

Solved Exercise 2.9

Q1. Determine the interval in which f is increasing or decreasing for the domain mentioned in each case.

i. $f(x) = \sin x$

$X \in [-\pi, \pi]$

Solution

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

As $f'(x) = \cos x$ — ive $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So, f is decreasing on intervals $\left[-\pi, -\frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$

But $f'(x) = \cos x$ is + ive for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So, f is increasing on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Ans

ii. $f(x) = \cos x \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Solution

$$f'(x) = -\sin x$$

As $\sin x$ is — ive for $-\frac{\pi}{2} < x < 0$

So f is increasing on the interval $\left[-\frac{\pi}{2}, 0\right]$

$\Rightarrow f'(x) = -\sin x$ is negative for $0 < x < \frac{\pi}{2}$

So f is decreasing on the interval $\left[0, \frac{\pi}{2}\right]$

Ans

iii. $f(x) = 4 - x^2$ $x \in [-2, 2]$

Solution

$$f(x) = 4 - x^2$$

$$f'(x) = -2x$$

As $f'(x)$ is +ve when x is -ive

$\Rightarrow f$ is increasing on the interval $[-2, 0]$

As $f'(x)$ is +ve when x is positive

$\Rightarrow f$ is decreasing on the interval $[0, 2]$ **Ans**

iv. $f(x) = x^2 + 3x + 2$

Solution

$$f'(x) = 2x + 3$$

As $f'(x) = 2x + 3$ is negative for $-4 < x < -\frac{3}{2}$

So f is decreasing on the interval $[-4, -\frac{3}{2}]$

As $f'(x) = 2x + 3$ is +ive for $-\frac{3}{2} < x < 1$

Thus, f is increasing on the interval $[-\frac{3}{2}, 1]$ **Ans**

Q2. Find the extreme value for the following function defined as

i. $f(x) = 1 - x^3$

Solution

$$f(x) = 1 - x^3$$

$$\Rightarrow f'(x) = -3x^2$$

$$\text{Put } f'(x) = 0$$

$$-3x^2 = 0$$

$$\Rightarrow x = 0$$

$$f''(x) = -6x$$

$$f''(0) = 0$$

The 2nd derivation does not help in determining the extreme values

$$y = f(x) = 1 + x^3$$

$$= 1 - 0 = 1$$

As 1st derivative does not change at $x=0$, or $x=0$ (0,1) is the pt. of inflexion

ii. $f(x) = x^2 - x - 2$

Solution

$$f'(x) = 2x - 1$$

$$f'(x) = 0$$

$$\Rightarrow 2x - 1 = 0$$

$$\Rightarrow x = \frac{1}{2}$$

$$\text{Let } x = \frac{1}{2} - t \text{ on } t \rightarrow 0$$

$$f\left(\frac{1}{2} - t\right) = 2\left(\frac{1}{2} - t\right) - 1$$

$$= 1 - 2t - 1 = -2t < 0$$

$$f\left(\frac{1}{2} + t\right) = 2\left(\frac{1}{2} + t\right) - 1$$

$$= 1 + 2t - 1 = 2t > 0$$

$$f'(x) < 0 \text{ before } x = \frac{1}{2} \text{ and}$$

$$f'(x) > 0 \text{ after } x = \frac{1}{2}$$

Hence f has relative minima at $x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right) - 2 = \frac{1}{4} - \frac{1}{4} - 2$$

$$= \frac{1-2-8}{4} = -\frac{9}{4}$$

Note that $f''(x) = 2 > 0$ **Ans**

$$\text{iii. } f(x) = 5x^2 - 6x + 2$$

Solution

$$\Rightarrow f'(x) = 10x - 6$$

$$f''(x) = 10 > 0$$

$$f'(x) = 10x - 6 = 0$$

$$10x = 6$$

$$x = \frac{6}{10}$$

$$x = \frac{3}{5}$$

$$\text{Let } x = \frac{3}{5} - \epsilon \text{ as } \epsilon > 0$$

$$f\left(\frac{3}{5} - \epsilon\right) = 10\left(\frac{3}{5} - \epsilon\right) - 6$$

$$= 6 - 10\epsilon < -6$$

$$= -10\epsilon < 0$$

$$\text{Put } x = \frac{3}{5} + \epsilon$$

$$\begin{aligned}f'\left(\frac{3}{5}+\epsilon\right) &= 10\left(\frac{3}{5}+\epsilon\right) - 6 \\&= 6 + 10\epsilon - 6 \\&= 10\epsilon > 0\end{aligned}$$

