

Solved Exercise 2.4

Q1. Find $\frac{dy}{dx}$ making suitable substitutions in the following function defined as:

i. $y = \sqrt{\frac{1-x}{1+x}}$

Solution

$$y = \sqrt{\frac{1-x}{1+x}}$$

Let $u = \frac{1-x}{1+x}$ (1) then

$$y = u^{\frac{1}{2}}$$
(2)

Diff. w.r.t x , we name

$$\frac{du}{dx} = \frac{(-1)(1+x) - (1)(1-x)}{(1+x)^2}$$

$$= \frac{-1-x-1+x}{(1+x)^2}$$

$$= \frac{-2}{(1+x)^2}$$

Diff. w.r.t u

$$\frac{dy}{du} = \frac{1}{2} u^{\frac{1}{2}-1} \Rightarrow \frac{1}{2} u^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{u}}$$

By Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{x}} \cdot \frac{-2}{(1+x)^2}$$

$$\begin{aligned}\text{But } u &= \frac{1-x}{1+x} \Rightarrow \sqrt{u} = \sqrt{\frac{1+x}{1-x}} \\ &= \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \times \left(\frac{-2}{(1+x)^2} \right) \\ &= \frac{-\sqrt{1+x}}{(\sqrt{1-x})(1+x)^2} \\ &= \frac{-1}{\sqrt{1-x} (1+x)^{1/2}}\end{aligned}$$

ii. $y = \sqrt{x + \sqrt{x}}$

Solution

$$\text{Let } u = \sqrt{x + \sqrt{x}} \dots\dots\dots (1)$$

$$\text{Then } y = \sqrt{u} = u^{\frac{1}{2}} \dots\dots\dots (2)$$

$$\begin{aligned}\text{From (1) } \frac{du}{dx} &= 1 + \frac{1}{2\sqrt{u}} \\ &= \frac{2\sqrt{x+1}}{2\sqrt{x}}\end{aligned}$$

Also from (2)

$$\frac{dy}{du} = \frac{1}{2} u^{\frac{1}{2}-1} = \frac{1}{2\sqrt{u}}$$

By Chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{2\sqrt{u}} \cdot \frac{2\sqrt{x+1}}{2\sqrt{x}}\end{aligned}$$

$$\text{But } u = x + \sqrt{x}$$

$$\begin{aligned}&= \frac{1}{2\sqrt{x+\sqrt{x}}} \times \frac{2\sqrt{x+1}}{2\sqrt{x}} \\ &= \frac{2\sqrt{x+1}}{4\sqrt{x}\sqrt{x+\sqrt{x}}}\end{aligned}$$

$$\text{iii. } Y = x \sqrt{\frac{a+x}{a-x}} = \sqrt{\frac{x^2(a+x)}{a-x}} = \sqrt{\frac{(ax^2+x^2)}{a-x}}$$

Solution

$$\text{Let } t = \frac{ax^2+x^3}{a-x} \text{ therefore } y = \sqrt{t} = t^{1/2}$$

$$\begin{aligned} \frac{dt}{dx} &= \frac{d}{dx} \left(\frac{ax^2+x^3}{a-x} \right) \\ &= \frac{(a-x)d/dx(ax^2+x^3) - (ax^2+x^2)d/dx(a-x)}{(a-x)^2} \\ &= \frac{(a-x)(2ax+3x^2) - (ax^2+x^2)(0-1)}{(a-x)^2} \\ &= \frac{2a^2x+3ax^2-2ax^2-3x^3+ax^2+x^3}{(a-x)^2} \\ &= \frac{2a^2x+2ax^2-2x^3}{(a-x)^2} \\ &= \frac{2x(a^2+ax-x^2)}{(a-x)^2} \end{aligned}$$

$$\begin{aligned} \text{and } \frac{dy}{dx} &= \frac{d}{dt} (t)^{1/2} \\ &= \frac{1}{2} t^{-1/2} \\ &= \frac{1}{2\sqrt{t}} \\ &= \frac{1}{2\sqrt{\frac{ax^2+x^3}{a-x}}} \\ &= \frac{\sqrt{a-x}}{2x\sqrt{a+x}} \end{aligned}$$

