

Solved Exercise 2.5

Q.1. Differentiate the following trigonometric functions from the first Principle.

i. $\sin 2x$

Solution

Let $y = \sin 2x$

$$y + \delta y = \sin 2(x + \delta x)$$

$$\Rightarrow \delta y = \sin 2(x + \delta x) - \sin 2x$$

$$= 2 \cos \left[\frac{2x + 2\delta x + 2x}{2} \right] \sin \left(\frac{2x + 2\delta x - 2x}{2} \right)$$

$$= 2 \cos \left(\frac{4x + 2\delta x}{2} \right) \sin \left(\frac{2\delta x}{2} \right)$$

$$= 2 \sin \delta x \cdot \cos(2x + \delta x) \quad (1)$$

Dividing both sides by δx and take limit as $\delta x \rightarrow 0$

$$\Rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0}$$

$$= 2 \lim_{\delta x \rightarrow 0} \cos(2x + \delta x) \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}$$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = 2 \lim_{\delta x \rightarrow 0} \cos(2x + \delta x)$$

$$\frac{dy}{dx} = 2(1) \cos 2x$$

$$= 2 \cos 2x$$

Ans

ii. $\tan 3x$

Solution

Let $y = \tan 3x$

$$\begin{aligned}
 y + \delta y &= \tan x (x + \delta x) \\
 \Rightarrow \delta y &= \tan 3(x + \delta x) - \tan 3x \\
 \Rightarrow &= \frac{\sin 3(x + \delta x)}{\cos 3(x + \delta x)} - \frac{\sin 3x}{\cos 3x} \\
 &= \frac{\sin 3(x + \delta x) \cos 3x - \cos 3(x + \delta x) \sin 3x}{\cos 3(x + \delta x) \cos 3x} \\
 \Rightarrow \frac{\sin \delta x}{\delta x} \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \rightarrow 0} \frac{3}{\cos 3(x + \delta x) \cos 3x} \times \lim_{\delta x \rightarrow 0} \frac{3\delta x}{3\delta x} \\
 \Rightarrow \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \left[\frac{3}{\cos 3(x + \delta x) \cos 3x} \right] \times \lim_{\delta x \rightarrow 0} \frac{3\delta x}{3\delta x} \\
 \Rightarrow \frac{dy}{dx} &= \frac{3}{\cos 3x \cos 3x} \\
 &= 3 \sec^2 3x \quad \text{Ans}
 \end{aligned}$$

iii. $\sin 2x + \cos 2x$

Solution

Let $y = \sin 2x + \cos 2x$ then

$$\begin{aligned}
 \Rightarrow \delta y &= (\sin 2(x + \delta x) - \sin 2x) + (\cos 2(x + \delta x) - \cos 2x) \\
 &= 2 \cos \left(\frac{2(x + \delta x) + 2x}{2} \right) \sin \left(\frac{2(x + \delta x) - 2x}{2} \right) \\
 &= 2 \sin \left(\frac{2(x + \delta x) + 2}{2} \right) \cdot \sin \left(\frac{2(x + \delta x) - 2x}{2} \right) \\
 &= 2 [(\cos(2x + \delta x) \cdot \sin \delta x - \sin(2x + \delta x) \cos \delta x)] \\
 \frac{\delta y}{\delta x} &= 2 [(\cos(2x + \delta x) \cdot \sin \delta x - \sin(2x + \delta x) \cos \delta x)]
 \end{aligned}$$

$$\begin{aligned}
 \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \rightarrow 0} 2 \cos(2x + \delta x) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x} \\
 &= \lim_{\delta x \rightarrow 0} 2 \cos(2x + \delta x) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x} \\
 &= \left[2 \left\{ \cos(2x + \delta x) - \lim_{\delta x \rightarrow 0} (2x + \delta x) \right\} \right] \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}
 \end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= 2(\cos 2x - \sin 2x) \times 1 = 2(\cos 2x - \sin 2x) \\ &= 2 \cos 2x - 2 \sin 2x\end{aligned}$$