$f'(x) < 0$ before $x = \frac{3}{5}$ and $f'(x) > 0$ after $x = \frac{3}{5}$

Hence f has relative minima at $x = \frac{3}{5}$

$$\begin{aligned}f\left(\frac{3}{5}\right) &= 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2 \\&= 5 \times \frac{9}{25} - \frac{18}{5} + 2 = \frac{9}{5} - \frac{18}{5} + 2 \\&= \frac{9-18+10}{5} = \frac{19-18}{5} \\&= \frac{1}{5} \quad \text{Ans}\end{aligned}$$

iv. $f(x) = 3x^2$

Solution

$\Rightarrow$

$$f(x) = 3x^2$$

$$f'(x) = 6x$$

$$\text{Put } f'(x) = 0$$

$\Rightarrow$

$$6x = 0$$

$$x = 0$$

where ϵ is very small + ive number

$$\text{Now let } x = 0 - \epsilon$$

Then

$$f'(0 - \epsilon) = 6(0 - \epsilon) = -6\epsilon < 0$$

$$f'(0 + \epsilon) = 6(0 + \epsilon) = 6\epsilon > 0$$

$f'(x) < 0$ before $x = 0$ and

$$f'(x) < 0 \text{ before } x = 0$$

Hence f has relative Minima at $x = 0$

$$f(0) = 3(0)^2 = 0 \text{ has relative minima at } x = 0. \quad \text{Ans}$$

$$\text{v. } f(x) = 3x^2 - 4x + 5$$

Solution

$$f(x) = 3x^2 - 4x + 5$$

$$\Rightarrow f'(x) = 6x - 4$$

$$\text{Now } f'(x) = 0$$

$$\Rightarrow 6x - 4 = 0$$

$$x = \frac{2}{3}$$

But $f'' = 6$ and $6 > 0$ which shows that f has relative minima at $x = \frac{2}{3}$

$$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 5$$

$$= 3 \times \frac{4}{3} + 5 = \frac{4}{3} - \frac{8}{3} + 5 = \frac{4-8+15}{3}$$

$$= \frac{11}{3} \quad \text{Ans}$$

$$\text{vi. } f(x) = 2x^3 - 2x^3 - 36x + 3$$

Solution

$$\Rightarrow f'(x) = 6x^2 - 4x - 36$$

$$\Rightarrow f'(x) = 0$$

$$6x^2 - 4x - 36 = 0$$

$$3x^2 - 2x - 18 = 0$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(-18)}}{6}$$

$$x = \frac{2 \pm \sqrt{220}}{6}$$

$$= \frac{1 \pm \sqrt{55}}{3}$$

$$f''(x) = 12x - 4 = 4(3x - 1)$$

$$f''(x) = \frac{1 + \sqrt{55}}{3} = 4 \left[3 \left(\frac{1 + \sqrt{55}}{3} \right) - 1 \right]$$

$$= 4(1 + \sqrt{55} - 1)$$

$$= 4\sqrt{55} > 0$$

which shows that f has relative minima at $x = \frac{1 + \sqrt{55}}{3}$, e

$$f\left(\frac{1 + \sqrt{55}}{3}\right) = 2\left(\frac{1 + \sqrt{55}}{3}\right)^3 - 2\left(\frac{1 + \sqrt{55}}{3}\right)^2 - 36\left(\frac{1 + \sqrt{55}}{3}\right) + 3$$

$$= \frac{2}{27}(1 + 3\sqrt{55} + 3 + 55 + 55\sqrt{55})$$

$$- \frac{2}{9}(1 + 2\sqrt{55} + 55) - 12(1 + \sqrt{55}) + 3$$

$$= \frac{2}{27}(166 + 58\sqrt{55}) - \frac{2}{9}(56 + 2\sqrt{55}) - 12(1 + \sqrt{55}) + 3$$

$$= \frac{322}{27} + \frac{116}{27}\sqrt{55} - \frac{4}{9}\sqrt{55} - 12 - 12\sqrt{55} + 3$$

$$= \left(\frac{332}{27} - \frac{112}{9} - 9\right) + \left(\frac{116}{27} - \frac{4}{9} - 12\right)\sqrt{55}$$

$$= -\frac{1}{2}(247 + 220\sqrt{55})$$

$$\text{At } f(x) = \frac{1 - \sqrt{55}}{3} \Rightarrow f(x) = 4 \left(3 \left(\frac{1 - \sqrt{55}}{3} \right) - 1 \right) = -4\sqrt{55} < 0$$