By chain rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ &= \frac{a-x}{2x\sqrt{a+x}} \times \frac{2x(a^2+ax-x^2)}{(a-x)^2} \\ &= \frac{a^2+ax-x^2}{\sqrt{a+x}(a-x)^2(a-x)^{1/2}} \end{aligned}$$

$$= \frac{a^2 + ax - x^2}{\sqrt{a+x}(a-x)^{1/2}} \quad \text{Ans}$$

iv. $y = (3x^2 - 2x + 7)^6$

Solution

Let $u = 3x^2 - 2x + 7 \dots\dots\dots(1)$

Then $y = u^6 \dots\dots\dots(2)$

From (1)

$$\frac{du}{dx} = \frac{d}{du}(u)^6 = 6u^5$$

Thus

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Chain rule

$$\begin{aligned} &= 6u^5(6x - 2) = 6(3x^2 - 2x + 7)^5 - 2(3x - 1) \\ &= 12(3x - 1)(3x^2 - 2x + 7)^5 \end{aligned}$$

v. $y = \sqrt{\frac{a^2 + x^2}{a^2 - x^2}}$

Solution

$$y = \sqrt{\frac{a^2 + x^2}{a^2 - x^2}}$$

Let $y = u^{1/2} \dots\dots\dots(1)$

Where $u = \frac{a^2 + x^2}{a^2 - x^2} \dots\dots\dots(2)$

From (1) $\frac{dy}{du} = \frac{d}{du}(u^{1/2}) = \frac{1}{2}u^{-1/2}$

$$= \frac{1}{2} \left[\frac{a^2+x^2}{a^2-x^2} \right]^{-1/2} = \frac{1}{2} \left[\frac{a^2-x^2}{a^2+x^2} \right]^{-1/2}$$

From (2) $\frac{dy}{dx} = \frac{(a^2-x^2)(2x) - (a^2+x^2)(-2x)}{(a^2-x^2)^2}$

$$= \frac{4a^2x}{(a^2-x^2)^2}$$

By chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2} \frac{(a^2-x^2)^{1/2}}{(a^2+x^2)^{1/2}} \cdot \frac{4a^2x}{(a^2-x^2)^2}$$

$$= \frac{2a^2x}{\sqrt{(a^2+x^2)} (a^2-x^2)^{3/2}}$$

Ans

Q2. Find $\frac{dy}{dx}$ if

(i) $3x + 4y + 7 = 0$

Solution

$$3x + 4y + 7 = 0$$

Diff.w.r.t. x

$$3 + 4 \frac{dy}{dx} = 0$$

$$4 \frac{dy}{dx} = -3$$

$$\frac{dy}{dx} = \frac{-3}{4}$$

Ans

ii. Find $\frac{dy}{dx}$ if $xy + y^2 = 2$

Solution

Diff.w.r.t. x

$$\frac{d}{dx}[xy] + \frac{d}{dx}y^2 = \frac{d}{dx}(2)$$

$$x \frac{dy}{dx} + y(1) + 2y \frac{dy}{dx} = 0$$

$$x \frac{d}{dx} + y + 2y \frac{dy}{dx} = 0$$

$$(x + 2y) \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x+2y} \quad \text{Ans}$$

iii. $x^2 - 4xy - 5y = 0$

Solution

Diff.w.r.t. x

$$\frac{d}{dx}(x^2) - 4 \frac{d}{dx}(xy) - 5 \frac{d}{dx}(y) = \frac{d}{dx}(0)$$

$$2x - 4x \left[\frac{dy}{dx} + (1)y \right] - 5 \frac{dy}{dx} = 0$$

$$(4x + 5) \frac{dy}{dx} = \frac{2x-4y}{4x+5} = \frac{2(x-2y)}{4x+5} \quad \text{Ans}$$

