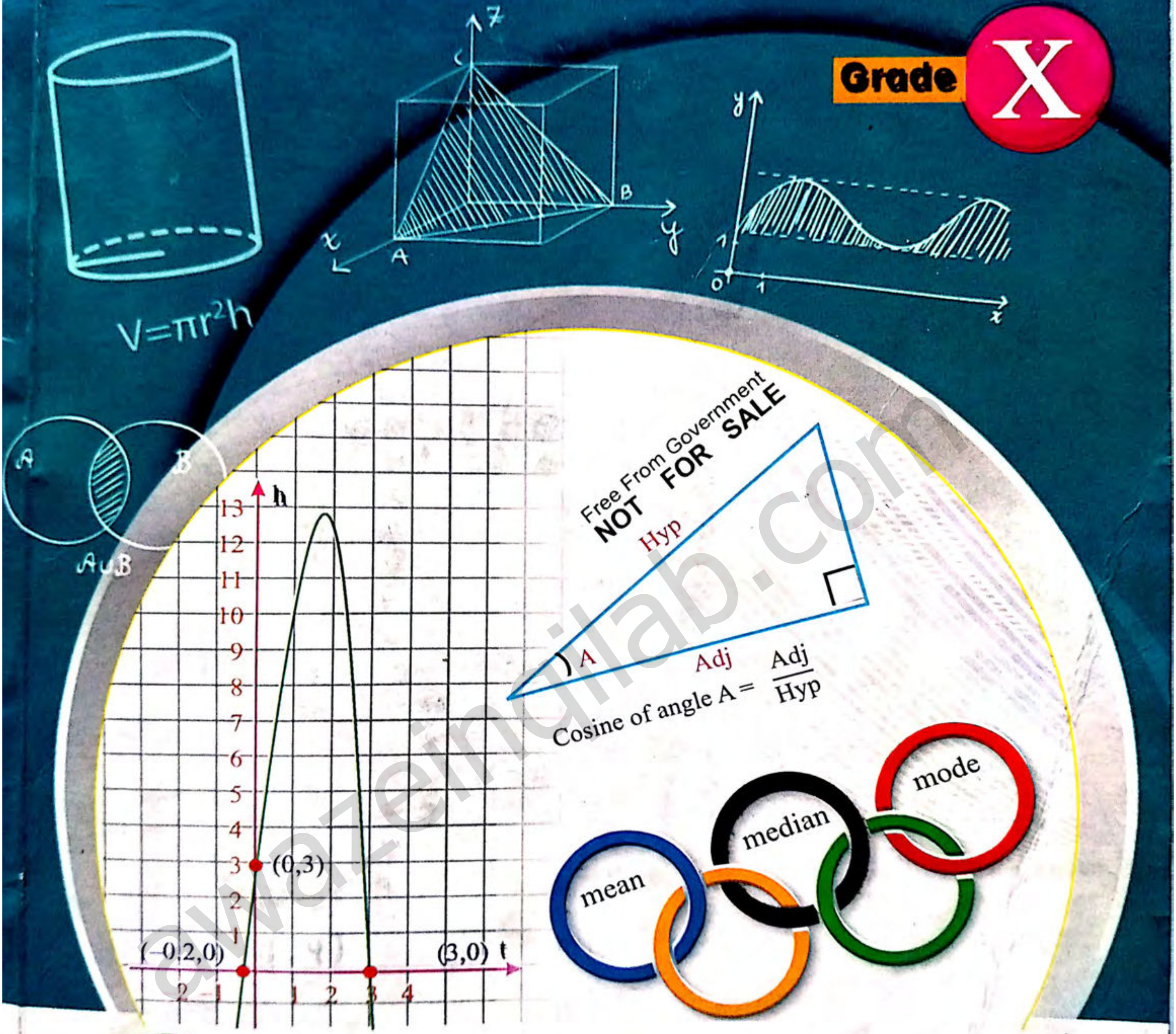
A TEXTBOOK OF

Mathematics

Grade X



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Unit 1

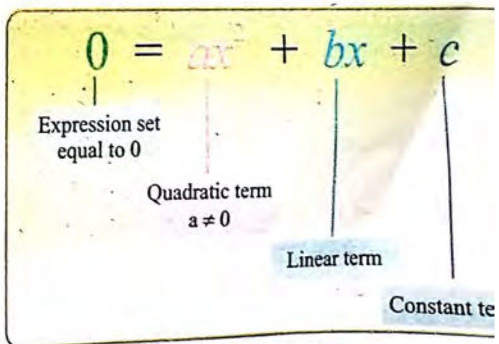
QUADRATIC EQUATIONS

In this unit the students will be able to

- Define a quadratic equation.
- Solve a quadratic equation in one variable by
  - Factorization
  - Completing square method
- Use method of completing square to derive the quadratic formula.
- Use the quadratic formula to solve quadratic equations.
- Solve equations, reducible to quadratic form, of the type: $ax^4+bx^2+c=0$.
- Solve the equations of the type: $ap(x)+\frac{b}{p(x)}=c$.
- Solve reciprocal equations of the type $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$.
- Solve exponential equations in which the variables occur in exponents.
- Solve equations of the type $(x+a)(x+b)(x+c)(x+d)=k$, where $a+b=c+d$.
- Solve equations of the type:
  - $\sqrt{ax+b}=cx+d$.
  - $\sqrt{x+a}+\sqrt{x+b}=\sqrt{x+c}$.
  - $\sqrt{x^2+px+m}+\sqrt{x^2+px+n}=q$.

$$ax^2+bx+c=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$



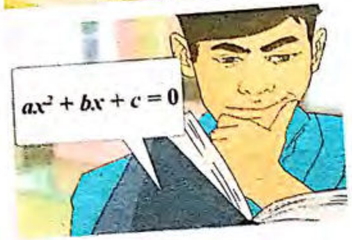
Why it's important

Balls, Arrows, Missiles and Stones

When you throw a ball (or shoot an arrow, fire a missile or throw a stone) it goes up into the air, slowing as it travels, then comes down again faster and faster ...

... and a Quadratic Equation tells you its position at all times!

Quadratic Equations are useful in many other areas:
For a parabolic mirror, a reflecting telescope or a satellite dish, the shape is defined by a quadratic equation.



Quadratic equations are also needed when studying lenses and curved mirrors.
Many questions involving time, distance and speed need quadratic equations.

1.1 Quadratic equation

A quadratic equation in one variable is an equation that can be written in the form $ax^2 + bx + c = 0$, where a, b, and c, are real numbers and $a \neq 0$.

$5x^2 - 3x + 3 = 0$ — this makes it Quadratic

Tidbit
The name Quadratic comes from "quad" meaning square because the highest power of the variable is 2.

Examples of quadratic equations include
$x^2 + x + 1 = 0$, $2x^2 - 3 = 0$, $x^2 + 2x = 0$, and $x^2 = 3x + 1$.

1.2 Solution of quadratic equations

All those values of the variable for which the given equation is true are called solution or roots of the equation, and the set of all solutions is called solution set.

For example, the quadratic equation $x^2 - 9 = 0$ is true only for $x = 3$ and $x = -3$, hence $x = 3$ and $x = -3$ are the solutions or roots of the quadratic equation $x^2 - 9 = 0$ and $\{3, -3\}$ is the solution set.

We may solve quadratic equations by the following methods.

(a) by factorization
(b) by completing the square
(c) by quadratic formula

(a) Solution of a quadratic equation by factorization to its

A quadratic equation can easily be solved by splitting it in factors. The factorization method is illustrated in the following examples:

Example 1 Solving a quadratic equation with factorization.

Solve each quadratic equation by factorization. Check your results

(i) $2x^2 + 2x - 11 = 1$ **(ii)** $12t^2 = t + 1$

Solution

(i) Start by writing the equation in the form $ax^2 + bx + c = 0$.

$2x^2 + 2x - 11 = 1$		given equation
$2x^2 + 2x - 12 = 0$		subtract 1 from each side
$x^2 + x - 6 = 0$		divide each side by 2 (optional step)
$(x + 3)(x - 2) = 0$		factor
$x + 3 = 0$ or	$x - 2 = 0$	zero-product property
$x = -3$ or	$x = 2$	solve

These solution can be checked by substituting them in the given equation.

$2(-3)^2 + 2(-3) - 11 = 1$ $\quad$ $2(2)^2 + 2(2) - 11 = 1$
$1 = 1$ (true) $\quad$ $1 = 1$ (true)

(ii) Write the equation in the form $at^2 + bt + c = 0$.

$12t^2 = t + 1$		given equation
$12t^2 - t - 1 = 0$		subtract t and 1
$(3t - 1)(4t + 1) = 0$		factorize
$3t - 1 = 0$ or	$4t + 1 = 0$	zero-product property
$t = \frac{1}{3}$ or	$t = \frac{-1}{4}$	solve

To check these solutions, substitute them into the given equation.

$12\left(\frac{1}{3}\right)^2 = \frac{1}{3} + 1$ $\quad$ $12\left(-\frac{1}{4}\right)^2 = -\frac{1}{4} + 1$

$\frac{4}{3} = \frac{4}{3}$ (true) $\quad$ $\frac{3}{4} = \frac{3}{4}$ (true)

Try This
Solve the following equations
(i). $a^2 + 7a - 8 = 0$
(ii). $6p^2 + 7p - 20 = 0$

A ball is thrown straight up, from 3 m above the ground with a velocity of 14 m/s. When does it hit the ground?

Solution

We can work out its height by adding up these three things:

The height starts at 3 m: 3

It travels upwards at 14 meters per second (14 m/s): 14t

Gravity pulls it down, changing its position by *about* 5 m per second squared: ...$-5t^2$

Add them up and the height h at any time t is:

> **Note**
> t is time in seconds

$$h = 3 + 14t - 5t^2$$

And the ball will hit the ground when the height is zero:

$$3 + 14t - 5t^2 = 0$$

Or $$5t^2 - 14t - 3 = 0.$$

Which is a quadratic equation.

Let us solve it ...

Replace the middle term with −15 and 1

i.e $$5t^2 - 15t + t - 3 = 0$$

$$5t(t-3) + 1(t-3) = 0$$

$$(5t+1)(t-3) = 0$$

And the two solutions are:

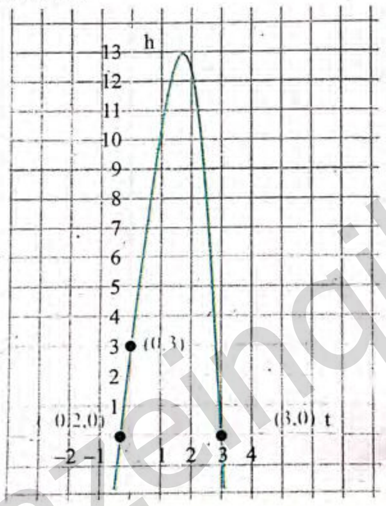
$$5t + 1 = 0 \quad \text{or} \quad t - 3 = 0$$

$$t = -0.2 \quad \text{or} \quad t = 3$$

The "$t = -0.2$" is a negative time which is impossible in our case.

The "$t = 3$" is the answer we want:

The ball hits the ground after 3 seconds!



> **Note**
> (0,3) says when $t = 0$ (at the start) the ball is at 3 m.
> (3,0) says that at 3 seconds the ball is at ground level.

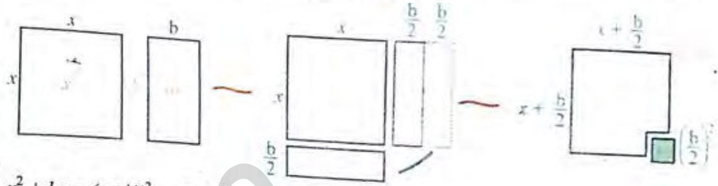
Study Tip

The examples in the text are carefully chosen to prepare you for success with the exercise sets. Study the step-by-step solutions of the examples, noting the substitutions and explanations. The time you spend studying the examples will save your valuable time when you do your homework.

(b) Solution of a quadratic equation by completing square

Some quadratic equations are not easily factorized and hence it is not easy to find their solution by factorization.

For example, the quadratic equation $x^2 + bx = -c$ cannot be easily factorized. Such quadratic equation can be solved by completing the square method



$$x^2 + bx = (x + ½)^2 - (½ b)^2$$

$$\left(x + \frac{1}{2}b\right)^2 = \left(\frac{1}{2}b\right)^2 - c.$$

$$x + \frac{1}{2}b = \pm\sqrt{\left(\frac{1}{2}b\right)^2 - c}$$

$$x = -\frac{1}{2}b \pm \sqrt{\left(\frac{1}{2}b\right)^2 - c}$$

Try This

What must be added to obtain a perfect square? (i). $x^2 + 5x$ (ii). $q^2 - 4q$

Steps involved in completing the square

(i). Write the quadratic equation in its standard form.
(ii). Divide both sides of the equation by the co-efficient of x^2 if it is other than 1.
(iii). Shift the constant term to the right-hand side of the equation.
(iv). Square half the co-efficient of x and add the square to both sides.
(v). Write the left-hand side of the equation as a perfect square and simplify the right-hand side.
(vi). Take square root of both sides of the equation and solve the resulting equation to find the solutions of the equation.

Example 3 Completing the square.

Solve each equation

$x^2 - 8x + 9 = 0$

Solution

Start by writing the equation in the form $x^2 + kx = d$.

$x^2 - 8x + 9 = 0$	given equation
$x^2 - 8x = -9$	subtract 9 from each side
$x^2 - 8x + 16 = -9 + 16$	add $\left(\frac{x}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = 16$
$(x-4)^2 = 7$	the perfect square
$x - 4 = \pm\sqrt{7}$	square root property
$x = 4 \pm \sqrt{7}$	add 4 to each side

Avoid Errors

When completing the square to solve an equation make sure that
- the coefficient of x^2 is 1
- you add the term $(b/2)^2$ to both sides of the equation.

1.3 Quadratic formula

The general form of quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are real numbers and "a" is not equal to zero. By using completing the square method we can derive the quadratic formula for the solution of all quadratic equations.

$ax^2 + bx + c = 0$

$ax^2 + bx = -c$ shift the constant term to the right of the equation

$x^2 + \frac{b}{a}x = -\frac{c}{a}$ divide all terms of both sides by a.

Add $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ to both sides of the equation.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$\Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Which is the required quadratic formula.

Example 4 Using the quadratic formula.

Solve the equation $3x^2 - 6x + 2 = 0$

Solution

Let a = 3, b = –6 and c = 2

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ quadratic formula

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)}$ substitute for a, b and c

$x = \frac{6 \pm \sqrt{12}}{6}$ simplify

$x = 1 \pm \frac{1}{\sqrt{3}}$

Activity

Use an online calculator To solve the following quadratic equations step by step

$2x^2 - 9x + 4 = 0$

Try This

Solve

(i). $4x^2 + 3x - 2 = 0$

(ii). $9x^2 - 42x + 49 = 0$

(iii). $5x^2 - 10x + 13 = 0$

Example 5

A company is making frames as part of a new product they are launching. The frame will be cut out of a piece of steel. To keep the weight down, the final area should be 28 cm². The inside of the frame has to be 11 cm by 6 cm. What should the width x of the metal be?

Solution Area of steel before cutting:

Area = $(11 + 2x) \times (6 + 2x)$ cm² $= 4x^2 + 34x + 66$cm²

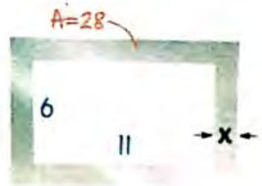
Area of steel after cutting out the 11 × 6 middle:

Area = $4x^2 + 34x + 66 - 66 = 4x^2 + 34x$

Since the area equals 28 cm².

$4x^2 + 34x = 28$.