Which show f has relative maxima at $\frac{1 - \sqrt{55}}{3}$

$$f''(-2) = 12(-\sqrt{2})^2 - 8$$

$$= 24 - 8$$

$$= 16$$

Which shows that f has relative minima at $x = -\sqrt{2}$

$$f(-2) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2 = 4 - 8 = -9 \quad \text{Ans}$$

vii. $f(x) = x^4 - 4x^2$

Solution

$$f'(x) = 4x^3 - 8x$$

$$\text{let } f''(x) = 0$$

$$\Rightarrow 4x^3 - 8x = 0$$

$$4x = 0 \text{ or } x^2 - 2 = 0$$

$$x = 0 \text{ or } x = \pm\sqrt{2}$$

$$f''(x) = 12x^2 - 8$$

$$f''(x) = 12(0) - 8 = -8 < 0$$

$f(x)$ is maxima at $x = 0$

$$f(0) = 0 - 0 + 0$$

Now

$$f''(\sqrt{2}) = 12(\sqrt{2})^2 - 8 + 16 > 0$$

$$f(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 = -4$$

$$f''(-\sqrt{2}) = 12(-\sqrt{2})^2 - 8(\sqrt{2})^2 = 16 > 0$$

Hence $f(x)$ has a minima at $x = -\sqrt{2}$ i. e

$$f(\sqrt{2}) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2$$

$$= 4 - 8 = -4 \quad \text{Ans}$$

viii. $f(x) = (x - 2)^2(x - 1)$

Solution

$$\Rightarrow f'(x) = 2(x - 2)(x - 1) + (x - 2)^2 = (x - 2)[2x - 2 + x - 2]$$

$$f'(x) = (x - 2)(3x - 4)$$

$$f'(x) = 0 \Rightarrow (x - 2)(3x - 4) = 0$$

$$\Rightarrow x = 2, \quad x = \frac{4}{3}$$

$$\text{Also } f''(x) = 1 + (3x - 4) + (x - 2)(3) = 3x - 4 + 3x - 6$$

$$= 6x - 10$$

$$x = 2$$

$$f''(x) = 6(2) - 10 = 12 - 10 = 2 > 0$$

$\Rightarrow f$ has relative minima at $x = 2$

$$f(2) = (2 - 2)^2(2 - 1) = 0 \times 1 = 0$$

$$f\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right) - 10 = 8 - 10 = -2 < 0$$

$$f''(x) < 0 \text{ where } x = \frac{4}{3}$$

Hence f has relative maxima at $x = \frac{4}{3}$

$$f\left(\frac{4}{3}\right) = \left(-\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)$$

$$= \left(\frac{4}{9}\right) \left(\frac{1}{3}\right)$$

$$= \frac{4}{27}$$

Ans

ix. $f(x) = 5 + 3x - x^3$

Solution

$$\Rightarrow f'(x) = 3 - 3x^2$$

$$f'(x) = 0$$

$$\Rightarrow 3 - 3x^2 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

$$\text{Now } f''(x) = -6x$$

$$\text{at } x = 1$$

$$f''(1) = -6(1) = -6 < 0$$

$$\Rightarrow f \text{ has relative maxima at } x = 1$$

$$\text{Now } f''(-1) = -6(-1) = 6 > 0$$

$$\Rightarrow f \text{ has relative minima at } x = -1$$

$$f(-1) = 5 + 3(-1) - (-1)^3 = 5 - 3 + 1 = 3$$

Ans

Q3. Find the maximum and minimum value of the function defined by the following equations occurring in the interval $[0, 2\pi]$, $f(x) = \sin x + \cos x$

Solution

$$f(x) = \sin x + \cos x$$

$$f'(x) = \cos x - \sin x$$

$$\text{Now } f'(x) = 0$$

$$\Rightarrow \cos x - \sin x = 0$$

$\Rightarrow \cos x = \sin x$ it is only the above equation is specified when $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$ for $x \in (0, 2\pi)$.