iv. Find = if $4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Solution

Diff.w.r.t. x

$$\frac{d}{dx}(4x^2 + 2hxy + by^2 + 2gx + 2fy + c) = \frac{d}{dx}(0)$$

$$8x + 2h \left[x \frac{ay}{ax} + y \right] + b \left[2y \frac{dy}{dx} \right] + 2g + 2f \frac{dy}{dx} = 0$$

$$2(hx + by + f) \frac{dy}{dx} = -2(4x + hy + g)$$

$$\frac{dy}{dx} = \frac{-2(4x+hy+g)}{2(hx+by+f)}$$

$$\frac{dy}{dx} = \frac{4x+hy+g}{hx+by+f} \quad \text{Ans}$$

v. if $x\sqrt{1+y} + y\sqrt{1+x} = 0$

Solution

Diff.w.r.t. x

$$\frac{d}{dx}(x \cdot \sqrt{1+y} \cdot 1 + \frac{d}{dx}(y\sqrt{1+x})) = 0$$

$$(\sqrt{1+y})(1) + \frac{x}{2\sqrt{1+y}} \frac{dy}{dx} + \sqrt{1+x} \frac{dy}{dx} + \frac{1}{2\sqrt{1+x}} = 0$$

$$\left[\frac{x}{2\sqrt{1+y}} + \sqrt{1+x} \right] \frac{dy}{dx} = \frac{-y}{2\sqrt{1+x}} - \left[\frac{y+2\sqrt{1+x}(\sqrt{1+y})}{2\sqrt{1+x}} \right]$$

$$\frac{dy}{dx} = \frac{(y+2\sqrt{1+x}\sqrt{1+y})}{2\sqrt{1+x}} \cdot \frac{2\sqrt{1+y}}{x+2\sqrt{1+x}\sqrt{1+y}}$$

$$\frac{dy}{dx} = \frac{\sqrt{1+y}(y+2\sqrt{(1+x)(1+y)})}{\sqrt{1+x}(x+2\sqrt{(1+x)(1+y)})} \quad \text{Ans}$$

vi. If $y(x^2 - 1) = x\sqrt{x^2 + 4}$

Solution

Diff.w.r.t. x

$$\frac{d}{dx}(y(x^2 - 1)) \frac{d}{dx}(x\sqrt{x^2 + 4})$$

$$\frac{dy}{dx}(x^2 - 1) + y(2x) = 1 \cdot \sqrt{x^2 + 4} + x \cdot \frac{2x}{2\sqrt{x^2 + 4}}$$

$$(x^2 - 1) \frac{dy}{dx} = \sqrt{x^2 + 4} + \frac{x^2}{\sqrt{x^2 + 4}} - 2xy$$

$$\frac{dy}{dx} = \frac{x^2 + 4 + x^2}{\sqrt{x^2 + 4}} - 2xy$$

$$= \frac{2(x^2+2)-2xy\sqrt{x^2+4}}{\sqrt{x^2+4}}$$

$$\frac{dy}{dx} = \frac{2(x^2+2)-2xy\sqrt{x^2+4}}{(x^2-1)\sqrt{x^2+4}}$$

$$= \frac{-2(3x^2+2)}{\sqrt{x^2+4}(x^2+1)^2} \quad \text{Ans}$$

Q3. Find $\frac{dy}{dx}$ of the following parametric functions.

i. $\theta + \frac{1}{\theta}, y = \theta + 1$

ii. $x = \frac{a(1-t)}{1+t^2}, y = \frac{2bt}{1+t^2}$

i. $\theta + \frac{1}{\theta}, y = \theta + 1$

Solution

Take $x = \theta + \theta^{-1}$

Diff. w.r.t θ

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\theta + \theta^{-1})$$

$$= \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(\theta^{-1})$$

$$= 1 - \theta^{-2}$$

$$= 1 - \frac{1}{\theta^2}$$

$$\frac{dx}{d\theta} = \frac{\theta^2-1}{\theta^2}$$

$$y = \theta + 1$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta + 1)$$

$$= \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(1)$$

$$= 1 + 0$$

$$\frac{dy}{d\theta} = 1$$

By chain rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= 1 \times \frac{\theta^2}{\theta^2 - 1} = \frac{\theta^2}{\theta^2 - 1} \quad \text{Ans}\end{aligned}$$