Or $2x^2 + 17x - 14 = 0$. Here a = 2, b = 17 and c = –14



$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ quadratic formula

$x = \frac{-(17) \pm \sqrt{(17)^2 - 4(2)(-14)}}{2(2)}$ substitute the values

$x = \frac{-17 \pm \sqrt{401}}{4}$ simplify

x is about –9.3 or 0.8

The negative value of x make no sense, so the answer is:

$x = 0.8$ cm (approx.)

Exercise 1.1

1. Solve each of the following equations by factorization.
   - (i). $x^2+5x+4=0$
   - (ii). $(x-3)^2=4$
   - (iii). $x^2+3x-10=0$
   - (iv). $6x^2-13x+5=0$
   - (v). $3(x^2-1)=4(x+1)$
   - (vi). $x(3x-5)=(x-6)(x-7)$
2. Solve each of the following equations by completing the square.
   - (i). $x^2+6x-40=0$
   - (ii). $x^2-10x+11=0$
   - (iii). $4x^2+12x=0$
   - (iv). $5x^2-10x-840=0$
   - (v). $9x^2-6x+\frac{5}{9}=0$
   - (vi). $(x-1)(x+3)=5(x+2)-3$
3. Solve each of the following equations by quadratic formula.
   - (i). $x^2-8x+15=0$
   - (ii). $x^2-2x-4=0$
   - (iii). $4x^2+3x=0$
   - (iv). $3x(x-2)+1=0$
   - (v). $6x^2-17x+12=0$
   - (vi). $\frac{x^2}{3}-\frac{x}{12}=\frac{1}{24}$
4. Find all solutions to the following equations
   - (i). $t^2-8t+7=0$
   - (ii). $72+6x=x^2$
   - (iii). $r^2+4r+1=0$
   - (iv). $x(x+10)=10(-10-x)$
5. The equation $(y+13)(y+a)$ has no linear term. Find value of a.
6. The equation $ax^2+5x=3$ has $x=1$ as a solution. What is the other solution?
7. What is the positive difference of the roots of $x^2-7x-9=0$?

Activity

Verify the answers of Exercise 1.1 by using an online calculator.

Activity

Solve the following quadratic equation $x^2-8x+15=0$, by

(i) Factorization (ii) Completing square

Verify the answer by using an online calculator.

1.4 Solution of equations reducible to quadratic form

The following types of equations can be reduced to the quadratic form and can be solved by any of the method learnt above.

Type Equation of the form $ax^4+bx^2+c=0$.

Example 6 Solve $12x^4-11x^2+2=0$

Solution By making substitution $y=x^2$ the equation becomes $12y^2-11y+2=0$, which is a quadratic equation in terms of y and can be solved by factorization,

$$(3y-2)(4y-1)=0$$
$$3y-2=0 \text{ or } 4y-1=0$$
$$\Rightarrow \quad y=\frac{2}{3} \text{ or } y=\frac{1}{4}$$

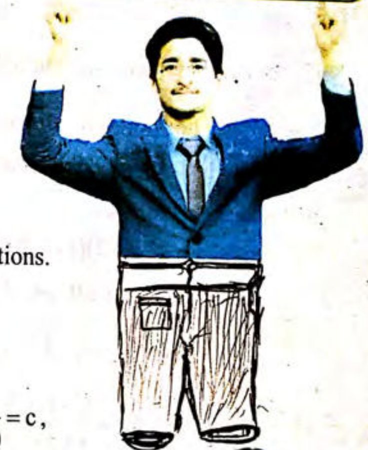
To find x, use the fact that $y=x^2$, therefore,

$$x^2=\frac{2}{3} \text{ or } x^2=\frac{1}{4}$$
$$\Rightarrow \quad x=\pm\sqrt{\frac{2}{3}} \text{ or } x=\pm\sqrt{\frac{1}{4}}$$
$$\Rightarrow \quad x=\pm\sqrt{\frac{2}{3}} \text{ or } x=\pm\frac{1}{2}$$

The given equation $12x^4-11x^2+2=0$ has four solutions.

Thus the solution set is $\left\{\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{2}\right\}$.

Polynomial of degree four is called biquadratic.



Type Equation of the form $a\,p(x)+\frac{b}{p(x)}=c$,

Example 7 Solve $2x+\frac{4}{x}=9$

Solution

$$2x+\frac{4}{x}=9$$

Multiplying both sides by x, we get

$$x\left(2x+\frac{4}{x}\right)=9x$$
$$\Rightarrow \quad 2x^2+4=9x$$

$$\Rightarrow \quad 2x^2-9x+4=0$$

which is a quadratic equation in x and can be solved by factorization

$$(2x-1)(x-4)=0$$

$$\Rightarrow \quad x=\frac{1}{2} \text{ or } x=4$$

Hence the solutions are $\frac{1}{2}, 4$ and the solution set is $\left\{\frac{1}{2}, 4\right\}$.

Example 8 Solve $\frac{x-1}{x+3}+\frac{x+3}{x-1}=\frac{13}{6}$

> **WARNING**
> When you perform a substitution of variable, you must remember to go back and to express the answers in terms of the original variable

Solution

$$\frac{x-1}{x+3}+\frac{x+3}{x-1}=\frac{13}{6}$$

Let $y=\frac{x-1}{x+3}$. Then $\frac{1}{y}=\frac{x+3}{x-1}$

Therefore, the given equation reduces to $y+\frac{1}{y}=\frac{13}{6}$.

Multiplying both sides by $6y$, we get

$6y^2-13y+6=0$, which is a quadratic equation in terms of x and can be solved by factoring as follows:

$$(2y-3)(3y-2)=0$$

$$\Rightarrow \quad 2y-3=0 \text{ or } 3y-2=0$$

$$\Rightarrow \quad y=\frac{3}{2} \text{ or } y=\frac{2}{3}$$

If $y=\frac{3}{2}$, then $\frac{x-1}{x+3}=\frac{3}{2}$

$$\Rightarrow \quad 2x-2=3x+9$$

$$\Rightarrow \quad x=-11$$

If $x=\frac{2}{3}$, then $\frac{x-1}{x+3}=\frac{2}{3}$

$$\Rightarrow \quad 3x-3=2x+6$$

$$\Rightarrow \quad x=9$$

Thus the solution set is $\{-11,9\}$.

Type III Reciprocal equation of the form: $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$

Example 9 Solve the following equations.

(i) $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0$ (ii) $8\left(x^2+\frac{1}{x^2}\right)-42\left(x-\frac{1}{x}\right)+29=0$

Solution

(i) The given equation is $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0$

Let $x+\frac{1}{x}=y$, square both sides $x^2+\frac{1}{x^2}+2=y^2$ or $x^2+\frac{1}{x^2}=y^2-2$

Therefore, the given equation reduces as follows:

$$2(y^2-2)-9y+14=0$$

$$\Rightarrow \quad 2y^2-4-9y+14=0$$

$$\text{or} \quad 2y^2-9y+10=0$$

The equation can be factorized

$$\therefore \quad (2y-5)(y-2)=0$$

$$\therefore \quad y=\frac{5}{2} \text{ or } y=2$$

Now $y=\frac{5}{2}$

$$\Rightarrow \quad x+\frac{1}{x}=\frac{5}{2}$$

$$\Rightarrow \quad 2x^2+2=5x$$

$$\text{or} \quad 2x^2-4x-x+2=0$$

$$\Rightarrow \quad 2x^2-4x-x+2=0$$

$$\Rightarrow \quad 2x(x-2)-1(x-2)=0$$

$$\Rightarrow \quad (x-2)(2x-1)=0$$

and $y=2$

$$\Rightarrow \quad x+\frac{1}{x}=2$$

$$\Rightarrow \quad x^2+1=2x$$

$$\Rightarrow \quad x^2-2x+1=0$$

$$\Rightarrow \quad (x-1)^2=0$$

$$\therefore \quad x=1,1$$

Given $x=2$ or $x=\frac{1}{2}$

Hence the solutions are $2, \frac{1}{2}, 1$ and 1. The solution set is $\left\{2, \frac{1}{2}, 1\right\}$.

(ii) The given equation is $8\left(x^2+\frac{1}{x^2}\right)-42\left(x-\frac{1}{x}\right)+29=0$

Let $x-\frac{1}{x}=y$. Then $\left(x-\frac{1}{x}\right)^2=y^2 \Rightarrow x^2+\frac{1}{x^2}-2=y^2$

$$\Rightarrow x^2+\frac{1}{x^2}=y^2+2.$$

Therefore, the given equation reduces to

$$8(y^2+2)-42y+29=0$$
$$8y^2+16-42y+29=0$$
$$\Rightarrow \quad 8y^2-42y+45=0$$
$$\text{or} \quad 8y^2-42y+30y+45=0$$
$$\Rightarrow \quad 4y(2y-3)-15(2y-3)=0$$
$$\Rightarrow \quad (2y-3)(4y-15)=0$$
$$\Rightarrow \quad 2y-3=0 \text{ or } 4y-15=0$$

give $y=\frac{3}{2}$ *or* $y=\frac{15}{4}$

Now $y=\frac{3}{2}$

$\Rightarrow \quad x-\frac{1}{x}=\frac{3}{2}$

$\therefore \quad \frac{x^2-1}{x}=\frac{3}{2}$

or $\quad 2(x^2-1)=3x$

$\Rightarrow \quad 2x^2-3x-2=0$

$(x-2)(2x+1)=0$

$(x-2)=0$ or $(2x+1)=0$

$x=2$ or $x=-\frac{1}{2}$

and $y=\frac{15}{4}$

$\Rightarrow \quad x-\frac{1}{x}=\frac{15}{4}$

$\therefore \quad \frac{x^2-1}{x}=\frac{15}{4}$

or $\quad 4(x^2-1)=15x$

$\Rightarrow \quad 4x^2-15x-4=0$

$(x-4)(4x+1)=0$

$(x-4)=0 \quad (4x+1)=0$

$x=4$ or $x=-\frac{1}{4}$

Hence the required solution set is 2, $\left\{2, \frac{-1}{2}, 4, \frac{-1}{4}\right\}$.

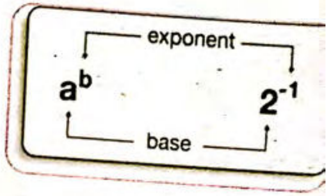
Type IV

Equations that involve terms of the form a^x where $a>0$, $a \neq 1$ are called **exponent equations.** These equations can be reduced to quadratic equations by making the substituti $y=a^x$, which changes the equations into quadratic equations in term of y.

Example 10 Solve $4.2^{2x}-10.2^x+4=0$

Solution

$$4.2^{2x}-10.2^x+4=0$$



We may write the given equation as

$$4.(2^x)^2-10.2^x+4=0$$

Let $2^x=y$. The above equation reduces to

$$4y^2-10y+4=0$$
$$\Rightarrow \quad 2y^2-5y+2=0 \qquad \textit{(Taking 2 common)}$$
$$\Rightarrow \quad (2y-1)(y-2)=0 \qquad \textit{(Factorizing)}$$
$$\Rightarrow \quad 2y-1=0 \quad or \quad y-2=0$$
$$gives \quad y=\frac{1}{2} \quad or \quad y=2$$
$$\Rightarrow \quad 2^x=\frac{1}{2} \quad or \quad 2^x=2$$
$$\Rightarrow \quad 2^x=2^{-1} \quad or \quad 2^x=2^1$$
$$\Rightarrow \quad x=-1 \quad or \quad x=1$$

Thus the solution set is $\{-1,1\}$.

Example 11 Solve the equation $2^{2+x}+2^{2-x}=10$

One-to-One Property of Exponential Functions

If $b^n=b^m$ then $n=m$

Solution

$$2^{2+x}+2^{2-x}=10$$
$$\Rightarrow \quad 2^2\cdot 2^x+2^2\cdot 2^{-x}-10=0$$

Let $\quad 2^x=y$. Then $2^{-x}=\frac{1}{y}$

The above equation reduces to

$$4y+\frac{4}{y}-10=0$$
$$\Rightarrow \quad 4y^2-10y+4=0 \qquad \text{taking 2 common}$$
$$\Rightarrow \quad 2y^2-5y+2=0$$

$\Rightarrow \quad (2y-1)(y-2)=0$ factorizing

$\Rightarrow \quad 2y-1=0 \quad or \quad y-2=0$

$\Rightarrow \quad y=\frac{1}{2} \quad or \quad y=2$

$\Rightarrow \quad 2^x=\frac{1}{2} \quad or \quad 2^x=2$

$\Rightarrow \quad 2^x=2^{-1} \quad or \quad 2^x=2^1$

$\Rightarrow \quad x=-1 \quad or \quad x=1$

Thus the solution set is $\{-1,1\}$.

Type V Equations of the form $(x+a)(x+b)(x+c)(x+d)=k$ where $a+b=c+d$

Example 12 Solve $(x+1)(x+3)(x-2)(x-4)=24$

Solution As $1+(-2)=3+(-4)$

So re-arranging the factors on the left side of the given equation, we have

$$[(x+1)(x-2)][(x+3)(x-4)]=24$$

$$\Rightarrow \quad (x^2-x-2)(x^2-x-12)-24=0$$

Let $x^2-x=y$. The above equation becomes

$$(y-2)(y-12)-24=0$$

$$\Rightarrow \quad y^2-14y+24-24=0$$

$$\Rightarrow \quad y^2-14y=0$$

$$\Rightarrow \quad y(y-14)=0$$

$$\Rightarrow \quad y=0 \quad or \quad y=14$$

If $y=0$, then $x^2-x=0$

$\Rightarrow \quad x(x-1)=0$

gives $x=0$ or $x=1$. If $y=14$, then $x^2-x=14$

$\Rightarrow x^2-x-14=0$

$$\Rightarrow x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-14)}}{2(1)}=\frac{1\pm\sqrt{1+56}}{2}=\frac{1\pm\sqrt{57}}{2}=\frac{1+\sqrt{57}}{2},\frac{1-\sqrt{57}}{2}$$

Thus the solution set is $\left\{0, 1, \frac{1+\sqrt{57}}{2}, \frac{1-\sqrt{57}}{2}\right\}$.

Exercise 1.2

1. Solve the following equations.

(i). $x^4-5x^2+4=0$ (ii). $x^4-7x^2+12=0$

(iii). $6x^4-13x^2+5=0$ (iv). $x+2-\frac{1}{x+2}=\frac{3}{2}$

(v). $x-\frac{4}{x}=2$ (vi). $\frac{x+2}{x-2}-\frac{x-2}{x+2}=\frac{5}{6}$

(vii). $3\left(x^2+\frac{1}{x^2}\right)-16\left(x+\frac{1}{x}\right)+26=0$ (viii). $\left(x+\frac{1}{x}\right)^2-10\left(x+\frac{1}{x}\right)+16=0$

(ix). $\left(x^2+\frac{1}{x^2}\right)-\left(x-\frac{1}{x}\right)-4=0$ (x). $3^{2x}-10.3^x+9=0$

(xi). $3.3^{2x+1}-10.3^x+1=0$ (xii). $5^{x+1}+5^{2-x}=126$

(xiii). $(x-3)(x+9)(x+5)(x-7)=385$ (xiv). $(x+1)(x+2)(x+3)(x+4)+1=0$

(xv). $(x+1)(x+3)(x+5)(x+7)+16=0$

2. Solve the equation $x^4-x^3-2x^2+2x+1=0$

Math Fun

Take any four digit number, follow these steps, and you'll end up with 6174.

1. Choose a four digit number (the only condition is that it has at least two different digits).
2. Arrange the digits of the four digit number in descending then ascending order.
3. Subtract the smaller number from the bigger one.
4. Repeat.

Eventually you'll end up at 6174, which is known as Kaprekar's constant. If you then repeat the process you'll just keep getting 6174 over and over again.