iv. $\cos x^2$

Solution

Let $y = \cos x^2$

$$\begin{aligned}\delta y &= \cos(x + \delta x)^2 - \cos x^2 \\ &= -2 \sin \frac{(x + \delta x)^2 + x^2}{2} \sin \frac{(x + \delta x)^2 - x^2}{2} \\ &= -2 \sin \left(\frac{x + \delta x}{2x} \right) \cdot \sin \frac{(x + \delta x)^2 + \delta x}{2}\end{aligned}$$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = - \lim_{\delta x \rightarrow 0} 2 \times \sin(x^2 + x\delta x) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin x \delta x}{x \delta x}$$

$$\begin{aligned}\frac{dy}{dx} &= -2x \sin \frac{2x^2}{2} \\ &= -2x \sin x^2 \quad \text{Ans}\end{aligned}$$

v. $\tan^2 x$

Solution

Let $y = \tan^2 x$

$$y + \delta y = \tan^2(x + \delta x)$$

$$\Rightarrow \delta y = \tan^2(x + \delta x) - \tan^2 x$$

$$\delta y = [\tan(x + \delta x) - \tan x][\tan(x + \delta x) + \tan x]$$

$$= \left[\frac{\sin(x + \delta x) - \sin x}{\cos(x + \delta x) \cos x} \right] \left[\frac{\sin(x + \delta x)}{\cos(x + \delta x)} + \frac{\sin x}{\cos x} \right]$$

$$\delta y = \left[\frac{\sin(x + \delta x) \cos x - \sin x \cos(x + \delta x)}{\cos(x + \delta x) \cos x} \right]$$

$$\begin{aligned} & \times \left[\frac{\sin(x+\delta x) \cos x + \cos(x+\delta x) \sin x}{\cos(x+\delta x) \cos x} \right] \\ & = \frac{\sin(x+\delta x-x)}{\cos(x+\delta x) \cos x} \frac{\sin(x+\delta x+x)}{\cos(x+\delta x) \cos x} \\ \delta y & = \frac{\sin \delta x \sin(2x+\delta x)}{\cos^2(x+\delta x) \cos^2 x} \end{aligned}$$

Dividing by δx and taking limit $\delta x \rightarrow 0$, we have

$$\begin{aligned} \frac{\delta y}{\delta x} & = \lim_{\delta x \rightarrow 0} \left[\frac{\sin(2x+\delta x)}{\cos^2(x+\delta x) \cos^2 x} \right] \\ & = \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x} \lim_{\delta x \rightarrow 0} \left[\frac{\sin(2x+\delta x)}{\cos^2(x+\delta x) \cos^2 x} \right] \\ \frac{dy}{dx} & = 1 \cdot \frac{\sin 2x}{\cos^2 x \cos^2 x} \\ & = \frac{2 \sin x \cos x}{\cos^2 x \cos^2 x} = \frac{2 \sin x}{\cos x} \\ & = \frac{2 \sin x}{\cos x} \cdot \frac{1}{\cos^2 x} \\ & = 2 \tan x \times \sec^2 x \end{aligned}$$

Thus $\frac{dy}{dx} [\tan^2 x] = 2 \tan x \times \sec^2 x$ **Ans**

vi. $\sqrt{\tan x}$

Solution

Let $y = \sqrt{\tan x}$

$$y + \delta y = \sqrt{\tan(x + \delta x)}$$

$$\Rightarrow \delta y = \sqrt{\tan(x + \delta x)} - \sqrt{\tan x}$$

$$\begin{aligned} & = \frac{\sqrt{\sin(x+\delta x)}}{\sqrt{\cos(x+\delta x)}} - \frac{\sqrt{\sin x}}{\sqrt{\cos x}} \\ & = \frac{\sqrt{\sin(x+\delta x)\cos x} - \sqrt{\cos x(x+\delta x)\sin x}}{\sqrt{\cos(x+\delta x)}\sqrt{\cos x}} \end{aligned}$$

