x \to a} f(x)$ is exist .

(c) $\lim_{x \to a} f(x) = f(a)$.

A function $f(x)$ is continues in the open interval $(a\ ,\ b)$ if it is continues at each point in the interval $(a\ ,\ b)$ .

If the function is not continues at $x = a$ is called discontinues

**Example. 37**: Discuss the continuity of

$$f(x) = x^3 - 2x^2 - 3x + 5 \quad \text{at} \quad x = 1$$

**Solution**: $f(x) = x^3 - 2x^2 - 3x + 5$ .....(i) , put $x = 1$

(a) $f(1) = (1)^3 - 2(1)^2 - 3(1) + 5 = 1 - 2 - 3 + 5 = 1$

Which is defined.

(b) $\lim_{x \to 1} f(x) = \lim_{x \to 1}(x^3 - 2x^2 - 3x + 5)$

$$= (1)^3 - 2(1)^2 - 3(1) + 5 = 1 - 2 - 3 + 5 = 1$$

(c ) since $\lim_{x \to 1} f(x) = f(1)$

Thus $f(x)$ is continous at $x = 1$ .

**Note**: A function $f(x)$ is continous function in an interval when the function is defined at every point in that interval and no jumps or breaks involves .

**Example 38** : Discuss the continuity of $f(x) = \dfrac{x^2 - 4}{x + 2}$ ,

at $x = -2$

**Solution**: $f(x) = \dfrac{x^2 - 4}{x + 2}$, at $x = -2$

Put $x = -2$

$$f(-2) = \frac{(-2)^2 - 4}{-2 + 2} = \frac{0}{0} = \text{Undefined .}$$

Thus f(x) is discontinous at x = -2 .

**Example 39**: First class postage in 1995 was 0.32 dollars for the first ounce (or any fraction there of) and 0.23 dollars for each additional ounce (for fraction thereof) up to 11 ounces . If p(x) is the amount of postage for a letter weighing in x ounces , then we write :

$$P(x)=\begin{cases} 0.32 \text{ dollars} & \text{if } \quad 0<x\le 1 \\ 0.55 \text{ dollars} & \text{if } \quad 1<x\le 2 \\ 0.78 \text{ dollars} & \text{if } \quad 2<x\le 3 \\ \quad \text{and so on} & \end{cases}$$

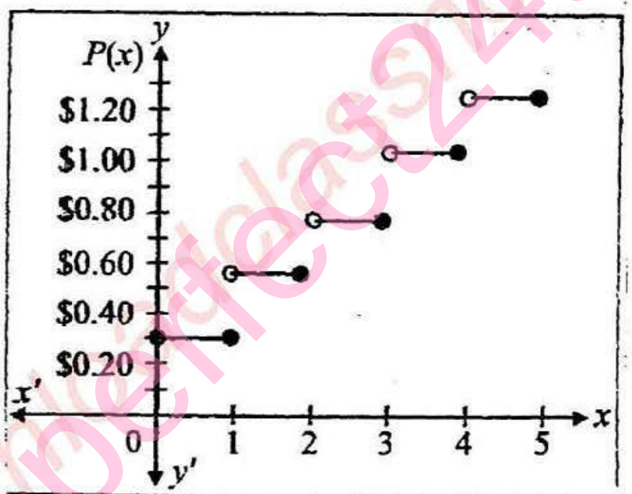
**(a)** **Graph of** $P(x)$ **for** $0<x\le 5$

**(b)** **Find** $\lim\limits_{x\to 1^-} p(x)$ , $\lim\limits_{x\to 1^+} p(x)$ **and** $\lim\limits_{x\to 1} p(x)$

**(c)** **Find** $\lim\limits_{x\to 4.5} p(x)$ **and** $p(4.5)$

**Solution: (a)**

**The graph of f(x) is**
**Shown in the figure.**




**(b)** $\lim\limits_{x\to 1^-} p(x) = 0.32$ **dollars**

$\lim\limits_{x\to 1^-} p(x) = 0.55$ **dollars**

$\lim\limits_{x\to 1} p(x)$ **does not exist .**

$p(1) = 0.32$ *dollars.*

**(c)** $\lim\limits_{x\to 4.5} p(x) = 1.24$ **dollars**

**And** $p(4.5) = 1.24$ **dollars.**

**Example .40: Use MAPLE command "iscount" to check**

the continuity of the function $f(x)=x^2+4$ in
(a) Internal from 0 to 1. (b) closed interval $[0\ ,\ 1]$
(c) open interval $(0\ ,\ 1)$.