1.5 Radical equations

An equation in which the variable appears in one or more radicands is called a radical equation.

For example,

$\sqrt{x+2}=3,\ \sqrt{2x+3}=2x+5,\ \sqrt{x+5}=\sqrt{2x-1},\ 3\sqrt{x^2+x+1}=2$ are radical equations.

To solve radical equations, we transform the given equation into an equation that contains no radicals by squaring it.
A solution of the transformed equation that does not satisfy the original radical equation is called an extraneous solution.

Note
Check all roots (solutions) of the transformed equation in the original equation to exclude extraneous roots.

Type I Equation of the form $\sqrt{ax+b}=cx+d$

Example 13 Solve $\sqrt{27-3x}=x-3$

Solution

$$\sqrt{27-3x}=x-3$$

Squaring both sides, we get $\left(\sqrt{27-3x}\right)^2=(x-3)^2$

Tidbit
We can get rid of a square root by squaring.

$$27-3x=x^2-6x+9$$

$\Rightarrow\quad 0=x^2-6x+9-27+3x$

or $\quad x^2-3x-18=0$

$\quad (x-6)(x+3)=0 \quad$ factorizing

$\Rightarrow\quad x-6=0 \quad or \quad x+3=0$

gives $\quad x=6 \quad$ or $\quad x=-3$

Now it is necessary to check the solutions in the original equation.

$$\sqrt{27-3x}=x-3$$

If $x=6$, then $\sqrt{27-3x}=x-3$

$\sqrt{27-3(6)}=6-3$

$\sqrt{27-18}=3$

$\sqrt{9}=3$

$3=3$ (true)

If $x=-3$, then $\sqrt{27-3x}=x-3$

$\sqrt{27-3(-3)}=-3-3$

$\sqrt{27+9}=-6$

$\sqrt{36}=-6$

$6=-6$ (false)

On checking, we find that $x=-3$ is an extraneous root.
Thus the solution set is $\{6\}$.

Type II Equation of the form $\sqrt{x+a}+\sqrt{x+b}=\sqrt{x+c}$

Example 14 Solve $\sqrt{x+2}+\sqrt{x+7}=\sqrt{x+23}$

Solution

$$\sqrt{x+2}+\sqrt{x+7}=\sqrt{x+23}$$

Squaring both sides, we get

$$\left(\sqrt{x+2}+\sqrt{x+7}\right)^2=\left(\sqrt{x+23}\right)^2$$

$$x+2+x+7+2\sqrt{x+2}\cdot\sqrt{x+7}=x+23$$

$\Rightarrow\quad 2x+9+2\sqrt{x+2}\cdot\sqrt{x+7}=x+23$

$\Rightarrow\quad 2\sqrt{(x+2)(x+7)}=14-x$

Squaring both sides again, we get

$\Rightarrow\quad 4(x+2)(x+7)=(14-x)^2$

$\Rightarrow\quad 4(x^2+9x+14)=196-28x+x^2$

$\Rightarrow\quad 4x^2+36x+56=196-28x+x^2$

$\Rightarrow\quad 4x^2-x^2+36x+28x+56-196=0$

$\Rightarrow\quad 3x^2+64x-140=0$

$\Rightarrow\quad 3x^2-6x+70x-140=0$

$\Rightarrow\quad 3x(x-2)+70(x-2)=0$

$\Rightarrow\quad (x-2)(3x+70)=0$

$\Rightarrow\quad x-2=0 \quad$ or $\quad 3x+70=0$

$\quad x=2 \quad$ or $\quad x=-\frac{70}{3}$

On checking, we find that $-\frac{70}{3}$ is an extraneous root.
Thus the solution set is $\{2\}$.

WARNING
You can perform addition only with identical radical forms. Adding unlike radicals is one of the most common mistakes made by students in algebra! You can easily verify that

$$\sqrt{9}+\sqrt{16}=3+4=7$$
$$\sqrt{9+16}=\sqrt{25}=5$$
$$7=\sqrt{9}+\sqrt{16}\neq\sqrt{9+16}=5$$

Type III Equation of the form $\sqrt{x^2+px+m}+\sqrt{x^2+px+n}=q$

Example 13

Solve the equation $\sqrt{x^2+3x+5}+\sqrt{x^2+3x+1}=2$

Solution

$$\sqrt{x^2+3x+5}+\sqrt{x^2+3x+1}=2$$

The given equation can be written as

$$\sqrt{x^2+3x+5}=2-\sqrt{x^2+3x+1}$$

Squaring both sides, we get

$$x^2+3x+5=4+x^2+3x+1-2(2)\sqrt{x^2+3x+1}$$

$$\Rightarrow \quad x^2+3x+5=x^2+3x+5-4\sqrt{x^2+3x+1}$$

$$\Rightarrow \quad 0=-4\sqrt{x^2+3x+1}$$

$$\Rightarrow \quad \sqrt{x^2+3x+1}=0 \qquad \text{squaring both sides}$$

$$\Rightarrow \quad x^2+3x+1=0$$

$$\Rightarrow \quad x=\frac{-3\pm\sqrt{(3)^2-4(1)(1)}}{2(1)}$$

$$=\frac{-3\pm\sqrt{9-4}}{2}$$

$$=\frac{-3\pm\sqrt{5}}{2}$$

Thus the solution set is $\left\{\frac{-3\pm\sqrt{5}}{2}\right\}$.

Did You Know?

Beautiful Number Relationships.

$135=1^1+3^2+5^3$

$175=1^1+7^2+5^3$

$518=5^1+1^2+8^3$

$598=5^1+9^2+8^3$

Exercise 1.3

1. Solve the following equations.

(i). $\sqrt{5x+21}=x+3$ (ii). $\sqrt{2x-1}=x-2$

(iii). $\sqrt{4x+5}=2x-5$ (iv). $\sqrt{29-4x}=2x+3$

(v). $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$ (vi). $\sqrt{x}+\sqrt{3x+1}=\sqrt{5x+1}$

(vii). $\sqrt{6x+40}-\sqrt{x+21}=\sqrt{x+5}$ (viii). $\sqrt{2x-3}+\sqrt{2x+4}=\sqrt{6x+13}$

(ix). $\sqrt{x^2+2x+4}+\sqrt{x^2+2x+9}=5$ (x). $\sqrt{2x^2+3x+5}+\sqrt{2x^2+3x+1}=2$

2. Find $2x+5$ if x satisfies $\sqrt{40-9x}-2\sqrt{7-x}=\sqrt{-x}$

Review Exercise 1

1. At the end of each question, four circles are given. Fill in the correct circle only.

(i). If $(x+1)(x-5)=0$, then the solutions are

○ $x=1,-5$ ○ $x=1, 5$ ○ $x=-1,-5$ ○ $x=-1, 5$

(ii). If $x^2-x-1=0$, then $x=$

○ $\frac{-1\pm\sqrt{5}}{2}$ ○ $-1\pm\frac{\sqrt{5}}{2}$ ○ $\frac{1\pm\sqrt{5}}{2}$ ○ $1\pm\frac{\sqrt{5}}{2}$

(iii). $\frac{-1\pm\sqrt{5}}{2}$ in simplified form is

○ $1\pm\sqrt{24}$ ○ $1\pm\sqrt{6}$ ○ $2\pm\sqrt{6}$ ○ cannot be simplified

(iv). To apply the quadratic formula to $2x^2-x=3$

○ a = 2, b = −1, c = 3 ○ a = 2, b = 1, c = 3

○ a = 2, b = −1, c = −3 ○ a = 2, b = −1, c = 0

(v). If $x^2-3x-4=0$, then the solutions are

○ $x=4,-1$ ○ $x=-4, 1$ ○ $x=4, 1$ ○ $x=-4,-1$

(vi). If $2x^2+4x-9=0$, the solutions are

○ $x=\frac{2\pm\sqrt{22}}{2}$ ○ $x=\frac{-2\pm\sqrt{22}}{2}$ ○ $x=2\pm\frac{\sqrt{22}}{2}$ ○ $x=-2\pm\frac{\sqrt{22}}{2}$

(vii). $x^2-\frac{1}{4}=0$, the solution are

○ $x=\pm\frac{1}{2}$ ○ $x=\pm\frac{1}{4}$ ○ $x=\pm\frac{1}{8}$ ○ $x=\pm\frac{1}{16}$

(viii). What are the solutions of the equation $x^2 + 7x - 18 = 0$?

○ 2 or –9 ○ –2 or 9 ○ –2 or –9 ○ 2 or 9

(ix). Which of the following values of x are roots of the equation $x^2 - 8x + 15 = 0$?

○ $x = 1$ or $x = -7$ ○ $x = 2$ or $x = 4$

○ $x = -2$ or $x = 4$ ○ $x = 3$ or $x = 5$

2. Solve $2w^4 - 5w^2 + 2 = 0$.
3. Find the constants a and b such that $x = -1$ and $x = 1$ are both solutions to the equation $ax^2 + bx + 2 = 0$.
4. Find all values of x such that $x^2 + 5x + 6$ and $x^2 + 19x + 34$ are equal.

Challenge !

5. Find the solutions to the equation: $49x^2 - 316x + 132 = 0$
If you can factorize this successfully, you have probably mastered the art of factorizing.

PROJECT

Find some internet sites which solve quadratic equations. Which site do you think is better and why?

Activity

Juwaria tried to solve the quadratic equation $x^2 + 5x - 2 = 0$ by completing the square, but she made a mistake. In which line of her working, shown below, did she make the mistake?

$x^2 + 5x - 2 = 0$

[Line 1] $\Rightarrow x^2 + 5x = 2$

[Line 2] $\Rightarrow x^2 + 5x + (5/2)^2 = 2 + (5/2)^2$

[Line 3] $\Rightarrow (x + 5/2)^2 = 2 + 10/4$

[Line 4] $\Rightarrow (x + 5/2)^2 = 18/4$

[Line 5] $\Rightarrow (x + 5/2) = \pm\sqrt{(18/4)} = \pm(\sqrt{18})/(\sqrt{4}) = \pm\sqrt{18}/2$

[Line 6] $\Rightarrow x = \pm\sqrt{18}/2 - 5/2$

[Line 7] $\Rightarrow x = -4.62$ or -0.38 to 2 decimal places.

Summary

The following table summarizes important topics related to quadratic equations.

Concept	Explanation	[Exam]ples
Quadratic equation	$ax^2 + bx + c = 0$, where a, b, and c are constants with $a \neq 0$.	A quadratic equation can have zero, one, or two real solutions. $x^2 = -5$ No real solutions $(x-2)^2 = 0$ One real solution $x^2 - 4 = 0$ Two real solution
Factorizing	A symbolic technique for solving equations, based on the zero-product property: if $ab = 0$, then either $a = 0$ or $b = 0$.	$x^2 - 3x + 2 = 0$ $x^2 - 3x + 2 = 0$ $(x-1)(x-2) = 0$ $x - 1 = 0$ or $x - 2 = 0$ $x = 1$ or $x = 2$
Square root property	The solutions to $x^2 = k$ are $x = \pm\sqrt{k}$, where $k \geq 0$.	$x^2 = 9$ is equivalent to $x = \pm 3$. $x^2 = 11$ is equivalent to $x = \pm\sqrt{11}$.
Completing the square	To solve $x^2 + kx = d$ symbolically, add $\left(\frac{k}{2}\right)^2$ to each side to obtain to perfect square trinomial. Then apply the square root property.	$x^2 - 6x = 1$ $x^2 - 6x + 9 = 1 + 9$ $\left(\frac{-6}{2}\right)^2 = 9$ $x^2 - 6x + 9 = 1 + 9$ $(x-3)^2 = 10$ $x - 3 = \pm\sqrt{10}$ $x = 3 \pm \sqrt{10}$
Quadratic formula	The solutions to $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Always gives the exact solutions.	To solve $2x^2 - x - 4 = 0$, let $a = 2$, $b = -1$, and $c = -4$. $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-4)}}{2(2)}$ $= \frac{1 \pm \sqrt{33}}{4} \approx 1.69, -1.19$

The following table outlines important concepts in this section.

Solving radical equations	The solutions to $a = b$ are among the solutions to $a^n = b^n$ when n is a positive integer. Check your results.	Solve $\sqrt{2x + 3} = x$ $2x + 3 = x^2$ Square each able. $x^2 - 2x - 3 = 0$ Rewrite equations $x = -1$ or $x = 3$ Factor and adobe. Checking reveals that 3 is the only solution.

To complete the square of a quadratic equation

Add and Subtract

$$\left(\frac{\text{Coefficient of "x"}}{2}\right)^2$$

Unit 2

THEORY OF QUADRATIC EQUATIONS

In this unit the students will be able to

- Define discriminant ($b^2 - 4ac$) of the quadratic expression $ax^2 + bx + c$.
- Find discriminant of a given quadratic equation.
- Discuss the nature of roots of a quadratic equation through discriminant.
- Determine the nature of roots of a given quadratic equation and verify the result by solving the equation.
- Determine the value of an unknown involved in a given quadratic equation when the nature of its roots is given.
- Find cube roots of unity.
- Recognize complex cube roots of unity as ω and ω^2.
- Prove the properties of cube roots of unity.
- Use properties of cube roots of unity to solve appropriate problems.
- Find the relation between the roots and the coefficients of a quadratic equation.
- Find the sum and product of roots of a given quadratic equation without solving it.
- Find the value(s) of unknown(s) involved in a given quadratic equation when
  - sum of roots is equal to a multiple of the product of roots,
  - sum of the squares of roots is equal to a given number,
  - roots differ by a given number,
  - roots satisfy a given relation (e.g. the relation $2\alpha + 5\beta = 7$ where α and β are the roots of given equation),
  - both sum and product of roots are equal to a given number.
- Define symmetric functions of roots of a quadratic equation.
- Evaluate a symmetric function of the roots of a quadratic equation in terms of its coefficients.
- Establish the formula, $x^2 - (\text{Sum of roots})x + (\text{Product of roots}) = 0$, to find a quadratic equation from the given roots.
- Form the quadratic equation whose roots, for example, are of the type:
  - $2\alpha + 1, 2\beta + 1$,
  - α^2, β^2,
  - $\frac{1}{\alpha}, \frac{1}{\beta}$,
  - $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$,

  - $\alpha+\beta, \frac{1}{\alpha}+\frac{1}{\beta}$,

  where α, β are the roots of a given quadratic equation.
- Describe the method of synthetic division.
- Use synthetic division to
  - find quotient and remainder when a given polynomial is divided by a linear polynomial,
  - find the value(s) of unknown(s) if the zeros of a polynomial are given,
  - find the value(s) of unknown(s) if the factors of a polynomial are given,
  - solve a cubic equation if one root of the equation is given,
  - solve a biquadratic (quartic) equation if two of the real roots of the equation are given.
- Solve a system of two equations in two variables when
  - one equation is linear and the other is quadratic,
  - both the equations are quadratic.
- Solve the real life problems leading to quadratic equations.

Why it's important

The discriminant tells you the number and types of answers (roots) you will get. The discriminant can be +, –, or 0 which actually tells you a lot! Since the discriminant is under a radical, think about what it means if you have a positive or negative number or 0 under the radical.

2.1 The discriminant of a quadratic equation

In the quadratic formula, the expression $b^2 - 4ac$, is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

The value of the discriminant is used to determine the number of solutions of a quadratic equation and number of x intercepts of the graph of the related function. An x intercept of the graph is the x – coordinate of a point where the graph crosses the x – axis.

Cases	Case (i)	Case (ii)	Case (iii)
Discriminant	$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
Nature of roots	The roots are unequal and real. Roots are rational if $b^2 - 4ac$ is perfect square otherwise they are irrational.	The root are equal.	The roots are unequal and imaginary.