By rationalizing we get

$$\begin{aligned}\delta y &= \frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{(\sqrt{\sin(x+\delta x)\cos x} + \sqrt{\cos(x+\delta x)\sin x})\sqrt{\cos(x+\delta x)\cos x}} \\&= \frac{\sin(x+\delta x-x)}{[\sqrt{\sin(x+\delta x)\cos x} + \sqrt{\cos(x+\delta x)\sin x}]\sqrt{\cos(x+\delta x)\cos x}} \\ \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= \frac{\sin(\delta x)}{[\sin(x+\delta x)] + \sqrt{\cos(x+\delta x)\sin x}\sqrt{\cos(x+\delta x)\cos x}} \\&= \frac{1}{[\sqrt{\sin x \cos x} + \sqrt{\cos x \sin x}]\sqrt{\cos x \sin x}\sqrt{\cos x \cos x}} \\&= \frac{1}{2\sqrt{\sin x \cos x}\sqrt{\cos x}} \\ \frac{dy}{dx} &= \frac{\sqrt{\cos x}}{2\sqrt{\sin x}} x \sec x \frac{d}{dx} [\sqrt{\tan x}] \\&= \frac{\sec^2 x}{2\sqrt{\tan x}}\end{aligned}$$

vii. $\cos\sqrt{x}$

solution

let $y = \cos\sqrt{x}$

$$y + \delta y = \cos\sqrt{x + \delta x}$$

$$\Rightarrow \delta y = \cos\sqrt{x + \delta x} - \cos\sqrt{x}$$

$$= -2 \sin \left[\frac{\sqrt{x+\delta x} + \sqrt{x}}{2} \right] \sin \left[\frac{\sqrt{x+\delta x} - \sqrt{x}}{2} \right]$$

$$\frac{\delta y}{\delta x} = \frac{\sin \left[\frac{\sqrt{x+\delta x} + \sqrt{x}}{2} \right] \sin \left[\frac{\sqrt{x+\delta x} - \sqrt{x}}{2} \right]}{2x(\sqrt{x+\delta x} + \sqrt{x}) \frac{(x+\delta x) - x}{2}}$$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{-\sin \left[\frac{\sqrt{x+\delta x} + \sqrt{x}}{2} \right]}{\sqrt{x+\delta x} + \sqrt{x}} \sin x \frac{\left[\frac{\sqrt{x+\delta x} + \sqrt{x}}{2} \right]}{\frac{\sqrt{x+\delta x} - \sqrt{x}}{2}}$$

$$(-1) \frac{\sin \frac{2\sqrt{x}}{2}}{2\sqrt{x}} \left[\frac{\sqrt{x+\delta x} - \sqrt{x}}{2} \rightarrow 0 \text{ when } \delta x \rightarrow 0 \right]$$

$$= \frac{-1 \sin \sqrt{x}}{2 \sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-\sin \sqrt{x}}{2 \sqrt{x}}$$

Q2. Differentiate the following w.r.t the variables involved

i. $x^2 \sec 4x$

Solution

$$\text{Let } y = x^2 \sec 4x$$

$$\frac{dy}{dx} = \frac{d}{dx} [x^2 \sec 4x]$$

$$= x^2 \frac{d}{dx} [\sec 4x] + \sec 4x \frac{d}{dx} (x^2)$$

$$= x^2 (\sec 4x + \tan 4x) \times 4 + (\sec 4x) \times 2x$$

$$\frac{d}{dx} [x^2 \sec 4x] = 2x \sec 4x \times [2 \tan 4x + 1] \quad \text{Ans}$$

ii. $\tan^2 \theta \sec^2 \theta$

Solution

Differentiate w.r.t ' θ ' we have

$$\frac{dy}{d\theta} = \frac{d}{d\theta} [\tan^2 \theta \sec^2 \theta]$$

$$= \sec^2 \theta \left[\frac{d}{d\theta} (\tan^2 \theta) \right] + \tan^2 \theta \frac{d}{d\theta} (\sec^2 \theta)$$

$$= \sec^2 \theta \left[3 \tan^2 \theta \frac{d}{d\theta} (\tan \theta) \right] + \tan^2 \theta \left[2 \sec \theta \frac{d}{d\theta} (\sec \theta) \right]$$

$$= \sec^2 \theta \left[3 \tan^2 \theta \frac{d}{d\theta} (\tan \theta) \right] + \tan^2 \theta [2 \sec \theta \sec \theta \tan \theta]$$

$$= 3 \tan^2 \theta \sec^4 \theta + 2 \sec^2 \theta \tan^4 \theta$$

$$\frac{d}{d\theta} (\tan^3 \theta \sec^2 \theta) = \sec^2 \theta \tan^2 \theta \tan [3 \sec^2 \theta + 2 \tan^2 \theta] \quad \text{Ans}$$