**Solution**: (a) Open the MAPLE programme and then open the new document and use key board and write $isocont(x\hat{}2+4, x=0..1)$; and press enter key get

**Command:**

$> iscont(x^2+4, x=0..1);$

$$true$$

(b) $> iscont(x^2+4, x=0..1, closed);$

$$true$$

(c) $> iscont(x^2+4, x=0..1, open);$

$$true$$

## Exercise 2.4:

**Q.1**: Use properties of continuous function to test the continuity and discontinuity of the following functions.

(a) $f(x)=2x-3$

**Solution**: As $f(x)$ is defined for every real number.
Thus $f(x)$ is continues for all $x$.

(b) $h(x)=\dfrac{2}{x-5}$

**Solution**: As $h(x)$ is undefined for $x=5$ i.e. if put $x=5$

Then $h(5)=\dfrac{2}{5-5}=\dfrac{2}{0}=\infty$.

And $h(x)$ is defined for every real number other than $5$.
Thus $h(x)$ is continuous for all $x$ except $5$.
And $h(x)$ is discontinuous at $x=5$.

(c) $g(x)=\dfrac{x-5}{(x-3)(x+2)}$

**Solution**:
$g(x)$ is undefined for $x=-2$ and $x=3$ and defined

for every real number except -2 and 3.

Thus g(x) is continues for all x except - 2 and 3.

**Q.2**: `Show that function $f(x)=\begin{cases}\dfrac{\sin x}{x}, & if \quad x \neq 0 \\ 1 & if \quad x=0\end{cases}$

is continous at $x=0$

**Solution**: (a) Given that $f(0)=1$, which is defined.

(b) $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

(c) Also $\lim\limits_{x\to 0} f(x)=f(0)$

Thus $f(x)$ is continouse at $x=1$.

**Q.3**: Use the graph of the function $g(x)$ to answer the following questions.

(a) Is $g(x)$ continuous on the open interval $(-1,2)$?

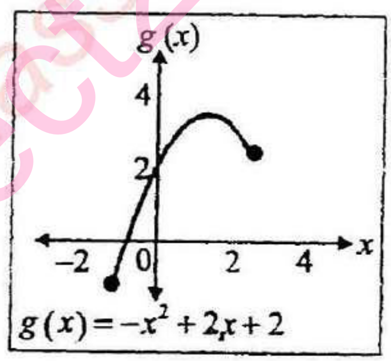
(b) Is $g(x)$ continuous from the right at $x=-1$?

Is $\lim\limits_{x\to -1^+} g(x)=g(-1)$ ?

(c) Is $g(x)$ continuous from the left at $x=2$ ?

Is $\lim\limits_{x\to 2^-} g(x)=g(2)$ ?

(d) Is $g(x)$ continuous on the close interval $[-1, 2]$?




**Solution:** (a) yes $g(x)$ is continuous on the open interval $(-1,2)$ because from the graph it is clear that $g(x)$ is defined for every x in the open interval $(-1,2)$.

(b) yes $g(x)$ is continuous from the right at $x=-1$

Because from the graph it is clear that $\lim\limits_{x\to 1^+} g(x)=-1$

Also $g(1) = -1$ . Thus $\lim_{x \to -1^+} g(x) = g(-1)$ .

(c) yes $g(x)$ is continuous from the left at $x = 2$

Because from the graph it is clear that $\lim_{x \to 2^-} g(x) = 2$

Also $g(2) = 2$ . Thus $\lim_{x \to 2^-} g(x) = g(2)$ .

(d) yes $g(x)$ is continuous on the close interval $[-1 , 2]$.

Because $g(x)$ is defined for every x in the close interval $[-1 , 2]$ .

**Q.4**: Use the graph of the function $f(x)$ to answer the following questions .

(a) Is $f(x)$ continuous on the open interval $(0 , 3)$?