Example 1

Find discriminant of the quadratic equation $x^2+9x+2=0$.

Solution

Comparing the coefficient $x^2+9x+2=0$ with $ax^2+bx+c=0$, we have

$a=1, b=9, c=2$

$\therefore$ Discriminant $= b^2-4ac=(9)^2-4(1)(2)=8-18=73$

2.1.1 Nature of roots of a quadratic equation through discriminant

Example 2

Examine the nature of the roots of the following quadratic equations.

(i) $x^2-8x+16=0$ (ii) $x^2+9x+2=0$

(iii) $6x^2-x-15=0$ (iv) $4x^2+x+1=0$

Solution

(i) Comparing $x^2-8x+16=0$ with $ax^2+bx+c=0$, we have

$a=1, b=-8, c=16.$

Discriminant $= b^2-4ac=(-8)^2-4(1)(16)=64-64=0$

Since discriminant = 0, therefore, the roots of the given equation are real (rational) and equal.

(ii) In x^2+9x+2, we have,

Here $a=1, b=9, c=2$

Discriminant $= b^2-4ac=(9)^2-4(1)(2)=81-8=73$

Since discriminant > 0, but not a perfect square, therefore, the roots are real, unequal and irrational.

(iii) Here $a=6, b=-1, c=-15$

Discriminant $= b^2-4ac=(-1)^2-4(6)(-15)=1+360=361=(19)^2$

As the discriminant is a perfect square, therefore, the roots are real, unequal and rational.

(iv) Here $a=4, b=1, c=1$

Discriminant $= b^2-4ac=(1)^2-4(4)(1)$

$=1-16$

$=-15$

As the discriminant is negative, therefore, the roots are imaginary and unequal.

2.1.2 Determining and verifying nature of roots.

Example 3 Determine the nature of roots of the following equations and verify the results by solving them by factorization.

(i) $x^2-6x+9=0$ (ii) $x^2+5x+6=0$

Solution

(i) $x^2-6x+9=0$

Here $a=1, b=-6, c=9$. The discriminant is given by

$b^2-4ac=(-6)^2-4(1)(9)=36-36=0$

Since the discriminant is zero, the roots are real and equal.

$x^2-6x+9=0$

$(x-3)(x-3)=0$

$x=3, 3$

Which are real and equal. Hence the result is verified.

(ii) $x^2+5x+6=0$

Here $a=1, b=5, c=6$. The discriminant is given by

$b^2-4ac=(5)^2-4(1)(6)=25-24=1=(1)^2$

Since the discriminant is a perfect square, the roots are unequal and rational.

$x^2+5x+6=0$

$(x+2)(x+3)=0$

$x=-2$ and $x=-3$

Which are real, unequal and rational. Hence the result is verified.

Example 4 Without solving, determine the nature of the roots of the quadratic equation.

$3x^2-4x+6=0.$

Solution We evaluate b^2-4ac using $a=3$, $b=-4$, and $c=6$:

$b^2-4ac=(-4)^2-4(3)(6)=16-72=\ 56$

The discriminant is negative, so the equation has two complex roots.

Example 5 Without solving, determine the nature of the roots of the equation.

$2x^2-7x=-1$

Solution We rewrite the equation in the standard form

$2x^2-7x+1=0$

and then substitute $a=2$, $b=-7$, and $c=1$ in the discriminant. Thus,

$b^2-4ac=(-7)^2-4(2)(1)=49-8=41$

The discriminant is positive and is not a perfect square; thus, the roots are real, unequal, and irrational.

2.1.3 Determining and verifying the value of an unknown nature of roots.

Example 6 Determine the set of values of k for which the given quadratic equations have real roots.

(i) $kx^2+4x+1=0$ (ii) $2x^2+kx+3=0$

Solution

(i) Comparing $kx^2+4x+1=0$ with the quadratic equation $ax^2+bx+c=0$, we have

$a=k,\ b=4,\ c=1$

Discriminant $= b^2-4ac$

$= 4^2-4k$

Since roots are real, so $b^2-4ac \geq 0$

$\Rightarrow \quad 4^2-4k \geq 0$

$\Rightarrow \quad 16-4k \geq 0$

$\Rightarrow \quad 16 \geq 4k$

$\Rightarrow \quad 4 \geq k$

or $\quad k \leq 4$

(ii) Comparing $2x^2+kx+3=0$ with the quadratic equation $ax^2+bx+c=0$, we have

$a=2,\ b=k,\ c=3$

Since roots are real, so $b^2-4ac \geq 0$

$\Rightarrow \quad k^2-4(2)(3) \geq 0$

Discriminant $= b^2-4ac = k^2-4(2)(3) \geq 0$

$\Rightarrow \quad k^2-24 \geq 0$

or $\quad k^2 \geq 24$

$\sqrt{k^2} \geq \sqrt{24}$

$|k| \geq 2\sqrt{6}$

$\pm k \geq 2\sqrt{6}$

$k \geq 2\sqrt{6},\ -k \geq 2\sqrt{6}$

$k \geq 2\sqrt{6}$ or $k \leq -2\sqrt{6}$

Exercise 2.1

1. Find the discriminant of the following quadratic equations:

   (i). $x^2-4x+13=0$ (ii). $4x^2-5x+1=0$ (iii). $x^2+x+1=0$

2. Examine the nature of the roots of the following equations:

   (i). $3x^2-5x+1=0$ (ii). $6x^2+x-2=0$ (iii). $3x^2+2x+1=0$

3. For what value of k the roots of the following equations are equal.

   (i). $x^2+kx+9=0$ (ii). $12x^2+kx+3=0$ (iii). $x^2-5x+k=0$

4. Determine whether the following quadratic equations have real roots and if so, find the roots.

   (i). $x^2+5x+5=0$ (ii). $4x^2+12x+9=0$ (iii). $6x^2+x-2=0$

5. Determine the nature of roots of the following quadratic equations and verify the results by solving them.

   (i). $3x^2-10x+3=0$ (ii). $x^2-6x+4=0$ (iii). $x^2-3=0$

6. For what value of k the roots of the following equations are:

   (a) real (b) imaginary

   (i). $2x^2+3x+k=0$ (ii). $kx^2+2x+1=0$ (iii). $x^2+5x+k=0$

Math Fun

$1 \times 8 + 1 = 9$

$12 \times 8 + 2 = 98$

$123 \times 8 + 3 = 987$

$1234 \times 8 + 4 = 9876$

$12345 \times 8 + 5 = 98765$

$123456 \times 8 + 6 = 987654$

$1234567 \times 8 + 7 = 9876543$

$12345678 \times 8 + 8 = 98765432$

$123456789 \times 8 + 9 = 987654321$

2.2 Cube roots of unity and their properties

2.2.1 Cube root of unity

Let x be a cube root of unity,

then $x = \sqrt[3]{1} = (1)^{\frac{1}{3}}$

$\Rightarrow \quad x^3 = 1$

$\Rightarrow \quad x^3 - 1 = 0$

$\Rightarrow \quad (x-1)(x^2+x+1) = 0$

$\Rightarrow \quad x - 1 = 0 \quad or \quad x^2 + x + 1 = 0$

gives $x = 1$ or $x = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2}$

where $i = \sqrt{-1} \Rightarrow i^2 = -1$

Thus the cube roots of unity are 1, $\dfrac{-1+i\sqrt{3}}{2}$ and $\dfrac{-1-i\sqrt{3}}{2}$.

Here 1 is the real root and $\dfrac{-1+i\sqrt{3}}{2}$ and $\dfrac{-1-i\sqrt{3}}{2}$ are complex roots.

Let one of the complex roots be denoted by the Greek letter ω (read as omega).

Suppose $\omega = \dfrac{-1+i\sqrt{3}}{2}$

Then $\omega^2 = \left(\dfrac{-1+i\sqrt{3}}{2}\right)^2$

$= \dfrac{(-1)^2 + (i\sqrt{3})^2 + 2(-1)(i\sqrt{3})}{4}$

$= \dfrac{1-3-2i\sqrt{3}}{4}$

$= \dfrac{-2-2i\sqrt{3}}{4} = \dfrac{-1-i\sqrt{3}}{2}$

Similarly, if $\omega = \dfrac{-1-i\sqrt{3}}{2}$, then $\omega^2 = \dfrac{-1+i\sqrt{3}}{2}$

2.2.2 Properties of the cube roots of unity

1. The sum of the cube roots of unity is zero i.e. $1 + \omega + \omega^2 = 0$.

If $\omega = \dfrac{-1+i\sqrt{3}}{2}$ and $\omega^2 = \dfrac{-1-i\sqrt{3}}{2}$

Then we have $1 + \omega + \omega^2 = 1 + \dfrac{-1+i\sqrt{3}}{2} + \dfrac{-1-i\sqrt{3}}{2}$

$= \dfrac{2-1+i\sqrt{3}-1-i\sqrt{3}}{2}$

$= \dfrac{0}{2}$

$= 0$

Thus, the sum of the cube roots of unity is zero.

2. The product of the cube roots of unity is 1, i.e. $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$

If $\omega = \dfrac{-1+i\sqrt{3}}{2}$ and $\omega^2 = \dfrac{-1-i\sqrt{3}}{2}$,

then $1 \cdot \omega \cdot \omega^2 = 1 \cdot \left(\dfrac{-1+i\sqrt{3}}{2}\right)\left(\dfrac{-1-i\sqrt{3}}{2}\right)$

$= \dfrac{(-1)^2 - (i\sqrt{3})^2}{4} = \dfrac{1-3i^2}{4}$

$= \dfrac{1-(-3)}{4} \quad (\because i^2 = -1)$

$= \dfrac{1+3}{4}$

$= \dfrac{4}{4}$

$= 1$

Thus, the product of the cube roots of unity is 1.

3. Each complex cube root of unity is reciprocal of the other i.e. $w = \dfrac{1}{w^2}$ and $w^2 = \dfrac{1}{w}$

By property 2, we have $w^3 = 1$.

$\Rightarrow \quad w \cdot w^2 = 1$

$\Rightarrow \quad w = \dfrac{1}{w^2}$

$\Rightarrow \quad w^2 = \dfrac{1}{w}$ Thus, $w = \dfrac{1}{w^2}$ and $w^2 = \dfrac{1}{w}$.

2.2.3 Using properties of cube roots of unity to solve problems

Example 7 Show that $x^3+y^3=(x+y)(x+wy)(x+w^2y)$

Solution R.H.S $=(x+y)(x+wy)(x+w^2y)$

$=(x+y)\left[(x+wy)(x+w^2y)\right]$

$=(x+y)\left[x^2+(w+w^2)xy+w^3y^2\right]$

$=(x+y)(x^2-xy+y^2)$ $\quad(\because 1+w+w^2=0 \text{ and } w^3=1)$

$=x^3+y^3$

$=L.H.S$

Example 8 Evaluate $w^{15},\ w^{24},\ w^{90},\ w^{101},\ w^{-2},\ w^{-13}$

Solution

$w^{15}=(w^3)^5=(1)^5=1$

$w^{24}=(w^3)^8=(1)^8=1$

$w^{90}=(w^3)^{30}=(1)^{30}=1$

$w^{101}=w^{99}\cdot w^2=(w^3)^{33}\cdot w^2=(1)^{33}\cdot w^2=1\cdot w^2=w^2$

$w^{-2}=\frac{1}{w^2}=w$ $\quad$ (by property 3)

$w^{-13}=\frac{1}{w^{13}}=\frac{1}{w^{12}\cdot w}=\frac{1}{(w^3)^4\cdot w}=\frac{1}{(1)^4\cdot w}=\frac{1}{1\cdot w}=\frac{1}{w}=w^2$

Example 9 Show that $\left(-1+i\sqrt{3}\right)^3+\left(-1-i\sqrt{3}\right)^3=16$

Solution Since $w=\frac{-1+i\sqrt{3}}{2}$ and $w^2=\frac{-1-i\sqrt{3}}{2}$

Then $2w=-1+i\sqrt{3}$ and $2w^2=-1-i\sqrt{3}$

L.H.S $=\left(-1+i\sqrt{3}\right)^3+\left(-1-i\sqrt{3}\right)^3$

$=(2w)^3+(2w^2)^3$

$=8w^3+8w^6$

$=8w^3+8(w^3)^2$

$=8(1)+8(1)^2$

$=8+8$

$=16$

$=R.H.S$

Exercise 2.2

1. Find the cube roots of the following numbers.
   (i). -1 (ii). 8 (iii). -27
2. Evaluate:
   (i). $w^{12}+w^{58}+w^{95}$ (ii). $(1+w-w^2)^7$ (iii). $(1+3w-w^2)(1+w-2w^2)$
3. Prove that:
   (i). $(1+2w)(1+2w^2)(1-w-w^2)=6$ (ii). $\left(-1+i\sqrt{3}\right)^4\left(-1-i\sqrt{3}\right)^5=512w^2$
4. Show that:
   (i). $x^3-y^3=(x-y)(x-wy)(x-w^2y)$ (ii). $(1+w)(1+w^2)(1+w^4)(1+w^8)=1$

2.3 Roots and coefficients of a quadratic equation

2.3.1 Relation between the roots and the co-efficient of a quadratic equation

We express the sum and the product of the roots of the quadratic equation in terms of its co-efficient. Let α,β be the roots of the quadratic equation $ax^2+bx+c=0;\ a\neq 0$ where

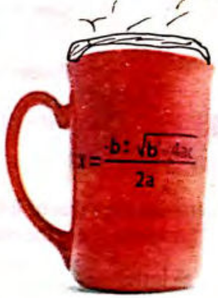
$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Then sum of the roots

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-2b}{2a}=\frac{-b}{a}$$



and product of the roots

$$\alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\cdot\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2}$$

$$=\frac{b^2-b^2+4ac}{4a^2}=\frac{c}{a}$$

Thus sum of the roots $=\alpha+\beta=\frac{-b}{a}=-\frac{co-efficient\ of\ x}{co-efficient\ of\ x^2}$

Product of the roots $=\alpha\beta=\frac{c}{a}=\frac{constant\ term}{co-efficient\ of\ x^2}$

2.3.2 The sum and product of roots of a given quadratic equation without solving it

Example 10

Without solving, find the sum and product of the roots of the equation.

(i) $2x^2-3x-4=0$ (ii) $3x^2+6x-2=0$

Solution

(i) In the equation $2x^2-3x-4=0$ $a=2,\ b=-3$ and $c=-4$

Sum of the roots $=\frac{-b}{a}=-\frac{(-3)}{2}=\frac{3}{2}$

Product of the roots $=\frac{c}{a}=\frac{-4}{2}=-2$

(ii) $3x^2+6x-2=0$

Here $a=3, b=6, c=-2$

Sum of the roots $=\frac{-b}{a}=\frac{-6}{3}=-2$

Product of the roots $=\frac{c}{a}=\frac{-2}{3}=-\frac{2}{3}$

2.3.3 The values of unknown(s) involved in a given quadratic equation

Example 11

Find the value of k so that the sum of the roots of the equation $2x^2+kx+6=0$ is equal to three times the product of its roots.

Solution

The given equation is

$2x^2+kx+6=0$

Here $a=2, b=k, c=6$

$\therefore$ sum of the roots $=\frac{-b}{a}=-\frac{k}{2}$

product of the roots $=\frac{c}{a}=\frac{6}{2}=3$

$\because$ sum of the roots = three times the product of roots

$\Rightarrow \frac{-k}{2}=3(3)=9$

$\Rightarrow k=-18$

Example 12

Find the value of a if the sum of the square of the roots of $x^2-3ax+a^2=0$ is 7.

Solution Let α, β be the roots of the equation $x^2-3ax+a^2=0$

then $\alpha+\beta=-\frac{(-3a)}{1}=3a$ and $\alpha\beta=\frac{a^2}{1}=a^2$.