iii. $(\sin 2\theta - \cos 3\theta)^2$

Solution

Let $y = (\sin 2\theta - \cos 3\theta)^2$

Differentiating w.r.t. ' θ ' we have,

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{d}{d\theta} [\sin 2\theta - \cos 3\theta]^2 \\ &= 2[\sin 2\theta - \cos 3\theta] \frac{d}{d\theta} (\sin 2\theta - \cos 3\theta) \\ &= 2[\sin 2\theta - \cos 3\theta] \left[\cos 2\theta (-\sin 3\theta) \frac{d}{d\theta} (3\theta) \right] \frac{d}{d\theta} \\ &\quad (\sin 2\theta - \cos 3\theta) \\ &= 2(\sin 2\theta - \cos 3\theta) [(\cos 2\theta) \cdot 2 + (\sin 3\theta) 3] \\ \frac{dy}{d\theta} &= 2(\sin 2\theta - \cos 3\theta) [2\cos 2\theta + 3\sin 3\theta]\end{aligned}$$

Ans

iv. $\cos \sqrt{x} + \sqrt{\sin x}$

Solution

Let $y = \cos \sqrt{x} + \sqrt{\sin x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\cos^{1/2} + \sin^{1/2} x] \\ &= \frac{d}{dx} (\cos x^{1/2}) + \frac{d}{dx} (\sin^{1/2} x) \\ &= -\sin x^{1/2} x^{-1/2} + \frac{1}{2} \sin^{-1/2} x \cos x \\ &= -\frac{\sin \sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}} \\ &= \frac{1}{2} \left(\frac{\sin \sqrt{x}}{\sqrt{x}} + \frac{\cos x}{\sqrt{\sin x}} \right)\end{aligned}$$

Ans

Q3. Find $\frac{dy}{dx}$ if

i. $y = x \cos y$ ii. $x = y \sin y$

i. $y = x \cos y$

Solution

Differentiating w.r.t. "x" we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[x \cos y] \\ &= x \frac{d}{dx}(\cos y) + \cos y \frac{d}{dx}(x) \\ &= x \left[-\sin y \frac{dy}{dx} \right] + \cos y\end{aligned}$$

$$\frac{dy}{dx} = -x \sin y \frac{dy}{dx} + \cos y$$

$$\frac{dy}{dx} + x \sin y \frac{dy}{dx} = \cos y$$

$$(1 + x \sin y) \frac{dy}{dx} = \cos y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos y}{1 + x \sin y} \quad \text{Ans}$$

ii. $x = y \sin y$

Solution

Differentiating w.r.t 'x' we have

$$\frac{dy}{dx}(x) = \frac{d}{dx}(y \sin y)$$

$$1 = y \frac{d}{dx}(\sin y) + \sin y \frac{dy}{dx}$$

$$1 = y \cos y \times \frac{dy}{dx} + \sin y \frac{dy}{dx}$$

$$1 = (y \cos y + \sin y) \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{y \cos y + \sin y} \quad \text{Ans}$$

Q4. Find the derivative w.r.t 'x'

i. $\sqrt{\frac{1+x}{1+2x}}$ ii. $\sin \sqrt{\frac{1+2x}{1+x}}$

i. $\sqrt{\frac{1+x}{1+2x}}$

Solution

Let $y = \cos u$ where $u = \sqrt{\frac{1+x}{1+2x}}$

$$\frac{dy}{dx} = \sin u \quad (1)$$

$$= \sin u \quad (1)$$

$$\frac{d}{dx}(u) = \frac{d}{dx} \left[\frac{1+x}{1+2x} \right]$$

$$= \frac{1}{2} \left(\frac{1+x}{1+2x} \right)^{-1/2} \frac{(1+2x) \frac{d}{dx}(1+x) - (1+x) \frac{d}{dx}(1+2x)}{(1+2x)^2}$$

$$= -\frac{1}{2} \sqrt{\frac{1+2x}{1+x}} \times \frac{(1+2x)(1) - (1+x)(2)}{(1+2x)^2}$$

$$\frac{du}{dx} = \frac{1}{2} \sqrt{\frac{1+2x}{1+x}} \times \frac{-1}{(1+2x)^2}$$