ii. $x = \frac{a(1-t)}{1+t^2}, y = \frac{2bt}{1+t^2}$

Solution

$$x = \frac{a(1-t^2)}{1+t^2}, y = \frac{2bt}{1+t^2}$$

$$\text{Take } \frac{dx}{dt} = \frac{d}{dt} \left[\frac{a(1-t^2)}{1+t^2} \right]$$

$$\begin{aligned}\text{And } \frac{dy}{dt} &= \frac{d}{dt} \left(\frac{2bt}{1+t^2} \right) \\ &= \frac{(1+t^2) \frac{d}{dt}(2bt) - 2bt \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \\ &= \frac{(1+t^2)(2b) - 2bt(0+2t)}{(1+t^2)^2} \\ &= \frac{2b+2bt^2-4bt^2}{(1+t^2)^2} \\ &= \frac{2b-2bt^2}{(1+t^2)^2} \\ &= \frac{2b(1-t^2)}{(1+t^2)^2}\end{aligned}$$

By chain rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ &= \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at} \\ &= -\frac{b(1-t^2)}{2at} \quad \text{Ans}\end{aligned}$$

Q4. Prove that $y \frac{dy}{dx} + x = 0$ if

$$\text{If } x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$$

Solution

$$\text{Take } y = \frac{2t}{1+t^2}$$

Diff.w.r.t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dt} \left(\frac{2t}{1+t^2} \right) \\ &= \frac{(1+t^2)2 - 2t(2t)}{(1+t^2)^2} \\ &= \frac{2+2t^2-4t^2}{(1+t^2)^2} \\ &= \frac{2(1-t^2)}{(1+t^2)^2} \end{aligned}$$

$$\text{Also, } x = \frac{1-t^2}{1+t^2}$$

Diff.w.r.t. x

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt} \left(\frac{1-t^2}{1+t^2} \right) \\ &= \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} \\ &= \frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2} \\ &= \frac{-4t}{(1+t^2)^2} \end{aligned}$$

By chain rule

$$\begin{aligned} \Rightarrow \frac{dy}{dt} &= \frac{dy}{dx} \times \frac{dx}{dt} \\ &= \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4t} \\ \frac{dy}{dx} &= \frac{t^2-1}{2t} \end{aligned}$$

Now we prove

$$\begin{aligned}
 y \frac{dy}{dx} + x &= 0 \\
 &= \left(\frac{2t}{1+t^2} \right) \left(\frac{t^2-1}{2t} \right) + \frac{1-t^2}{1+t^2} \\
 &= \frac{t^2-1}{1+t^2} + \frac{1-t^2}{1+t^2} \\
 &= \frac{t^2-1+1-t^2}{1+t^2} \\
 &= \frac{0}{1+t^2} \\
 &= 0 \\
 &= R.H.S \quad \text{Ans}
 \end{aligned}$$

Q5. Differentiate

i. $x^2 - \frac{1}{x^2}$ w.r.t. x^4

Solution

Let $y = x^2 - \frac{1}{x^2}$ (1)

and $u = x^4$ (2)

1. $\Rightarrow \frac{dy}{dx} = 2x + \frac{2}{x^3}$

2. $\Rightarrow \frac{du}{dx} = 4x^3$

By chain rule

$$\begin{aligned}
 \frac{dy}{du} &= \left(2x + \frac{2}{x^3} \right) \left(\frac{1}{4x^3} \right) \\
 &= \frac{x^4+1}{2x^4} \quad \text{Ans}
 \end{aligned}$$

ii. $(1+x^2)^n$ w.r.t. x^2

Solution

$$\text{Let } y = (1 + x^2)^n \dots\dots (1)$$

$$\text{and } u = x^2 \dots\dots (2)$$

$$(1) \Rightarrow \frac{dy}{dx} = n(1 + x^2)^{n-1} (2x) \\ = 2nx(1 + x^2)^{n-1}$$