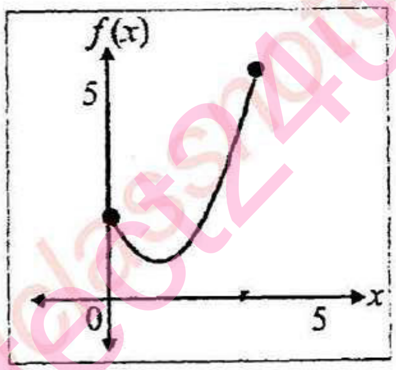
(b) Is $f(x)$ continuous from the right at $x = 0$ ?

Is $\lim_{x \to 0^+} f(x) = f(0)$ ?

(c) Is $f(x)$ continuous from the left at $x = 3$ ?

Is $\lim_{x \to 3^-} f(x) = f(3)$ ?

(d) Is $f(x)$ continuous on the close interval $[0 , 3]$?




**Solution**: (a) yes $f(x)$ is continuous on the open interval $(0 , 3)$ because from the graph it is clear that $f(x)$ is defined for every x in the open interval $(0 , 3)$.

(b) yes $f(x)$ is continuous from the right at $x = 0$

Because from the graph it is clear that $\lim_{x \to 0^+} f(x) = 2$

Also $f(0) = 2$ . Thus $\lim_{x \to -0^+} f(x) = f(0)$ .

(c) yes $f(x)$ is continuous from the left at $x = 3$

yes $\lim_{x \to 3^-} f(x) = f(3)$ .

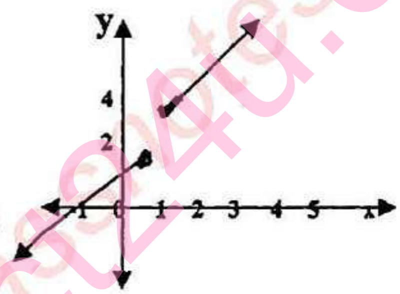
(d) yes $f(x)$ is continuous on the close interval $[0 , 3]$.

**Because** $f(x)$ **is defined for every x in the close interval** $[0\ ,3\ ]$.

**Q.5: Graph and locate all points of discontinuity of the following piece wise functions.**

(a) $f(x)=\begin{cases}1+x\ ,if & x<1\\ 5-x\ ,if & x\geq 1\end{cases}$ (b) $f(x)=\begin{cases}-x\ ,if & x<0\\ 1\ ,\ if & x=0\\ x\ ,\ if & x>0\end{cases}$

**Solution: (a) we check the discontinuity of the** $f(x)$ **at** $x=1$.




(a) $f(1)=5-1=4$ ,
**which is defined.**

(b) $\lim_{x\to 1^-} f(x)=\lim_{x\to 1}(1+x)=1+1=2$

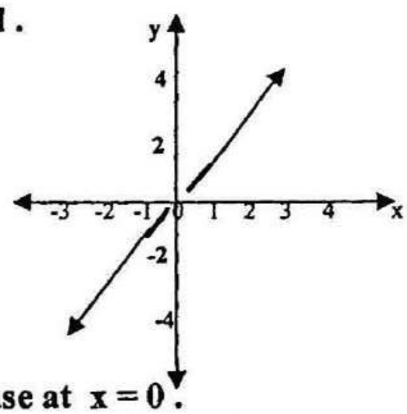
$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}(5-x)=5-1=4$

**As** $\lim_{x\to 1^-} f(x)\neq \lim_{x\to 1^+} f(x)$

**Thus** $\lim_{x\to 1} f(x)$ **does not exist.**

**Thus the function is discontinouse at x = 1.**

**(b) we check the discontinuity of the** $f(x)$ **at** $x=0$.




(a) $f(0)=1$ **, which is defined.**

(b) $\lim_{x\to 0^-} f(x)=\lim_{x\to 0}(-x)=0$

$\lim_{x\to 0^+} f(x)=\lim_{x\to 1}(x)=0$

**As** $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=0$

**Thus** $\lim_{x\to 0} f(x)=0$ **is exist.**

(c) **As** $\lim_{x\to 0} f(x)\neq f(0)$

**Thus the function is discontinouse at x = 0.**

**Q.6: Personal computer sales person recives a base salary of \$ 1,000 per month and a commission of 5% of all sales over \$ 10,000 during the month. If the monthly sales**

**are \$ 20 , 000 or more , the sale person is given an additional \$ 500 bonus . Let E(s) represents the person, earnings during the month as a function of the monthly sales .**