$$= x \frac{-1}{\sqrt{1+x}(1+2x)^{3/2}}$$

Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

$$= \sin u \times \frac{-1}{2\sqrt{1+x}(1+2x)^{3/2}}$$

$$= \frac{\sin u}{2\sqrt{1+x}(1+2x)^{3/2}}$$

$$\frac{dy}{dx} = \left(\cos \sqrt{\frac{1+x}{1+2x}} \right)$$

$$= \frac{\sin \sqrt{\frac{1+x}{1+2x}}}{2\sqrt{1+x}(1+2x)^{3/2}}$$

Ans

ii. $\sin \sqrt{\frac{1+2x}{1+x}}$

Solution

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sin \sqrt{\frac{1+2x}{1+x}} \right)$$

$$= \cos \sqrt{\frac{1+2x}{1+x}} \times \frac{d}{dx} \left[\sqrt{\frac{1+2x}{1+x}} \right]$$

$$= \cos \sqrt{\frac{1+2x}{1+x}} \times \frac{1}{2} \left[\frac{1+2x}{1+x} \right]$$

$$= \frac{(1+x) \frac{d}{dx}(1+2x) - (1+2x) \frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{1}{2} \left[\cos \sqrt{\frac{1+2x}{1+x}} \right] \sqrt{\frac{1+2x}{1+x}}$$

$$\frac{(1+x)(2) - (1+2x)(1)}{(-x)}$$

$$= \frac{1}{2} \cos \sqrt{\frac{1+2x}{1+x}} \times \frac{1}{\sqrt{1+2x}(1+x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{\cos \sqrt{\frac{1+2x}{1+x}}}{2\sqrt{1+2x}(1+x)^{3/2}}$$

Ans

Q5. Differentiate

i. $\sin x$ w.r.t. $\cot x$

Solution

Let $y = \sin x$ and $u = \cot x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x), \quad \frac{du}{dx} = \frac{d}{dx}(\cos x)$$

$$\frac{dy}{dx} = \cos x, \quad \frac{du}{dx} = -\operatorname{cosec}^2 x$$

$$\Rightarrow \frac{dx}{du} = \frac{1}{\operatorname{cosec}^2 x} \quad \Rightarrow \frac{dx}{du} = \sin^2 x$$

Now $\frac{dy}{du} \frac{dx}{du}$ by chain rule

$$\frac{dy}{du} = (\cos x)(-\sin^2 x)$$

$$= -(\cos x \sin^2 x) \quad \text{Ans}$$

ii. $\sin^2 x$ w.r.t. $\cos^4 x$

Solution

Let $y = \sin^2 x$ and $u = \cos^4 x$

we shall find $\frac{d}{dx}(u) = \frac{d}{dx} \cos^4 x$

$$= 4\cos^3 x \frac{d}{dx}(\cos x)$$

and $y = \sin^2 x$

$$\text{and } \frac{dy}{dx} = \frac{d}{dx}(\sin x)^2 \cdot 4\cos^3 x(-\sin x)$$

$$\frac{du}{dx} = -4\cos^3 x \sin x$$

$$\frac{du}{dx} = -4\cos^3 x \sin x$$

$$= 2 \left(\sin x \frac{d}{dx} \right) (\sin x)$$

$$\frac{dy}{dx} = 2\sin x \times \cos x \times \frac{dx}{du} = \frac{-1}{4\cos^3 x \sin x}$$

$$\Rightarrow \frac{dx}{du} = \frac{dy}{dx} \frac{dx}{du}$$

$$\begin{aligned}
 &= 2\sin x \times \cos x \times \frac{-1}{4\cos^3 x \sin x} \\
 &= \frac{-1}{2\cos^2 x} \\
 \frac{d(y)}{du} &= \frac{-1}{2} \sec^2 x
 \end{aligned}$$