$$(2) \Rightarrow \frac{dn}{du} = \frac{dy}{dx} \cdot \frac{dx}{du} \\ = \frac{2nx(1+x^2)^{n-1}}{2x} \\ = n(1 + x^2)^{n-1} \quad \text{Ans}$$

iii. $\frac{x^2+1}{x^2-1}$ w.r.t $\frac{x-1}{x+1}$

Solution

$$\text{Let } y = \frac{x^2+1}{x^2-1} \dots\dots (1)$$

$$\text{and } u = \frac{x-1}{x+1} \dots\dots (2)$$

$$(1) \Rightarrow \frac{dy}{dx} = \frac{2x(x^2-1) - (2x)(x^2+1)}{(x^2-1)^2} \\ = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2} \\ = \frac{-4x}{(x^2-1)^2}$$

$$\text{And } (2) \Rightarrow \frac{du}{dx} = \frac{1(x+1) - 1(x-1)}{(x+1)^2} \\ = \frac{x+1-x+1}{(x+1)^2} \\ = \frac{2}{(x+1)^2}$$

Now

$$\begin{aligned}\frac{dy}{du} &= \frac{dy}{dx} \times \frac{dx}{du} \\ &= \frac{-4x}{(x^2-1)^2} \times \frac{(x-1)^2}{2} \\ &= \frac{-2x(x+1)^2}{(x-1)^2(x+1)^2} \\ \frac{dy}{du} &= \frac{-2x}{(x-1)^2} \quad \text{Ans}\end{aligned}$$

iv. Let $y = \frac{ax+b}{cx+d}$ w.r.t $\frac{ax^2+b}{ax^2+d}$

Solution

$$\text{Let } y = \frac{ax+b}{cx+d} \quad (1)$$

$$\text{and } u = \frac{ax^2+b}{cx^2+d} \quad (2)$$

$$\begin{aligned}(1) \Rightarrow \frac{dy}{dx} &= \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} \\ &= \frac{acx+ad-acx-bc}{(cx+d)^2} \\ &= \frac{ad-bc}{(cx+d)^2}\end{aligned}$$

$$\begin{aligned}(2) \Rightarrow \frac{du}{dx} &= \frac{2ax(cx^2+d) - 2cx(ax^2+b)}{(cx^2+d)^2} \\ &= \frac{2acx^3+2adx-2acx^3-2bcx}{(cx^2+d)^2} \\ &= \frac{2adx-2bcx}{(cx^2+d)^2} \\ &= \frac{2ax(d-b)}{(cx^2+d)^2}\end{aligned}$$

Now

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$

$$\begin{aligned}\frac{dy}{du} &= \frac{ad-bc}{(cx+d)^2} \times \frac{(cx^2+d)^2}{2x(ad-bc)} \\ &= \frac{(ad-bc)}{(cx^2+d)^2} \times \frac{(ax^2+d)^2}{2ax(d-b)} \\ &= \frac{(ad-bc)(ax^2+d)^2}{2ax(d-b)^2(cx+d)^2}\end{aligned}$$

Ans

v. $\frac{x^2+1}{x^2-1}$ **w.r.t.** x^3

Solution

Let $y = \frac{x^2+1}{x^2-1} \dots\dots\dots (1)$

$u = x^3 \dots\dots\dots (2)$

Diff.(1) Both Sides

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2} \\ &= \frac{2x^3 - 2x + 2x^3(x^2+1)}{(x^2-1)^2} \\ &= \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2}\end{aligned}$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

and $\frac{dy}{dx} = 3x^3$

By chain rule

Now
$$\begin{aligned}\frac{dy}{du} &= \frac{dy}{dx} \times \frac{dx}{du} \\ &= \frac{-4x}{(x^2-1)^2} \times \frac{1}{3x^3}\end{aligned}$$

$$= \frac{-4}{3x(x^2-1)^2}$$

Ans

Perfect24u.com

Perfect24u.com

Perfect24u.com

Perfect24u.com

Perfect24u.com

Perfect24u.com

Perfect24u.com

Perfect24u.com

Perfect24u.com

Perfect24u.com

Perfect24u.com

Perfect24u.com

Perfect24u.com

Perfect24u.com