Given $\alpha^2+\beta^2=7$

$\Rightarrow (\alpha+\beta)^2-2\alpha\beta=7$

$\Rightarrow (3a)^2-2(a^2)=7$

$\Rightarrow 9a^2-2a^2=7$

$\Rightarrow 7a^2=7$

$\Rightarrow a^2=1$

$\Rightarrow a=\pm1$

Example 13 Find the value of k if the roots of $x^2-7x+k=0$ differ by unity.

Solution Let $\alpha, \alpha+1$ be the roots of $x^2-7x+k=0$

Then $\alpha+(\alpha+1)=\frac{-(-7)}{1}$

or $2\alpha+1=7$

or $\alpha=3$

and $\alpha(\alpha+1)=\frac{k}{1}=k$

$\Rightarrow 3(1+3)=k$

$\Rightarrow k=12$

Example 14 If α, β are the roots of $9x^2-27x+k=0$, find the value of k such that $2\alpha+5\beta=7$.

Solution If α, β are the roots of $9x^2-27x+k=0$, then

$\alpha+\beta=\frac{27}{9}=3$ (i)

and $\alpha\beta=\frac{k}{9}$ (ii)

also $2\alpha+5\beta=7$ (iii)

Solving (i) and (iii), we get $\alpha=\frac{8}{3},\ \beta=\frac{1}{3}$

Putting these values in (ii) we obtain $\left(\frac{8}{3}\right)\left(\frac{1}{3}\right)=\frac{k}{9}$

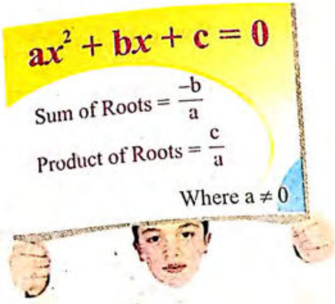
$\Rightarrow k=8$

Example 15 Find the value of m and n if both sum and product of roots of the quadratic equation $mx^2-5x+n=0$ are equal to 10.

Solution The given equation is $mx^2-5x+n=0$

Here sum of the roots $=\frac{5}{m}$

and product of the roots $=\frac{n}{m}$



According to given condition

$$\frac{5}{m}=10$$

$$\Rightarrow \quad m=\frac{1}{2}$$

and $$\frac{n}{m}=10$$

$$\Rightarrow \quad n=10\,m$$

$$=10\left(\frac{1}{2}\right) \qquad \left(\because m=\frac{1}{2}\right)$$

$$n=5$$

$$\therefore \quad m=\frac{1}{2} \text{ and } n=5.$$

Exercise 2.3

1. Without solving the equation, find the sum and products of the roots of the following quadratic equations.
   (i). $4x^2-4x-3=0$ (ii). $2x^2+5x+6=0$ (iii). $3x^2+2x-5=0$
2. Find the value of k if sum of the roots of $2x^2+kx+6=0$ is equal to the product of its roots.
3. Find the value of k if the sum of the square of the roots of $x^2-5kx+6k^2=0$ is equal to 13.
4. Find the value of k if the roots of $x^2-5x+k=0$ differ by unity.
5. Find the value of k if the roots of $x^2-9x+k+2=0$ differ by three.
6. If α, β are the roots of $x^2-5x+k=0$, find k such that $3\alpha+2\beta=12$.
7. Find the value of m and n if both sum and product of roots of the equation $mx^2-3x-n=0$ are equal to $\frac{3}{5}$.

2.4 Symmetric functions of roots of a quadratic equation

Let α, β be the roots of a quadratic equation, then the expressions of the form $\alpha+\beta$, $\alpha\beta$, $\alpha^2+\beta^2$ are called the functions of the roots of the quadratic equation. By symmetric function of the roots of an equation, we mean that the function remains invariant (unchanged) in values when the roots are interchanged. For example, the functions $\alpha+\beta$, $\alpha^2+\beta^2$, $\alpha^3+\beta^3$ are symmetric function of α and β.

2.4.1 Symmetric function of the roots of a quadratic equation in terms of its coefficients

Example 16 If α, β are the roots of $ax^2+bx+c=0$, then find the values of the symmetric functions of the roots of a given quadratic equation in terms of its coefficients.

(i) $\alpha+\beta$ (ii) $\alpha\beta$ (iii) $\alpha^2+\beta^2$ (iv) $\alpha^3+\beta^3$

(v) $\frac{1}{\alpha}+\frac{1}{\beta}$ (vi) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

Solution

(i) As α, β are the roots of the equation $ax^2+bx+c=0$,

$\therefore$ sum of the roots $=\alpha+\beta=-\frac{b}{a}$

(ii) Product of the roots $=\alpha\beta=\frac{c}{a}$

(iii) Since $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$=\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)$$

$$=\frac{b^2}{a^2}-\frac{2c}{a}$$

$$=\frac{b^2-2ac}{a^2}$$

(iv) Since $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$$=\left(-\frac{b}{a}\right)^3-3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)$$

$$=-\frac{b^3}{a^3}+3\frac{bc}{a^2}$$

$$=\frac{3bc}{a^2}-\frac{b^3}{a^3}=\frac{3abc-b^3}{a^3}$$

Did You Know?

Beautiful Number Relationships
$81=(8+1)^2=9^2$
$4913=(4+9+1+3)^3=17^3$

(v) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta}$

$=\frac{\alpha+\beta}{\alpha\beta}$

$=\frac{-\frac{b}{a}}{\frac{c}{a}} \qquad \left(\alpha+\beta=-\frac{b}{a},\ \alpha\beta=\frac{c}{a}\right)$

$=-\frac{b}{c}$

(vi) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$

$=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}=\frac{\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)}{\left(\frac{c}{a}\right)^2}$

$=\frac{\frac{b^2-2ac}{a^2}}{\frac{c^2}{a^2}} \qquad =\frac{b^2-2ac}{c^2}$

2.5 Formation of a quadratic equation whose roots are given

Let α, β be the roots of the quadratic equation $ax^2+bx+c=0$,

Then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

Now $ax^2+bx+c=0;\quad a\neq 0$

$\Rightarrow \quad x^2+\frac{b}{a}x+\frac{c}{a}=0$

$\Rightarrow \quad x^2-\left(-\frac{b}{a}\right)x+\frac{c}{a}=0$

$\Rightarrow \quad x^2-(\alpha+\beta)x+\alpha\beta=0$

$\Rightarrow \quad x^2-Sx+P=0$

Where $S=\alpha+\beta=$ sum of the roots

and $P=\alpha\beta=$ product of the roots

Thus, the formula for forming a quadratic equation whose roots are given is

x^2-(sum of the roots) $x+$ product of the roots $=0$

Example 17 Form a quadratic equation whose roots are $1+\sqrt{5}$ and $1-\sqrt{5}$.

Solution Roots of the required equation are $1+\sqrt{5}$ and $1-\sqrt{5}$

Sum of the roots $=1+\sqrt{5}+1-\sqrt{5}=2$

Product of the roots $=(1+\sqrt{5})(1-\sqrt{5})=1^2-(\sqrt{5})^2=1-5=-4$

$\therefore$ the required quadratic equation is

x^2-(sum of the roots) $x+$ product of the roots $=0$

$\Rightarrow \quad x^2-2x+(-4)=0$

$\Rightarrow \quad x^2-2x-4=0$

Forming an equation with the given roots is the reverse process of solving an equation.

Example 18 Form the quadratic equation whose roots are:

(i) $2a+1, 2b+1$ (ii) a^2, b^2 (iii) $\frac{1}{a}, \frac{1}{b}$ (iv) $\frac{2}{3}, \frac{3}{2}$

Solution (i) The roots of the required equation are $2a+1, 2b+1$

$\therefore$ sum of the roots $= 2a+1+2b+1$

$= 2a+2b+2$

and product of the roots $=(2a+1)(2b+1)$

$=4ab+2a+2b+1$

The required equation is given by

x^2-(sum of the roots) $x+$ product of the roots $=0$

$x^2-(2a+2b+2)x+(4ab+2a+2b+1)=0$

Which is the required quadratic equation.

(ii) The roots of the required quadratic equation are a^2, b^2 therefore,

$x=a^2$ or $x=b^2$

$x-a^2=0$ or $x-b^2=0$

$(x-a^2)(x-b^2)=0$

$x(x-b^2)-a^2(x-b^2)=0$

$x^2-xb^2-a^2x+a^2b^2=0$

$x^2-(a^2-b^2)x+a^2b^2=0$

Which is the required quadratic equation.

(iii) The roots of the required quadratic equation are $\frac{1}{a}, \frac{1}{b}$

Since $x = \frac{1}{a}$ or $x = \frac{1}{b}$

$ax = 1$ or $bx = 1$

$ax - 1 = 0$ or $bx - 1 = 0$

$(ax - 1)(bx - 1) = 0$

$ax(bx - 1) - 1(bx - 1) = 0$

$abx^2 - ax - bx + 1 = 0$

$abx^2 - (a - b)x + 1 = 0$

Which is the required quadratic equation.

(iv) The roots of the required equation are $\frac{2}{5}, \frac{5}{2}$

Therefore $x = \frac{2}{5}$ or $x = \frac{5}{2}$

$5x = 2$ or $2x = 5$

$5x - 2 = 0$ or $2x - 5 = 0$

$(5x - 2)(2x - 5) = 0$

$5x(2x - 5) - 2(2x - 5) = 0$

$10x^2 - 25x - 4x + 10 = 0$

$10x^2 - 29x + 10 = 0$

Which is the required quadratic equation.

$$9 - 24 = 25 - 40$$
$$9 - 24 + 16 = 25 - 40 + 16$$
$$(3 - 4)^2 = (5 - 4)^2$$
$$3 - 4 = 5 - 4$$
$$-1 = 1$$

NOT FOR SALE

Exercise 2.4

1. If α, β are the roots of $ax^2 + bx + c = 0$, find the values of
   (i). $\alpha^3\beta + \beta^3\alpha$ (ii). $(\alpha - \beta)^2$
2. Find the quadratic equation whose roots are
   (i). $1, \frac{1}{2}$ (ii). $-3, 4$ (iii). $3+\sqrt{2},\ 3-\sqrt{2}$ (iv). $a, -2a$
3. Form a quadratic equation whose roots are square of the roots of the equation $ax^2 + bx + c = 0;\ a \neq 0$.
4. If α, β are the roots of $2x^2 + 3x + 1 = 0$, then find the values of
   (i). $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (ii). $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (iii). $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$
5. If α, β are the roots of $3x^2 - 2x + 5 = 0$, find the equation whose roots are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$.
6. If α, β are the roots of $x^2 - 4x + 2 = 0$, find the equation whose roots are $\alpha + \frac{1}{\alpha},\ \beta + \frac{1}{\beta}$.

2.6 Synthetic Division

We are familiar with the process of long division for dividing one polynomial by another. We noticed that if the degrees of the polynomials in the numerator and denominator differ considerably, then the process of long division indeed becomes very long. However if the polynomial in the denominator is of the form $x - a$, then there is a shortcut method called synthetic division.

Steps of Synthetic Division:

Step 1. If the divisor is $x - r$, write r in the box. Arrange the coefficient of the dividend by descending powers of x, supplying a zero coefficient for every missing power.

Step 2. Copy the leading coefficient in the third row.

Step 3. Multiply the latest entry in the third row by the number in the box and write the result in the second row under the next coefficient. Add the number in that column.

Step 4. Repeat step 3 until there is an entry in the third row for each entry in the first row. the last numbers are the coefficient of the quotient in descending order.

WARNING

(a) Synthetic division can be used only when the divisor is a linear factor. Don't forget to write a zero for the coefficient of each missing term.

(b) When dividing by $x - r$. place r in the box. For example, when the divisor is $x + 3$, place -3 in the box, since $x + 3 = x - (-3)$. Similarly, when the divisor is $x - 3$, place +3 in the box, since $x - 3 = x - (+3)$.

NOT FOR SALE

(iii) The roots of the required quadratic equation are $\frac{1}{a}, \frac{1}{b}$

Since $x = \frac{1}{a}$ or $x = \frac{1}{b}$

$ax = 1$ or $bx = 1$

$ax - 1 = 0$ or $bx - 1 = 0$

$(ax - 1)(bx - 1) = 0$

$ax(bx - 1) - 1(bx - 1) = 0$

$abx^2 - ax - bx + 1 = 0$

$abx^2 - (a - b)x + 1 = 0$

Which is the required quadratic equation.

(iv) The roots of the required equation are $\frac{2}{5}, \frac{5}{2}$

Therefore $x = \frac{2}{5}$ or $x = \frac{5}{2}$

$5x = 2$ or $2x = 5$

$5x - 2 = 0$ or $2x - 5 = 0$

$(5x - 2)(2x - 5) = 0$

$5x(2x - 5) - 2(2x - 5) = 0$

$10x^2 - 25x - 4x + 10 = 0$

$10x^2 - 29x + 10 = 0$

Which is the required quadratic equation.

Math Fun

$9 - 24 = 25 - 40$

$9 - 24 + 16 = 25 - 40 + 16$

$(3 - 4)^2 = (5 - 4)^2$

$3 - 4 = 5 - 4$

$-1 = 1$

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1. If α, β are the roots of $ax^2 + bx + c = 0$, find the values of
   (i). $\alpha^3\beta + \beta^3\alpha$ (ii). $(\alpha - \beta)^2$
2. Find the quadratic equation whose roots are
   (i). $1, \frac{1}{2}$ (ii). $-3, 4$ (iii). $3 + \sqrt{2}, 3 - \sqrt{2}$ (iv). $a, -2a$
3. Form a quadratic equation whose roots are square of the roots of the equation $ax^2 + bx + c = 0; a \neq 0$.
4. If α, β are the roots of $2x^2 + 3x + 1 = 0$, then find the values of
   (i). $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (ii). $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (iii). $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$
5. If α, β are the roots of $3x^2 - 2x + 5 = 0$, find the equation whose roots are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$.
6. If α, β are the roots of $x^2 - 4x + 2 = 0$, find the equation whose roots are $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}$.

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Example 19 Use synthetic division to find the quotient $Q(x)$ and the remainder R when the polynomial $3x^3-2x^2-150$ is divided by $x-4$.

Solution Let $P(x) = 3x^3-2x^2-150$

$= 3x^3-2x^2+0x-150$

and $x-a=x-4 \Rightarrow a=4$

Then by synthetic, division we have

4	3	−2	0	−150
		12	40	160
	3	10	40	10

3, 10, 40: Coefficients of quotient; 10: Remainder

Therefore, $Q(x)=3x^2+10x+40$ and $R=10$

Example 20 Use synthetic division to find the value of k if 2 is a zero of the polynomial $2x^4+x^3+kx^2-8$.

Solution Let $P(x)=2x^4+x^3+kx^2-8$

$=2x^4+x^3+kx^2+0x-8$

Since 2 is a zero of $P(x) \Rightarrow P(2)=0$

$\therefore$ $x-2$ is a factor of the polynomial $P(x)$

Now use synthetic division to divide $P(x)$ by $x-2$.

2	2	1	k	0	−8
		4	10	$2k+20$	$4k+40$
	2	5	$k+10$	$2k+20$	$4k+32$

$4k+32$: Remainder

Since the remainder is $4k+32$.

$\Rightarrow$ $P(2)=4k+32$ by Remainder theorem

$\Rightarrow$ $4k+32=0$

$\Rightarrow$ $4k=-32=0$

$\Rightarrow$ $k=-\dfrac{32}{4}$

$\Rightarrow$ $k=-8$

Example 21 Use synthetic division to find the values of m and n if $x-1$ and $x+2$ are the factors of the polynomial $x^3-mx^2+nx+12$.