Ans

Q6. If $\tan y(1 + \tan x) = 1 - \tan x$, show that $\frac{dy}{dx} = -1$

$$\frac{dy}{dx} = -1$$

Solution

$$\tan y(1 + \tan x) = 1 - \tan x$$

Diff. w.r.t x

$$\sec^2 y \frac{dy}{dx} (1 + \tan x) + \tan y (\sec^2 x) = \sec^2 x$$

$$(1 + \tan^2 y)(1 + \tan x) \frac{dy}{dx} = \sec^2 x (1 - \tan y)$$

$$\left[1 + \left(\frac{1 - \tan x}{1 + \tan x} \right)^2 \right] (1 + \tan x) \frac{dy}{dx} = \sec^2 x$$

$$\left[1 + \frac{1 - \tan x}{1 + \tan x} \right] \left[\frac{1 - \tan x}{1 + \tan x} \right] \frac{dy}{dx} = \sec^2 x$$

$$\left[\frac{1 + \tan^2 x + 2 \tan x + 1 - 2 \tan x + \tan^2 x}{(1 + \tan x)^2} \right] (1 + \tan x) \frac{dy}{dx} = \sec^2 x$$

$$\frac{dy}{dx} = -\sec^2 x \left[\frac{1 + \tan x + 1 - \tan x}{1 + \tan x} \right]$$

$$\frac{2(1 + \tan^2 x) dy}{1 + \tan x dx} = -\sec^2 x \left[\frac{2}{1 + \tan x} \right]$$

$$\left[\frac{2 \sec^2 x}{1 + \tan x} \right] \frac{dy}{dx} = \frac{2 \sec^2 x}{1 + \tan x}$$

$$\Rightarrow \frac{dy}{dx} = -1 \quad \text{Ans}$$

Q7. $y = \sqrt{\sqrt{\tan x} + \sqrt{\tan x} + \sqrt{\tan x} + \dots \infty}$

Prove that $(2y - 1) \frac{dy}{dx} = \sec^2 x$

Solution

$$y = \sqrt{\tan x + (\sqrt{\tan x + \sqrt{\tan x} + \dots \infty})}$$

$$y^2 = \tan x + \sqrt{\tan x + \dots \infty}$$

$$y^2 = \tan x + y$$

Differentiating w.r.t. 'x'

$$2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$2y \frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$\Rightarrow (2y - 1) \frac{dy}{dx} = \sec^2 x \quad \text{Ans}$$

Q8. if $x = a \cos^3 \phi$ $y = b \sin^3 \phi$

show that $a \frac{dy}{dx} + b \tan \phi = 0$

Solution

$$x = a \cos^3 \phi$$

Differentiating w.r.t. ' ϕ ' we have

$$\frac{dx}{d\phi} = \frac{d}{d\phi} (a \cos^3 \phi)$$

$$= 3a \cos^2 \phi (-\sin \phi)$$

$$\Rightarrow \frac{dx}{d\phi} = -3a \cos^2 \phi \sin \phi$$

$$\frac{d\phi}{dx} = -\frac{1}{3a \cos^2 \phi \sin \phi}$$

Now $y = b \sin^3 \phi$

diff. w.r.t. ϕ we have

$$\frac{dy}{d\phi} = \frac{d}{d\phi} (b \sin^3 \phi)$$

$$= b(3\sin^2 \phi \cdot \cos \phi)$$

$$= 3b \sin^2 \phi \cdot \cos \phi$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{d\phi} \times \frac{d\phi}{dx}$$

$$= 3b \sin^2 \phi \cdot \cos \phi \times \frac{-1}{3a \cos^2 \phi \sin \phi}$$

$$\frac{dy}{dx} = -\frac{b}{a} \tan \phi$$

$$a \frac{dy}{dx} = -b \tan \phi$$

$$\Rightarrow a \frac{dy}{dx} + b \tan \phi = 0 \quad \text{Ans}$$

Q9. Find $\frac{dy}{dx}$ if $x = a(\cos t + \sin t)$

$$y = a(\sin t - t \cos t)$$

Solution

$$x = a(\cos t - \sin t) \dots\dots\dots(1)$$

$$y = a(\sin t - t \cos t) \dots\dots\dots(2)$$

Diff.(1) w.r.t t both sides

$$\frac{dx}{dt} = a[\cos t - (\cos t - t \sin t)]$$

$$= a[\cos t - \cos t + t \sin t]$$

$$= at \sin t.$$

Diff.(1) w.r.t

$$\frac{dx}{dt} = a(-\sin t + \cos t)$$

$$\text{and } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= at \sin t \times \frac{1}{a(\cos t - \sin t)}$$