**(a) Graph E(s) for** $0 \le s \le 30000$

**(b) Find** $\lim\limits_{s \to 10000} E(s)$ **and** $E(10000)$

**(c ) Find** $\lim\limits_{s \to 20000} E(s)$ **and** $E(20000)$

**(d) Is E continous at** $s = 10000$ **and at** $s = 20000$

**<u>Solution</u>:**

**Monthly base salary = E(s) = \$ 1000 ,** $s \le 10000$

**Commission = 5% of all sales over \$ 10,000 during the month , then earning** $E(s)$ **represented as**

$$E(s) = 1000 + 5\%(s - 10000), \quad 10000 \le s < 20000$$
$$= 1000 + 0.05(s - 10000), \quad 10000 \le s < 20000$$

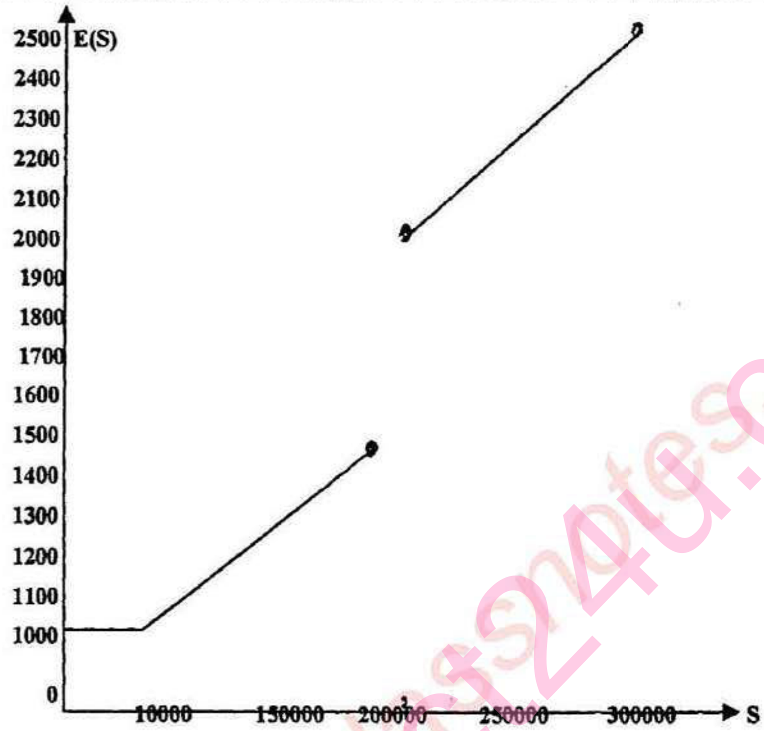
**He get bonus \$ 500 more if the sales are \$ 20000 or more .**

**Therefore for** $s \ge 20000$

$$E(s) = 1000 + 5\%(20000 - 10000) + 5\%(s - 20000) + 500$$
$$= 1000 + 0.05(10000) + 0.05(s - 20000) + 500$$
$$= 1000 + 500 + 0.05(s - 20000) + 500$$
$$= 2000 + 0.05(s - 20000)$$

**Thus the earning** $E(s)$ **function is**

$$E(s) = \begin{cases} 10000 \quad , & s \le 10000 \\ 1000 + 0.05(s - 10000) \quad , & 10000 < s < 20000 \\ 2000 + 0.05(s - 20000) \quad , & s \ge 20000 \end{cases}$$




**(b)** $\lim\limits_{s\to 10000} E(s) = \$\ 1000$ **and** $E(10000) = \$\ 1000$

**(c)** $\lim\limits_{s\to 20000^-} E(s) = \lim\limits_{s\to 20000} [1000 + 0.05(s-10000)]$

$= 1000 + 0.05(20000 - 10000)$

$= 1000 + 0.05(10000) = 1000 + 500 = \$1500$

$\lim\limits_{s\to 20000^+} E(s) = \lim\limits_{s\to 20000} [2000 + 0.05(s-20000)]$

$= 2000 + 0.05(200000 - 20000) = 2000 + 0$

$= \$\ 20000$

**Since** $\lim\limits_{s\to 20000^-} E(s) \neq \lim\limits_{s\to 20000^+} E(s)$

**Thus** $\lim\limits_{s\to 20000} E(s)$ **does not exist.**

**As** $E(s) = 2000 + 0.05(s - 20000)$ **, for** $s \geq 20000$

$E(20000) = 2000 + 0.05(20000 - 20000) = 2000 + 0$

=> $E(20000) = \$2000$

**(d) Yes E is continouse at s = 10000 . But does not continouse**

at $s = 20000$.