Solution Here $x-a=x-1 \Rightarrow a=1$

and $x-a=x+2 \Rightarrow a=-2$

Let $P(x)=x^3-mx^2+nx+12$

Now use synthetic division to divide $P(x)$ by $x-1$ and $Q(x)$ by $x+2$.

1	1	$-m$	n	12
		1	$1-m$	$1-m+n$
−2	1	$1-m$	$1-m+n$	$13-m+n$
		−2	$2+2m$	
	1	$-1-m$	$3+m+n$	

$13-m+n$: Remainder; $3+m+n$: Remainder

Since $x-1$ and $x+2$ are the factors of $P(x)$, we have

$13-m+n=0 = \text{(i)}$

$3+m+n=0 = \text{(ii)}$

Adding (i) and (ii), we get

$16+2n=0$

$\Rightarrow$ $2n=-16$

$\Rightarrow$ $n=-8$

Putting n = −8 in equation (i), we get

$13-m-8=0$

$\Rightarrow$ $-m+5=0$

$\Rightarrow$ $-m=-5$

$\Rightarrow$ $m=5$

Thus $m=5$ and $n=-8$.

Note

Synthetic division is a method of performing Euclidean division of polynomials with less writing and fewer calculations.
The advantages of synthetic division are that it allows one to calculate without writing variables, it uses few calculations, and it takes significantly less space on paper than long division. Also, the subtractions in long division are converted to additions by switching the signs at the very beginning, preventing sign errors.

Example 22 If −1 and 2 are roots of the quartic equation $x^4-5x^2+4=0$, use synthetic division to find the other roots.

Solution Let $P(x)=x^4-5x^2+4$

$$=x^4+0x^3-5x^2+0x+4$$

Since −1 and 2 are two roots of the equation $P(x)=0$, therefore $x+1$ and $x-2$ are factors of $P(x)$. To find the quotient we use synthetic division.

−1	1	0	−5	0	4
		−1	1	4	−4
2	1	−1	−4	4	0
		2	2	−4	
	1	1	−2	0	

Other factor is $x^2+x-2=Q(x)$

Other roots will be the roots of the equation $x^2+x-2=0$

$\Rightarrow\quad (x+2)(x-1)=0$

$\Rightarrow\quad x=-2 \text{ or } x=1$

Thus, the other roots of $P(x)=0$ are −2, 1.

Exercise 2.5

1. Use synthetic division to find the quotient $Q(x)$ and the remainder R when the first polynomial is divided by the second binomial in each case:
   (i). $3x^3+2x^2-x-1;\ x+3$ (ii). $2x^3-7x^2+12x-27;\ \ x-3$
   (iii). $2x^4-3x^2+5x-7;\ x+2$
2. Use synthetic division to find the value of k if −2 is a zero of the polynomial x^3+4x^2+kx+8.
3. Use synthetic division to find the values of p and q if $x+1$ and $x-2$ are the factors of x^3+px^2+qx+6.
4. If $x+1$ and $x-2$ are factors of the polynomial x^3+ax^2+bx+2, then using synthetic division, find the values of a and b.
5. One root of the cubic equation $x^3-7x-6=0$ is 3. Use synthetic division of find the to find the other roots.
6. If −1 and 2 are roots of the quartic equation $x^4-5x^3+3x^2+7x-2=0$, use synthetic use synthetic division to find other roots.

2.7 Simultaneous Equations

More than one equation which are satisfied by the same values of the variables involved are called simultaneous equations.

Note

A system of Linear equations consists of two or more linear equations in the same variables. A solution of system of linear equations in two variables is an ordered pair that satisfies each equation in the system.

2.7.1 Solution of one linear equation and one quadratic equation

Example 23

Solve the system.

$$2x+y=10$$
$$4x^2+y^2=68$$

Solution

$$2x+y=10 \quad \text{(i)}$$
$$4x^2+y^2=68 \quad \text{(ii)}$$

From equation (i) we have

$$y=10-2x \quad \text{(iii)}$$

Substituting this value of y in equation (ii) we have

$$4x^2+(10-2x)^2=68$$
$$4x^2+100+4x^2-40x=68$$

or $\quad 8x^2-40x+32=0$

$$x^2-5x+4=0 \quad \text{(Dividing by 8)}$$
$$(x-1)(x-4)=0$$

$\Rightarrow\quad x-1=0$ or $x-4=0$

which gives $x=1$, or $x=4$.

Substituting these values in (iii) we have

For $x=1,\ y=8$ and for $x=4,\ y=2$.

$\therefore$ The solutions of the given system are (1,8) and (4,2).

or solution set $=\{(1,8),(4,2)\}$

Example 24 Solve the system

$$x-y=7,\ x^2+3xy+y^2=-1$$

Solution $x-y=7$ (i)

$x^2+3xy+y^2=-1$ (ii)

From equation (i) we have

$x=7+y$ (iii)

Substituting this value of x in equation (ii) we have

$$(7+y)^2+3(7+y)y+y^2=-1$$

$$49+14y+y^2+21y+3y^2+y^2+1=0$$

or $5y^2+35y+50=0$

$y^2+7y+10=0$ (Dividing by 5)

$(y+2)(y+5)=0$

$\Rightarrow$ $y+2=0$ or $y+5=0$

$\Rightarrow$ $y=-2$ or $y=-5$

Substituting these values in (iii) we have for $y=-2$, $x=5$ and for $y=-5$, $x=2$.

$\therefore$ The solutions of the given system are $(5,-2)$ and $(2,-5)$

or solution set $=\{(5,-2),(2,-5)\}$

WARNING

The expression for x or y obtained from an equation must not be substituted in the same equation. From the first equation of the system

$$x+2y=-1$$
$$3x^2+y=2$$

We obtain

$$x=-1-2y$$

Substituting (incorrectly) in the same equation would result in

$$(-1-2y)+2y=-1$$
$$-1=-1$$

The substitution $x=-1-2y$ must be made in the second equation.

2.7.2 Solution when both equations are quadratic

Example 25 Solve the system

$$x^2+y^2=4$$
$$2x^2-y^2=8$$

Solution $x^2+y^2=4$ (i)

$2x^2-y^2=8$ (ii)

Add (i) and (ii) to eliminate y^2.

$$x^2+y^2=4$$
$$2x^2-y^2=8$$
$$3x^2=12$$
$$x^2=4$$
$$x=\pm 2$$

Substitute the values of x in (i) gives the corresponding values of y.

When $x=2$, we get $(2)^2+y^2=4$, $y^2=0$, $y=0$

When $x=-2$, we get $(-2)^2+y^2=4$, $4+y^2=4$, $y^2=0$, $y=0$

$\therefore$ the solutions of the given system are $(-2,0)$ and $(2,0)$ and the solution set is $\{(-2,0),(2,0)\}$

Exercise 2.6

1. Solve the following system of equations.

(i). $2x-y=3$, $x^2+y^2=2$

(ii). $x+2y=0$, $x^2+4y^2=32$

(iii). $2x-y=-8$, $x^2+4x=y$

(iv). $2x+y=4$, $x^2-2x+y^2=3$

(v). $4x^2+5y^2=4$, $3x^2+y^2=3$

(vi). $5x^2=y^2+9$, $x^2=-y^2+45$

(vii). $4x^2+3y^2-5=0$, $2x^2+3y^2-4=0$

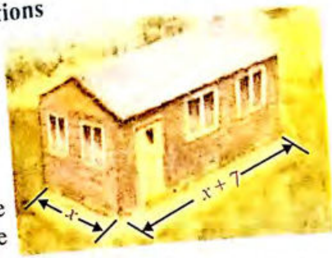
Challenge!

2. Solve the system of equations.

(i). $x+y=9$, $x^2+3xy+2y^2=0$

(ii). $y-x=4$, $2x^2+xy+y^2=8$

2.7.3 Real Life Application of Quadratic Equations



Example 26 Suppose a rectangular shed is being built that has an area of 120 square feet and is 7 feet longer than it is wide. Determine its dimensions.

Solution

Let x be the width of the shed. Then $x+7$ represents the length. Since area equals width times length, we solve the following equation.

$x(x+7)=120$ — area is equal to 120 square foot

$x^2+7x=120$ — distributive property

$x^2+7x-120=0$ — subtract 120 from each side

$(x+15)(x-8)=0$ — factorize

$x+15=0$ or $x-8=0$ — zero-product property

$x=-15$ or $x=8$ — solve

Since dimensions cannot be negative, the solution that has meaning is $x=8$. The length is 7 feet longer than the width, so the dimensions of the shed are 8 feet by 15 feet.

Example 27 A man purchased a number of shares of stock for an amount of Rs. 6000. If he had paid Rs. 20 less per share, the number of shares that could have been purchased for the same amount of money would have increased by 10. How many shares did he buy?

Solution Suppose the number of shares purchased $= x$

The amount paid per share $= y$ then $xy = \text{Rs. } 6000 \longrightarrow$ (i)

If the man had paid Rs. 20 less per share i.e. Rs. $(y-20)$, the number of shares would have been $x+10$

therefore $(x+10)(y-20)=Rs.\ 6000 \longrightarrow$ (ii)

From (i), we get $y=\dfrac{6000}{x}$. Substituting in (ii) we have

$$(x+10)\left(\frac{6000}{x}-20\right)=6000$$

$\Rightarrow \quad 6000x-20x^2+60000-200x=6000x$

$\Rightarrow \quad 20x^2+200x-60000=0$ (Dividing by 20)

$\Rightarrow \quad x^2+10x-3000=0$

$\Rightarrow \quad (x-50)(x+60)=0$

$\Rightarrow \quad x=50$ or $x=-60$

Since $x=-60$ is not admissible, so we neglect it. Thus the number of shares purchased is 50.

Exercise 2.7

1. Find two consecutive positive integers whose product is 72.
2. The sum of the squares of three consecutive integers is 50. Find the integers.
3. The length of a hall is 5 meters more than its width. If the area of the hall is 36 sq. m. Find the length and width of the hall.
4. The sum of two numbers is 11 and sum of their square is 65. Find the numbers.
5. The sum of the squares of two numbers is 100. One number is 2 more tha n the other. Find the numbers.
6. The area of a rectangular field is 252 square meters. The length of its side is 9 meter longer than its width. Find its sides.
7. One side of a rectangle is 3 centimeters less than twice the other. If the area of the rectangle is 54 square centimeters, then find the sides of the rectangle.
8. The length of one side of right triangle exceeds the length of the other by 3 centimeters. If the hypotenuse is 15 centimeters, then find the length of the sides of the triangle.
9. The sides of a right triangle in cm are $(x-1)$, x, $(x+1)$. Find the sides of the triangle.
10. A shepherd bought some goats for Rs. 9000. If he had paid Rs. 100 less for each, he would have got 3 goats more for the same amount of money. How many goats did he buy, when the rate in each case is uniform?

Review Exercise 2

1. At the end of each question, four circles are given. Fill in the correct circle only.

(i). If the sum of the roots of $(a+1)x^2+(2a+3)x+(3a+4)=0$

○ 0 ○ 1 ○ 2 ○ 3

(ii). The sum of the roots of a quadratic equation is 2 and the sum of the cubes of the roots in 98. The equations is

○ $x^2-2x-15=0$ ○ $x^2-2x+15=0$

○ $x^2-4x+15=0$ ○ none of these

(iii). If a, b, c are positive real number, then both the roots of the equation $ax^2+bx+c=0$, are always

○ real and positive ○ real and negative

○ rational and unequal ○ none of these

(iv). If a and b are the roots of $4x^2-3x+7=0$, then the value of $\frac{1}{a}+\frac{1}{b}$

○ $-\frac{3}{4}$ ○ $\frac{3}{7}$ ○ $-\frac{3}{7}$ ○ $\frac{4}{7}$

2. For what value of k the roots of the equation $3x^2-5x+k=0$ are equal.

3. Evaluate $\left(-1+\sqrt{-3}\right)^7+\left(-1-\sqrt{-3}\right)^7$.

4. Without solving the equation, find the sum and products of the roots of the following quadratic equations.

(i). $4x^2-1=0$ (ii). $3x^2+4x=0$

5. Find the value of k so that the sum of the roots of the equation $3x^2+(2k+1)x+k-5=0$ is equal to the product of roots.

6. Find the value of k if the roots of $x^2-3x+k+1=0$ differ by unity.

7. Find the quadratic equation whose roots are the multiplicative inverses of the roots of $12x^2-17x+6=0$.

8. If one of the roots of the quadratic equation $2x^2+kx+4=0$ is 2, find the other root. Also find the value of k.

9. One root of the cubic equation $x^3+6x^2+11x+6=0$ is -3. Use synthetic division to find the other roots.

10. Solve the following system of equations.

(i). $x+y=3, x^2-3xy+y^2=137$ (ii). $7x^2-4=5y^2, 3x^2+2=4y^2$

11. The area of a rectangle is 48 cm^2. If length and width are each increased by 4 cm. the area of the larger rectangle is 120 cm^2. Find the length and width of the original rectangle.

Activity

Find the error:

Abuzar is trying to solve the quadratic equation $x^2+x+11=0$. He finds two-real solutions to the quadratic equation. Habiba immediately points out that Abuzar's answers are wrong. Habiba explains that she doesn't know what the solutions are right away, but she knows they are not real numbers. Who is correct?

Activity

Using online calculator.

Divide x^5+x^2+5x+7 by $x+2$ and find Quotient and Remainder. Verify your answer by using long division method.

Summary

- The solution to an equation are called the **roots** of the equation.
- The quadratic formula is ⇒ $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
- The part of the quadratic formula underneath the square root sign is called the **discriminant.**
- Discriminant = b^2-4ac

Discriminant	$b^2-4ac>0$	$b^2-4ac=0$	$b^2-4ac<0$
Nature of roots	The roots are unequal and real. Roots are rational if b^2-4ac is perfect square otherwise they are irrational.	The roots are equal.	The roots are unequal and imaginary.

- A function of the roots of an equation, which remains unaltered when any two of the roots are interchanged is called **Symmetric function** of the roots.
- For finding the equation $ax^2+bx+c=0$, $(a \neq 0)$ when roots are given x^2-(sum of the roots) x + product of the roots = 0 where

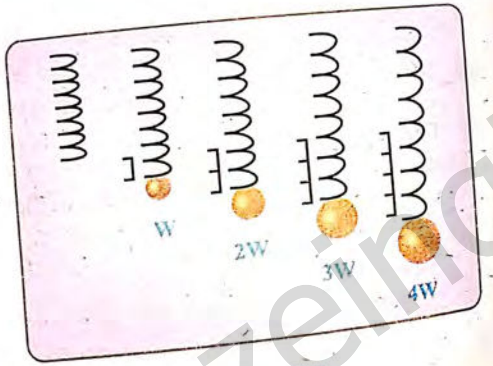
  Sum of roots = $-\frac{b}{a}$

  Product of roots = $\frac{c}{a}$
- **Synthetic division** is a method of performing Euclidean division of polynomials with less writing and fewer calculations.
- The advantages of synthetic division are that it allows one to calculate without writing variables, it uses few calculations, and it takes significantly less space on paper than long division. Also, the subtractions in long division are converted to additions by switching the signs at the very beginning, preventing sign errors.
- A system of Linear equations consists of two or more linear equations in the same variables.
- A solution of system of linear equations in two variables is an ordered pair that satisfies each equation in the system.