$$\text{Now } \frac{dy}{dx} = \frac{t \sin t}{\cos t - \sin t}$$

Ans

Q10. Differentiate w.r.t 'x'

i. $\cos^{-1} \frac{x}{a}$

Solution

$$\text{Let } y = \cos^{-1} \frac{x}{a}$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \cos^{-1} \frac{x}{a}$$

$$= -\frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \frac{d}{dx} \left(\frac{x}{a}\right)$$

$$= -\frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \frac{1}{a}$$

$$= -\frac{1}{\sqrt{\frac{a^2 - x^2}{a^2}}} \frac{1}{a}$$

$$= -\frac{a}{\sqrt{a^2 - x^2}} \frac{1}{a}$$

$$= -\frac{1}{\sqrt{a^2 - x^2}}$$

Ans

ii. $\cot^{-1} \frac{x}{a}$

Solution

$$\text{Let } y = \cot^{-1} \frac{x}{a}$$

Diff.w.r.t. x

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{1+\left(\frac{x}{a}\right)^2} \frac{1}{a} = \frac{1}{1+\frac{x^2}{a^2}} \frac{1}{a} \\ &= -\frac{1}{\frac{a^2+x^2}{a^2}} \frac{1}{a} \\ &= -\frac{a^2}{a^2+x^2} \frac{1}{a} \\ &= -\frac{a}{a^2+x^2} \quad \text{Ans}\end{aligned}$$

iii. $\frac{1}{a} \sin^{-1}\left(\frac{a}{x}\right)$

Solution

Let $y = \frac{1}{a} \sin^{-1} \frac{a}{x}$

Diff.w.r.t. x

$$\begin{aligned}\Rightarrow \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{1}{a} \sin^{-1} \frac{a}{x} \right) \\ &= \frac{1}{a} \left(\frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}} \frac{d}{dx} \left(\frac{a}{x} \right) \right) \\ &= \frac{1}{a} \left(\frac{1}{\sqrt{1-\frac{a^2}{x^2}}} \frac{-a}{x^2} \right) \\ &= \frac{1}{a} \left(\frac{x}{\sqrt{x^2-a^2}} \times \frac{-a}{x^2} \right) \\ &= \frac{-1}{x\sqrt{x^2-a^2}} \quad \text{Ans}\end{aligned}$$

iv. $\sin^{-1}\sqrt{1-x^2}$

Solution

Let $y = \sin^{-1}\sqrt{1+x^2}$

Diff.w.r.t. x

$$\begin{aligned}\Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1-(\sqrt{1-x^2})}} \frac{d}{dx} \sqrt{1-x^2} \\ &= \frac{1}{\sqrt{1-1+x^2}} \frac{1}{2\sqrt{1-x^2}} (-2x) \\ &= \frac{1}{\sqrt{x^2}} \frac{(-2x)}{2(\sqrt{1-x^2})} \\ &= \frac{-2}{2(x)\sqrt{1-x^2}} \\ &= \frac{-1}{\sqrt{1-x^2}}\end{aligned}$$

Ans

$$v. \sec^{-1} \left(\frac{x^2+1}{x^2-1} \right)$$

Solution

$$\text{Let } y = \sec^{-1} \left(\frac{x^2+1}{x^2-1} \right)$$

Diff.w.r.t. x

$$\begin{aligned}\Rightarrow &= \frac{1}{\frac{x^2+1}{x^2-1} \left(\sqrt{\left(\frac{x^2+1}{x^2-1} \right)^2 - 1} \right)} \frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right) \\ &= \frac{x^2-1}{x^2+1} \frac{1}{\sqrt{\frac{(x^2+1)^2}{(x^2-1)^2} - 1}} \left[\frac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2} \right] \\ \Rightarrow &= \frac{(x^2-1)^2}{(x^2+1)\sqrt{(x^2+1)^2 - (x^2-1)^2}} \left[\frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2} \right] \\ &= \frac{-4x}{(x^2+1)\sqrt{x^4 + 2x^3 + 1 - x^4 + 2x^2 - 1}} \\ &= \frac{-4x}{(x^2-1)\sqrt{4x^2}} \\ &= \frac{-4x}{(x^2+1)}\end{aligned}$$