**Q.7**: Use MAPLE command "iscont" to test the continuity of

$$f(x) = \frac{x^2+1}{x^2+2.7} \quad \text{in closed interval } [-5 \ , 5] \ .$$

**Solution:** > $iscont\left(\frac{x^2+1}{x^2+2.7}, x=-5..5, closed\right);$

*true*

## Review Exercise 2.

**Q.1**: Choose the correct option.

(i) The independent variable in the function $y = \frac{x^2+4x-3}{(x+3)^2}$ is

(a) $x$ (b) $x^2$ (c) $x^3$ (d) $y$

(ii) If $f(x) = \frac{3x^2-2}{3x+9}$ then $f(-3)$ is

(a) $\frac{25}{18}$ (b) $\frac{29}{18}$ (c) $\frac{3}{1}$ (d) undefined

(iii) The domain of $f(x) = \frac{3x^2-2}{3x+9}$ is

(a) $(-\infty, -3)U(-3 \ , \infty)$ (b) $(-\infty, 3)U(3 \ , \infty)$
(c) $[-\infty \ \ -3]$ (d) $(-\infty, \ 3)$

(iv) The domain of $f(x) = \frac{3x-2}{3x^2+9}$ is :

(a) $x \geq -3$ (b) $x \leq -3$ (c) $-\infty \leq x \leq \infty$ (d) $-\infty < x < \infty$

(v) If $f(x) = 2x+3$ then $f^{-1}(5)$ is

(a) 13 (b) – 13 (c) 1 (d) -1

(vi) If y is expresses in term of x as y = f (x) then y is called (a) Implicit function (b) explicit function (c) linear function (d) identity function

(vii) If $f(x) = 3x^2+2x-1$ and $g(x) = x+1$ then $f(g(x))$ is

(a) $3x^2+2x+1$ (b) $3x^2+8x+4$

(c) $3x^2+8x-1$ (d) $3x^2-8x-4$

(viii) **The value of e is**

(a) 3.142 (b) $\frac{22}{7}$ (c) 0 2.71 (d) 3.8

(ix) **If $5^x=7$ then the value of x is :**

(a) $\frac{\ln(7)}{\ln(5)}$ (b) $\frac{\ln(5)}{\ln(7)}$ (c) $\frac{\ln(x)}{\ln(5)}$ (d) $\frac{\ln(7)}{\ln(x)}$

(x) **Cothx = ……**

(a) $\frac{e^x-e^{-x}}{2}$ (b) $\frac{e^x+e^{-x}}{2}$ (c) $\frac{e^x+e^{-x}}{e^x-e^{-x}}$ (d) $\frac{e^x-e^{-x}}{e^x+e^{-x}}$

(xi) $\ln(m)-\ln(n)=$ ………

(a) $\ln(mn)$ (b) $\ln(m+n)$ (c) $\ln(m-n)$ (d) $\ln\frac{m}{n}$

(xii) **The function $f(x)=\frac{2}{x^2-9}$ is discontinous at point :**

(a) −1 , 1 (b) -2 , 2 (c) -3 , 3 (d) -4 , 4

(xiii) $\lim_{\theta\to 0}\frac{\sin 7\theta}{7\theta}=$……..

(a) 1 (b) 2 (c) 3 (d) 90

(xiv) **If $f(\theta)=\theta,\sec\theta$ then $f(0)=$……..**

(a) 0 (b) 1 (c) $\sqrt{2}$ (d) -1

(xv) **The inverse function of $\frac{e^x-e^{-x}}{e^x+e^{-x}}$ is :**

(a) $\frac{1}{2}\log\left(\frac{2-x}{2+x}\right)$ (b) $\frac{1}{2}\log\left(\frac{2+x}{2-x}\right)$

(c) $\frac{1}{2}\log\left(\frac{-1-x}{x-1}\right)$ (d) $\frac{1}{2}\log\left(\frac{-1-x}{1-x}\right)$

**<u>Answers</u>: (i) a (ii) d (iii) a (iv) d (v) c (vi) b (vii) b (viii) c (ix) a (x) c (xi) d (xii) c (xiii) a (xiv) a (xv) c**