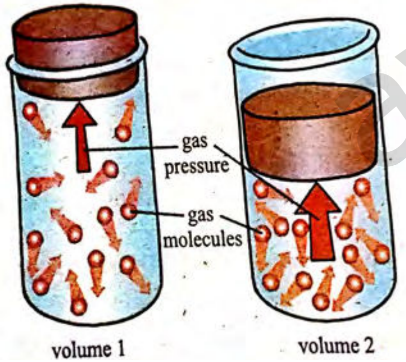
Unit 3 VARIATIONS

In this unit the students will be able to

- Define ratio, proportions and variations (direct and inverse).
- Find 3^{rd}, 4^{th} mean and continued proportion.
- Apply theorems of invertendo, alternendo, componendo, dividendo and componendo and dividendo to find proportions.
- Define joint variation.
- Solve problems related to joint variation.
- Use K-method to prove conditional equalities involving proportions.
- Solve real life problems based on variations.



This is Direct Variation



This is Inverse Variation

Why it's important

Direct variation is the relationship between two quantities, whereby if one quantity increases or decreases the other also increases or decreases. If one quantity increases, the other decreases or if one quantity decreases the other increases, it is called inverse variation. We often make use of one type of variation or another in our daily life. Thus knowing of variation is necessary for us. For example,

- If a person pushes a car, he has to apply more force.
- If a mother pushes the cart of her child, she has to apply less force.
- In a marathon race an athlete who has more speed than other participants will reach the finishing line in less time.
- On motorway from Peshawar to charsadda, from the same two cars if one car has to reach in less time, then its speed must be increased.

3.1 Ratio, Proportions and Variations

3.1.1 Ratio

A ratio is used to compare two or more quantities of the same kind which are measured in same unit. The ratio of two quantities a, b of the same unit can be shown as:

Example 1 Write the following ratio in simplified form:

(i) $3 : 12$ (ii) $6a : 18b$

Solution (i) $3 : 12 = 1 : 4$ (ii) $6a : 18b = a : 3b$

Example 2 Divide Rs. 5070 among three persons in the ratio $2 : 5 : 6$.

Solution Given ratio $= 2 : 5 : 6$

Sum of the ratios $= 13$

Share of 1^{st} person $= 5070 \times \frac{2}{13} =$ Rs. 780

Share of 2^{nd} person $= 5070 \times \frac{5}{13} =$ Rs. 1950

Share of 3^{rd} person $= 5070 \times \frac{6}{13} =$ Rs. 2340

Tidbit

A ratio is said to be in its simplest form a : b when a and b are integers with no common factors(other than 1)

3.1.2 Proportions

A proportion is an equation that states that two ratios are equivalent.
If a, b, c, d are four quantities then the general form of a proportion is given as

$\frac{a}{b} = \frac{c}{d}$ where b not equal to zero, d not equal to zero.

The above equations can also be written as:

$$a : b :: c : d$$
$$a : b = c : d$$

Note

Proportion is a comparison of the quantity of a part to the quantity of a whole.

The proportion $\frac{3}{4} = \frac{6}{8}$ can be written as $3:4 = 6:8$. In this form 4 and 6 are called means of the proportion and 3 and 8 are called the extremes of the proportion.

The cross product of a proportion are equal. i.e $4 \times 6 = 24$ and $3 \times 8 = 24$.

Example 3

$a^3 - b^3$, $a^2 - b^2$, $a^2 + ab + b^2$ and x are in a proportion. Find the value of x.

Solution

According to the question

$$\frac{(a^3 - b^3)}{(a^2 - b^2)} = \frac{(a^2 + ab + b^2)}{x}$$

$$\frac{(a-b)(a^2 + ab + b^2)}{(a-b)(a+b)} = \frac{(a^2 + ab + b^2)}{x}$$

or $$\frac{1}{a+b} = \frac{1}{x}$$

or $$x = a + b$$

3.1.3 Variations

In Mathematics, we usually deal with two types of quantities: Variable quantities (or variables) and Constant quantities (or constants). If the value of a quantity remains unchanged under different situations, it is called a constant. On the other hand, if the value of a quantity changes under different situations, it is called a variable.

For example: 4, 2.718, $\frac{22}{7}$ etc. are constants while speed of a train, demand of a commodity, population of a town etc. are variables.

The change of variable parameters is called as variation.

Kinds of Variation

There are two types of variation. Direct Variation and Inverse Variation.

a. Direct Variation

If we borrow books from the school library and are late in returning the books, we will be fined Rs. 15 per day for each overdue book. The table below shows the fine for an overdue book.

No. of days (x)	1	2	3	4	5
Fine (Rs. y)	15	30	45	60	75

From the table we notice that if the number of days (x) the book is overdue increases, the fine (Rs. y) will also increase proportionally, i.e. if x is doubled, y will also double; if x is halved, y will also halve. This is called direct proportion. We say that the fine (Rs. y) is directly proportional to the number of days (x) a book is overdue. We say that y varies directly as x, or y is directly proportional to x.

If y varies directly as x, this relation is written as

$$y \propto x \quad \text{or} \quad y = kx$$

It is clear from the above statement that $\frac{y}{x} = k$

Where k is a constant of a direct relation, and is called constant of variation.

Example 4

Given that y varies directly with x and $y = 27$ when $x = 3$. Find

(i) An equation connecting x and y. (ii) The value of y when $x = 11$.

Solution

Since y is directly proportional to x, then

$$y \propto x \Rightarrow y = kx$$

(i) Putting the given values of $x = 3$ and $y = 27$

$$\frac{27}{3} = k \Rightarrow k = 9$$

So the equation connecting x and y is $y = 9x$

(ii) Putting $x = 11$ in above equation.

$$y = 9(11) = 99$$

Which is the required value.

Example 5

If $y \propto x$, then complete the following table.

x	4	5	8		
y	6			18	22.5

Solution

$$y \propto x \quad \text{(i)}$$

or $\quad y = kx$

when $\quad x = 4\ ,\ y = 6$

Putting the values in (i)

$$6 = 4k \quad \text{or} \quad k = \frac{3}{2}$$

Equation (i) becomes $\quad y = \frac{3x}{2} \quad$ (ii)

Putting $x = 5$ in equation (ii)

$$y = \frac{3}{2} \times 5 = \frac{15}{2} = 7.5$$

Which is the corresponding value of y

Putting $x = 8$ in equation (ii)

$$y = \frac{3}{2} \times 8 = 12$$

Which is the corresponding value of y.

Now putting $y = 18$ in equation (ii)

$$18 = \frac{3x}{2} \quad \text{or} \quad x = 12$$

Which is the corresponding value of x.

Putting $y = 22.5$ in equation (ii)

$$22.5 = \frac{3x}{2} \quad \text{or} \quad x = 15$$

Which is the corresponding value of x.

So complete table is given as under.

x	4	5	8	12	15
y	6	7.5	12	18	22.5

b. Inverse Variation

The following table shows the time taken by a car to travel a distance of 120 km at different speeds.

Speed x km/h	10	20	30	40	60	120
Time taken (y hours)	12	6	4	3	2	1

You can notice that as the speed of car increases, the time taken decreases proportionally i.e, if x is doubled y will be halved. If x is tripled then y will be reduced to $\frac{1}{2}$ of its original value. Similarly as the speed of car decreases the time taken increases proportionally. This relationship is known as inverse proportion. We say that the speed of car, x km/h varies inversely to the time y hours taken.

If y varies inversely to x

i.e $\quad y \propto \frac{1}{x} \quad$ or $\quad y = \frac{k}{x} \quad$ or $\quad xy = k$

Where k is a constant and k is not equal to zero.

Example 6 If x varies inversely to y and $x = 3$, when $y = 12$. Find the value of y when $x = 6$.

Solution

$$y \propto \frac{1}{x} \quad \text{or} \quad y = \frac{k}{x} \quad \text{(i)}$$

Putting $x = 3$, $y = 12$ in equation (i)

$$12 = \frac{k}{3}$$

or $\quad 36 = k$

Putting the value of k in equation (i)

$$y = \frac{36}{x} \quad \text{(ii)}$$

Putting $x = 6$ in equation (ii)

$y = \frac{36}{6} = 6$ which is the required corresponding value of y.

> **Tidbit**
>
> **Inverse variation**
>
> when one quantity increases / decreases while another quantity decreases / increases.
>
> $xy = k$ or $y = \frac{k}{x}$

Example 7 Given that Pressure 'p' on the quantity of gas in a container varies inversely to its volume 'V', i.e. $P \propto \frac{1}{V}$. When pressure on gas in 10 N/m^2, its volume is 25m^3. Find the pressure when the volume is 20m^3.

Solution $P \propto \frac{1}{V}$ or $P = \frac{k}{V}$ (i)

Putting the values $P = 10\ N/m^2$ and $V = 25m^3$ in equation (i)

$$10 = \frac{k}{25} \quad \text{or} \quad k = 250$$

For equation (i)

$$P = \frac{250}{V} \qquad \text{(ii)}$$

Putting $V = 20m^3$ in equation (ii)

$$P = \frac{250}{20} = 12.5\,N/m^2$$

Tidbit

For direct proportion, $y_2/y_1 = x_2/x_1$, but for inverse proportion, we have $y_2/y_1 = x_1/x_2$ or $x_1y_1 = x_2y_2$. Note the order of x_1 and x_2.

Exercise 3.1

1. Which is the greater ratio, 5 : 7 or 151 : 208 ?
2. Gold and silver are mixed in the ratio 7:4. If 36 grams of silver is used. How much gold is used.
3. Divide the annual profit of Rs. 40,000 of a factory among 3 partners in the ratio of 5 : 8 : 12.
4. If 11 : x–1 = 22 : 27, find the value of x.
5. There is a direct variation between x^2 and y. When $x = 7$, $y = 49$ find:
   (i). y when $x = 9$ (ii). x when $y = 100$
6. There is an inverse variation between x and y, and when $x = 4$, $y = 6$, find:
   (i). y when $x = 12$ (ii). x when $y = 24$.
7. $r \propto \frac{1}{p^3}$ and $p = 9$ when $r = 2$. Find:
   (i). r when p = 3 (ii). p when $r = \frac{1}{4}$
8. If $y \propto x$, then complete the following table.

x	4	6		15
y	2		3.5	

3.2 Third, Fourth Mean and Continued Proportion

Three quantities are said to be in continued proportion if the ratio of the first term and second term are equal to the ratio of the second term and third term. If $a : b :: b : c$ then $ac = b^2$. If a, b and c are in continued proportion then b is called the mean proportional (or geometric mean) of a and c.

c is called the third proportional.

For example, the numbers 4, 6 and 9 are in continued proportion because

$$4 : 6 :: 6 : 9$$
$$4 \times 9 = 6^2$$
$$36 = 36$$

The numbers 2, 4 and 6 are not in continued proportion because

$$2 \times 6 \neq 4^2$$
$$12 \neq 16.$$

Example 8 Find the mean proportional of 5 and 15.

Solution Let x is the mean proportional

$$\frac{15}{x} = \frac{x}{5}$$

or $x^2 = 75$

or $x = \pm\sqrt{75}$ taking square roots on both sides

or $x = 5\sqrt{3}$

Example 9 Find the third proportion of a^2b^2 and abc.

Solution Let x be the third proportion.

Therefore, $\frac{a^2b^2}{abc} = \frac{abc}{x}$ or $x = c^2$.

Example 10 Find fourth proportional of $a^3 - b^3$, $a + b$ and $a^2 + ab + b^2$

Solution Let x be the fourth proportional,

then $(a^3 - b^3) : (a + b) :: (a^2 + ab + b^2) : x$

i.e., $x\,(a^3 - b^3) = (a + b)\,(a^2 + ab + b^2)$

$$x = \frac{(a+b)(a^2+ab+b^2)}{a^3-b^3} = \frac{(a+b)(a^2+ab+b^2)}{(a-b)(a^2+ab+b^2)}$$

$$x = \frac{a+b}{a-b}$$

3.3 Theorems on Proportion

1 Alternendo Property

If $a:b = c:d$ then $a:c = b:d$
that is, if the second and third term interchange their places, then also the four terms are in proportion.

Example If $3:5 = 6:10$ then $3:6 = 1:2 = 5:10$

2 Invertendo Property

If $a:b :: c:d$ then $b:a :: d:c$.
that is, if two ratios are equal, then their inverse ratios are also equal.

Example $6:10 = 9:15$
therefore, $10:6 = 5:3 = 15:9$

3 Componendo Property

If $a:b = c:d$ then $(a+b):b :: (c+d):d$

Example $4:5 = 8:10$
therefore, $(4+5):5 = 9:5 = 18:10$
$= (8+10):10$

4 Dividendo Property

If $a:b :: c:d$ then $(a-b):b :: (c-d):d$.

Example $5:4 = 10:8$
$(5-4):4 = 1:4 = (10-8):8$

5 Componendo-Dividendo Property

If $a:b :: c:d$ then $(a+b):(a-b) :: (c+d):(c-d)$.

Example $7:3 = 14:6$
$(7+3):(7-3) = 10:4 = 5:2$
Again, $(14+6):(14-6) = 20:8 = 5:2$
Therefore, $(7+3):(7-3) = (14+6):(14-6)$

Example 11 If $\frac{a}{b}=\frac{c}{d}$ then prove that $2a+3b : b = 2c+3d : d$

Solution

$\frac{a}{b}=\frac{c}{d}$ multiplying both sides by $\frac{2}{3}$

$\frac{2a}{3b}=\frac{2c}{3d}$

$\frac{2a+3b}{3b}=\frac{2c+3d}{3d}$ using componendo property

$\frac{2a+3b}{b}=\frac{2c+3d}{d}$

$2a+3b : b = 2c+3d : d$

Example 12 Prove that if $\frac{(3a-4b)}{(3a+4b)}=\frac{(3c-4d)}{(3c+4d)}$.

Then $\frac{a}{b}=\frac{c}{d}$

Solution $\frac{3a-4b}{3a+4b}=\frac{3c-4d}{3c+4d}$

Then $\frac{(3a-4b)+(3a+4b)}{(3a-4b)-(3a+4b)}=\frac{(3c-4d)+(3c+4d)}{(3c-4d)-(3c+4d)}$

(Componendo – Dividendo)

or $\frac{6a}{-8b}=\frac{6c}{-8d}$

or $\frac{a}{b}=\frac{c}{d}$

Example 13 If $\frac{(x+3)^2+(x-4)^2}{(x+3)^2-(x-4)^2}=\frac{13}{12}$ then find the value of x.

Solution $\frac{(x+3)^2+(x-4)^2}{(x+3)^2-(x-4)^2}=\frac{13}{12}$

By componendo – dividend property

$\frac{(x+3)^2+(x-4)^2+(x+3)^2-(x-4)^2}{(x+3)^2+(x-4)^2-\left[(x+3)^2-(x-4)^2\right]}=\frac{13+12}{13-12}$

or $\frac{\not{2}(x+3)^2}{\not{2}(x-4)^2}=\frac{25}{1}$

Taking square root of both sides $\frac{x+3}{x-4}=\pm 5$

When $\frac{x+3}{x-4}=5$ or, $x+3=5x-20$

or $x=\frac{23}{4}$

When $\frac{x+3}{x-4}=-5$ or, $x+3=-5x+20$

or $x=\frac{17}{6}$

Solution set $=\left\{\frac{23}{4},\frac{17}{6}\right\}$.

Exercise 3.2

1. Which of the following quantities are in continued proportion?

   (i). 4, 12, 36 (ii). 3, 12, 39 (iii). 72, 24, 8

2. Find the mean proportional of 12, 3.
3. If 5 : 15 : x are in continued proportion, find the value of x.
4. If $3x-1$, 4, 35 are continued proportion, find the value of x.
5. Find the mean proportional of $a^2 - b^2$ and $\frac{a+b}{a-b}$
6. If $\frac{a}{b}=\frac{c}{d}$ then prove that $\frac{ac+ad}{ac-bd}=\frac{a^2+b^2}{a^2-b^2}$
7. Solve the following equations.