Ans

vi. $\cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Solution

Let $y = \cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Diff.w.r.t. x

$$\begin{aligned}\Rightarrow \frac{dy}{dx} &= -\frac{1}{1+\left(\frac{2x}{1-x^2}\right)^2} \frac{d}{dx}\left(\frac{2x}{1-x^2}\right) \\ &= \frac{1}{1+\frac{4x^2}{(1-x^2)^2}} \left[\frac{2(1-x^2) - (-2x)(2x)}{(1-x^2)^2} \right] \\ &= -\frac{(1-x^2)^2}{(1-x^2)^2 + 4x^2} \left[\frac{2-2x^2+4x^2}{(1-x^2)^2} \right] \\ &= \frac{-(2x^2+2)}{1+x^4-2x^2+4x^2} \\ &= \frac{-(2x^2+2)}{1+2x^2+x^4} \\ &= \frac{-2(x^2+1)}{(x^2+1)^2} \\ &= \frac{-2}{1+x^2}\end{aligned}$$

Ans

vii. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution

Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Diff.w.r.t. x

$$\begin{aligned}\Rightarrow \frac{dy}{dx} &= \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \\ &= \frac{-1}{\sqrt{\frac{(1+x^2)^2 - (1-x^2)^2}{(1+x^2)^2}}} \left[\frac{-2x(1+x^2) - 2x(1-x^2)}{(1+x^2)^2} \right]\end{aligned}$$

$$\begin{aligned}
 &= \frac{(1+x^2)}{\sqrt{(1+x^2)^2 - (1-x^2)^2}} \left[\frac{-2x - 2x^3 - 2x + 2x^3}{(1+x^2)^2} \right] \\
 &= \frac{4x}{\sqrt{1+x^4+2x^2-1-x^4+2x^4}} \left[\frac{-4x}{(1+x^2)^2} \right] \\
 &= \frac{4x}{\sqrt{4x^2(1+x^2)}} \\
 &= \frac{2}{(1+x^2)}
 \end{aligned}$$

Ans

Q11. if $\frac{y}{x} = \tan^{-1}\left(\frac{y}{x}\right)$ show that $\frac{dy}{dx} = \frac{y}{x}$

Solution

$$\frac{y}{x} = x \tan^{-1}\left(\frac{y}{x}\right)$$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx} \left[x \tan^{-1}\left(\frac{y}{x}\right) \right]$$

$$\frac{dy}{dx} = 1 \times \tan^{-1}\left(\frac{y}{x}\right) + \left[\frac{1}{1+\left(\frac{y}{x}\right)^2} \right] \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\frac{y}{x} + \frac{y^2}{x^2+y^2} \times \frac{y-x\frac{dy}{dx}}{y^2}$$

$$= \frac{y}{x} + \frac{x}{x^2+y^2} \left[y - x \frac{dy}{dx} \right] \text{ from above}$$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{xy}{x^2+y^2} - \frac{x^2}{x^2+y^2} \frac{dy}{dx}$$

$$\left[1 + \frac{x^2}{x^2+y^2} \right] \frac{dy}{dx} = y \frac{(x^2+y^2)+x^2}{x(x^2+y^2)}$$

$$\left[1 + \frac{x^2+y^2+x^2}{x^2+y^2} \right] \frac{dy}{dx} = y \left[\frac{x^2+y^2+x^2}{x(x^2+y^2)} \right]$$

$$\left(\frac{2x^2+y^2}{x^2+y^2} \right) \frac{dy}{dx} = y \left[\frac{x^2+y^2+x^2}{x(x^2+y^2)} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \quad \text{Ans}$$

Q12. If $y = \tan(ptan^{-1}x)$ show that

$$(1 + x^2)y_1 - p(1 + y^2) = 0$$

Solution

$$y = \tan(ptan^{-1}x)$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}[\tan(ptan^{-1}x)] \dots\dots\dots (1)$$

$$\begin{aligned} y_1 &= \sec^2(ptan^{-1}) \cdot \frac{d}{dx}(ptan^{-1}x) \\ &= 1 + \tan^2(ptan^{-1}x) \cdot p \frac{1}{1+x^2} \end{aligned}$$

From(1)

$$(1 + x^2)y_1 = p(1 + y^2)$$

$$(1 + x^2)y_1 - p(1 + y^2) = 0$$

Hence proved.

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