Now $f'(x) = -\sin x - \cos x$

$$\begin{aligned}\text{When } x = \frac{\pi}{4}, f'(x) &= -\sin \frac{\pi}{4} - \cos \frac{\pi}{4} \\ &= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ &= -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0\end{aligned}$$

Hence f is maximum value at $x = \frac{\pi}{4}$

$$\begin{aligned}f\left(\frac{\pi}{4}\right) &= \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \sqrt{2}\end{aligned}$$

Now $x = \frac{5\pi}{4}$

$$\begin{aligned}f''\left(\frac{5\pi}{4}\right) &= -\sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right) \\ &= -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} > 0\end{aligned}$$

Thus f has relative minimum at $x = \frac{5\pi}{4} \in (0, 2\pi)$.

$$\begin{aligned}f\left(\frac{5\pi}{4}\right) &= \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right) \\ &= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} \\ &= -\sqrt{2}\end{aligned}$$

Q4. Show that $y = \frac{\ln x}{x}$ has maximum value at $x = e$

Solution

$$y = \frac{\ln x}{x}$$

Diff. w.r.t x

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x} - \frac{1}{x} + \ln x \left(\frac{1}{x^2} \right) \\ &= \frac{1}{x^2} - \frac{\ln x}{x^2} \end{aligned}$$

$$\frac{dy}{dx} = 0, \Rightarrow \frac{1}{x^2} - \frac{\ln x}{x^2} = 0$$

$$\Rightarrow \frac{1 - \ln x}{x^2} = 0$$

$$\Rightarrow 1 - \ln x = 0$$

$$\Rightarrow \ln x = 1$$

$$\ln x = \ln e$$

$$\Rightarrow x = e$$

$$\begin{aligned} y''(x) &= \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{1}{x^2} - \frac{\ln x}{x^2} \right) \\ &= -\frac{2}{x^3} - \left[\frac{1}{x^3} - \frac{2\ln x}{x^3} \right] \\ &= -\frac{2}{x^3} - \frac{1}{x^3} + \frac{2\ln x}{x^3} \\ &= \frac{-3}{x^3} + \frac{2x\ln x}{x^3} \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} = x = e &= \frac{-3}{e^3} + \frac{2x\ln x}{e^3} \\ &= \frac{-3}{e^3} + \frac{2}{e^3} \\ &= \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0 \end{aligned}$$

y has maximum value at $x = e$.

Q5. Show that $y = x^x$ has minimum value at $x = \frac{1}{e}$.

Solution

$$y = x^x$$

$$\Rightarrow \ln y = \ln(x)^x = x \ln x$$

Diff. w.r.t x we have

$$\frac{d}{dx}[\ln x] = \frac{d}{dx}[x \ln x]$$

$$\frac{1}{y} \frac{dy}{dx} = (\ln x + 1)$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

$$\frac{dy}{dx} = 0 \Rightarrow x[\ln x + 1] = 0$$

$$\Rightarrow [\ln x + 1] = 0 \quad (x \neq 0)$$

$$\Rightarrow \ln x = -1$$

$$\Rightarrow \ln x = -\ln e$$

$$\ln x = \ln e^{-1}$$

$$\Rightarrow x = e^{-1} \Rightarrow x = \frac{1}{e}$$

$$\text{Now } \frac{d^2y}{dx^2} = \frac{d}{dx}[x^x(1 + \ln x)]$$

$$= x^x(1 + \ln x)^2 + x^x \times \frac{1}{x}$$

$$= x^x \left[(1 + \ln x)^2 + \frac{1}{x} \right]$$

$$\frac{d^2y}{dx^2} x = \frac{1}{e} = \left(\frac{1}{e} \right)^{\frac{1}{2}} \left[\left(1 + \ln \frac{1}{e} \right) + \frac{1}{\frac{1}{e}} \right]$$

$$\frac{d^2y}{dx^2}x = \left(\frac{1}{e}\right)^{\frac{1}{e}} [(1-1)^2 + e] = \left(\frac{1}{e}\right)^{\frac{1}{e}} (e)$$

Thus, y has minimum value at $x = \frac{1}{e}$

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