   (i). $\frac{\sqrt{3x+2}+\sqrt{x}}{\sqrt{3x+2}-\sqrt{x}}=\frac{4}{1}$ (ii). $\frac{(x-1)^2+(x+2)^2}{(x-1)^2-(x+2)^2}=-\frac{17}{8}$

   (iii). $\frac{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}}=\frac{1}{3}$

Math Fun

6 x 7 = 42

66 x 67 = 4422

666 x 667 = 444222

6666 x 6667 = 44442222

66666 x 66667 = 4444422222

666666 x 666667 = 444444222222

6666666 x 6666667 = 44444442222222

66666666 x 66666667 = 4444444422222222

666666666 x 666666667 = 444444444222222222

3.4 Joint Variation

Joint variation is the same as direct variation with two or more quantities, i.e. Joint variation is a variation where a quantity varies directly as the product of two or more other quantities. If x is jointly proportional to y and z, we can write $xkyz$=for some constant k. We can also write this relationship as

$$\frac{x}{yz}=k.$$

For example,

Area of a triangle $=\frac{1}{2}bh$.

Here the constant k is $\frac{1}{2}$.

Area of a triangle varies jointly with base 'b' and height 'h'.

Area of a rectangle = L × W.

Here the constant k is 1.

Area of a rectangle varies jointly with length 'L' and width 'W'.

Newton's law of motion; Force = mass × acceleration.

The force exerted on an object varies jointly as the mass of the object and the acceleration produced.

Example 14 If y varies jointly as x and z, and y = 12 when x = 9 and z = 3, find z when y = 6 and x = 15.

Solution Write the equation. $y = kxz$ (i)

Substituting y = 12 , x = 9 , z = 3 in (i).

$$12=k(9)(3)$$

$$12=27k \Rightarrow \frac{4}{9}=k$$

So (i) becomes,

$$y=\frac{4}{9}xz \quad \text{(ii)}$$

Substituting $y=6$, $x=15$ in (ii).

$$6=\frac{4}{9}(15)(z)$$

$$6=\frac{60}{9}z \Rightarrow 54=60z \Rightarrow z=\frac{9}{10}$$

> **Try This**
> If z varies jointly as x and y and z = 24, when x = 2 and y = 4, find z when x = 2 and y = 5.

Types of Variation

y varies directly as *x*.
As *x* increases, *y* also increases.
As *x* decreases, *y* also decreases.
Equation: $y = kx$

y varies inversely as *x*.
As *x* increases, *y* also decreases.
As *x* decreases, *y* also increases.
Equation: $xy = k$ or $y = \frac{k}{x}$

y varies directly as *x*.
As *x* increases, *y* also increases.
As *x* decreases, *y* also decreases.
Equation: $y = kx$

y varies inversely as *x*.
As *x* increases, *y* also decreases.
As *x* decreases, *y* also increases.
Equation: $xy = k$ or $y = \frac{k}{x}$

Exercise 3.3

1. If y varies jointly as x and z, and $y = 33$ when $x = 9$ and $z = 12$, find y when $x = 16$ and $z = 22$.
2. If f varies jointly as g and the cube of h, and $f = 200$ when $g = 5$ and $h = 4$, find f when $g = 3$ and $h = 6$.
3. Suppose a is jointly proportional to b and c. If $a = 4$ when $b = 8$ and $c = 9$, then what is a when $b = 2$ and $c = 18$?
4. If p varies jointly as q and r squared, and $p = 225$ when $q = 4$ and $r = 3$, find p when $q = 6$ and $r = 8$.
5. If a varies jointly as b cubed and c, and $a = 36$ when $b = 4$ and $c = 6$, find a when $b = 2$ and $c = 14$.
6. If z varies jointly as x and y and $z = 12$, when $x = 2$ and $y = 4$, find the constant of variation.
7. If y varies jointly as x^2 and z and $y = 6$ when $x = 4$, $z = 9$. Write y as a function of x and z and determine the value of y, when $x = -8$ and $z = 12$.
8. If p varies jointly as q and r^2 and inversely as s and t^2, $p = 40$, when $q = 8, r = 5, s = 3, t = 2$. Find p in terms of q, r, s and t. Also find the value of p when $q = -2, r = 4, s = 3$ and $t = -1$.

Activity

Use online calculator.
Give examples of joint variation / direct and inverse variation from daily life. Write down these examples on flip chart / chart paper and present your work in class in groups.

3.5 K – Method

Let: $a : b : c : d$ be a proportion then $\frac{a}{b} = \frac{c}{d} = k(say)$

Thus $a = bk$, $c = dk$

These equation are used to evaluate certain expressions more easily. This method is called K-method.

K – Method is explained with the help of following examples.

Example 15

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ when b, d, f are non -zero numbers, then prove that each of the ratios is equal to the following ratios.

$$\frac{\ell a + mc + ne}{\ell b + md + nf}$$

Solution

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

$a = bk$; $c = dk$, $e = fk$

$$\frac{\ell a + m + cne}{\ell b + md + nf} = \frac{\ell bk + mdk + nfk}{\ell b + md + nf} = \frac{k(\ell b + md + nf)}{(\ell b + md + nf)} = k$$

Example 16

Prove that $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{a+c+e}{b+d+f}$

Solution

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

$a = bk, c = dk, e = fk$

or $a + c + e = bk + dk + fk$

or $a + c + e = k(b + d + f)$

$$\frac{a+b+c}{b+d+f} = k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$$

Did You Know?

Beautiful numbers relationship

$1^3 + 5^3 + 3^3 = 153$

$1 + 125 + 27 = 153$

Example 17 Prove that $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}}$

Solution $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$

Let $a=bk,\ c=dk,\ e=fk$

or $a^2=b^2k^2,\ c^2=d^2k^2,\ e^2=f^2k^2$

$$a^2+c^2+e^2=b^2k^2+d^2k^2+f^2k^2=k^2(b^2+d^2+f^2)$$

$$\frac{a^2+c^2+e^2}{b^2+d^2+f^2}=k^2$$

$$\sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$$

Example 18 If $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$ where a, b, c and x, y, z are non-zero numbers then prove that.

Solution

$$\frac{x^3}{a^3}=\frac{y^3}{b^3}+\frac{z^3}{c^3}=\frac{3xyz}{abc}$$

$a=xk,\ b=yk,\ c=zk$

or $\frac{x}{a}=\frac{1}{k},\ \frac{y}{b}=\frac{1}{k},\ \frac{z}{c}=\frac{1}{k}$ (i)

or $\frac{x^3}{a^3}=\frac{1}{k^3},\ \frac{y^3}{b^3}=\frac{1}{k^3},\ \frac{z^3}{c^3}=\frac{1}{k^3}$ (ii)

For equations in (ii) sum of L.H.S = Sum of R.H.S i.e

$$\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}=\frac{1}{k^3}+\frac{1}{k^3}+\frac{1}{k^3}=\frac{3}{k^3} \quad \text{(iii)}$$

For equations given in (i), Product of L.H.S = Product of R.H.S.

$$\frac{x}{a}\cdot\frac{y}{b}\cdot\frac{z}{c}=\frac{1}{k}\cdot\frac{1}{k}\cdot\frac{1}{k}=\frac{1}{k^3}$$

or $\frac{3xyz}{abc}=\frac{3}{k^3}$ (iv)

From (iii) and (iv)

$$\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}=\frac{3xyz}{abc}$$

Exercise 3.4

1. If $a:b=c:d$ then prove that
   (i) $\frac{2a+3b}{2a-3b}=\frac{2c+3d}{2c-3d}$. (ii) $pa+qb:ma-nb=pc+qd:mc-nd$.
2. Prove that $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\sqrt{\frac{pa^2+qc^2+e^2}{pb^2+qd^2+f^2}}$
3. If $\frac{x-y}{z}=\frac{y-z}{x}=\frac{z-x}{y}$ then prove that: $x=y=z$ where x, y, z are non-zero numbers and $x+y+z\neq 0$.
4. If $\frac{2y+2z-x}{a}=\frac{2z+2x-y}{b}=\frac{2x+2y-z}{c}$, then prove that
   $$\frac{x}{2b+2c-a}=\frac{y}{2c+2a-b}=\frac{z}{2a+2b-c}$$
5. Prove that each of the fraction in:
   $\frac{x+y}{a+b}=\frac{y+z}{b+c}=\frac{z+x}{c+a}$ is equal to $\frac{x+y+z}{a+b+c}$
6. If $\frac{bz+cy}{b-c}=\frac{cx+az}{c-a}=\frac{ay+bx}{a-b}$ then $(a+b+c)(x+y+z)=az+by+cz$.
7. If $\frac{x}{(b+c-a)}=\frac{y}{(c+a-b)}=\frac{z}{(a+b-c)}$, then $(b-c)x+(c-a)y+(a-b)z=0$.
8. If $2x+3y:3y+4z:4z+5x=4a-5b:3b-a:2b-3a$, then $7x+6y+8z=0$.

Challenge!

9. If $\frac{(a-b)}{(d-e)}=\frac{(b-c)}{(e-f)}$, then each of them is equal to $\frac{b(f-d)+(cd-af)}{e(f-d)}$.

Did You Know?

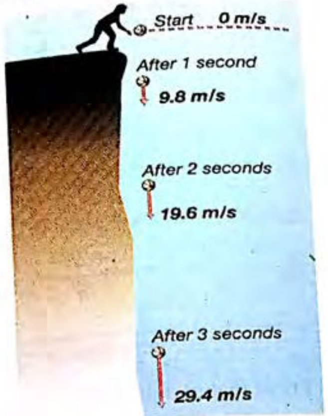
Do you notice anything interesting in the following multiplication?

$138\times 42=5796$

Answer: All digits are used

3.6 Real life problems based on variations

Example 19 A stone is dropped from the top of a hill. The distance it falls is proportional to the square of the time of fall. The stone falls 19.6 m after 2 seconds, how far does it fall after 3 seconds?



Solution We can use: $d = kt^2$

Where: d is the distance fallen and t is the time of fall

When $d = 19.6$ then $t = 2$

$$19.6 = k \times 2^2$$

$$19.6 = 4k \quad \Rightarrow k = 4.9$$

So now we know:

$$d = 4.9t^2$$

And when $t = 3$:

$$d = 4.9 \times 3^2 = 44.1$$

So it has fallen 44.1 m after 3 seconds.

Example 20 Height of an image y on a screen varies directly as distance x of the projector from the screen. Height of the image is 20cm when distance of the projector from the screen is 100cm. At what distance should the projector be kept from the screen so that the height of an image on the screen be 15cm.

Solution $y \propto x$ or $\dfrac{y}{x} = k$ (i)

when $y = 20\text{cm}$, $x = 100$ cm

Putting these values in (i)

$$\frac{20}{100} = k \text{ or } \frac{1}{5} = k$$

$$\frac{y}{x} = \frac{1}{5}$$

Putting $y = 15$ cm and $k = \frac{1}{5}$ in equation (i)

$$\frac{15}{x} = \frac{1}{5}$$

or $x = 75\text{cm}$

Hence the distance of projector from the screen = 75cm.

Example 21 The ratio of the mass of sand to cement in a particular type of concrete is 4.8 : 2. If 6 kg of sand are used, how much cement is needed?

Solution Let the amount required of cement be x kg. Then the ratio is,

sand : cement

4.8 : 2

6 : x

This is direct proportion.

$$\therefore \quad \frac{4.8}{6} = \frac{2}{x}$$

Cross-multiply: $4.8 \times x = 2 \times 6$, or more simply, $4.8x = 12$.

Solve this equation for x: $x = \dfrac{12}{4.8}$, which gives $x = 2.5$.

So 2.5 kg of cement are needed.

Example 22 4 people can paint a fence in 3 hours.

(i) How long will it take 6 people to paint it?

(ii) How many people are needed to complete the job in half an hour? (Assume everyone works at the same rate)

Solution It is an Inverse Proportion:

As the number of people goes up, the painting time goes down.

As the number of people goes down, the painting time goes up.

(i) We can use: $t = \dfrac{k}{n}$ Where: t = number of hours

k = constant of proportionality

n = number of people

"4 people can paint a fence in 3 hours" means that $t = 3$ when $n = 4$.

Therefore, $3 = \dfrac{k}{4} \Rightarrow k = 12$

So $t = \dfrac{12}{n}$ And when $n = 6$, $t = \dfrac{12}{6} = 2$ hours

So 6 people will take 2 hours to paint the fence.

(ii) $\dfrac{1}{2} = \dfrac{12}{n} \Rightarrow n = 24$

So it needs 24 people to complete the job in half an hour. (Assuming they don't all get in each other's way!)

Exercise 3.5

1. A hedge is made of wooden planks. The thickness (T) of the hedge varies directly as the number of planks(N). 4 planks make 12cm thick edge. Find
   (i) Thickness of the hedge when number of planks is 6.
   (ii) Number of planks when thickness of the hedge is 9cm.
2. In a fountain, the pressure 'P' of water at any internal point varies directly as depth 'd' from the surface. Pressure is 51 Newton/cm^2 when depth is 3cm. Find pressure when depth is 7cm.
3. Pressure P of gas in a container varies directly as temperature T. When pressure is 50 N/m^2, temperature is 75C°. Find pressure when temperature is 150°.
4. If 8 persons complete a work in 10 days then how many days would 10 persons take to complete the same work?
5. Volume of gas 'V' varies inversely as pressure 'P'. P = 300 N/m^2 when V = 4m^3. Find pressure when V = 3m^3
6. Attraction force 'F' between two magnets vary inversely as square of the distance 'd' between them. F is 18 Newton when d is 2cm. Find the distance when attraction force is 2 Newton.
7. The volume of a right circular cylinder varies jointly as the height and the square of the radius. The volume of a right circular cylinder, with radius 4 centimetres and height 7 centimetres, is 352 cm3. Find the volume of another cylinder with radius 8 centimetres and height 14 centimetres.

Review Exercise 3

1. At the end of each question, four circles are given. Fill in the correct circle only.

(i). Direct variation between a and b is expressed as.

○ $a=b$ ○ $a=\frac{1}{b}$ ○ $a\propto b$ ○ $a\propto\frac{1}{b}$

(ii). If $m\propto\frac{1}{n}$ then

○ m = kn ○ n = km ○ $\frac{m}{n}=k$ ○ mn = k

(iii). Identify the item that does not have the same ratio as the other three.

○ $\frac{30}{45}$ ○ 4 to 6 ○ 2:3 ○ 3 to 2

(iv). If $\frac{a}{b}=\frac{c}{d}$ then by alternendo property

○ $\frac{a-b}{b}=\frac{c-d}{d}$ ○ $\frac{a}{a+b}=\frac{c}{c+d}$
○ $\frac{a}{c}=\frac{b}{d}$ ○ $\frac{b}{a}=\frac{d}{c}$

(v). If 7 : 9 :: x : 27

○ $x=21$ ○ $x=3$ ○ $x=7$ ○ $x=81$

(vi). The third proportional of x and y is

○ xy ○ $\frac{x}{y}$ ○ $\frac{y^2}{x}$ ○ none of these

(vii). If $x\propto\frac{1}{y}$ and $y\propto\frac{1}{z}$ then

○ $y\propto\frac{1}{z}$ ○ $x\propto z$ ○ $xy\propto z$ ○ $xz\propto y$

(viii). If $2a+1:21::4:7$, then

○ $a=\frac{13}{2}$ ○ $a=\frac{11}{2}$ ○ $a=10$ ○ $a=\frac{9}{2}$

(ix). If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ then each fraction is equal to

○ $\frac{\ell a+mb+ne}{\ell d+me+nf}$ ○ $\frac{\ell a+mc+ne}{\ell b+md+nf}$
○ $\frac{\ell a+mc+ne}{md+nd+ef}$ ○ $\frac{\ell a+mb+nc}{\ell b+mc+nf}$

(x). Which of the follwing is a situation in which x varies directly as y?

○ $x=\frac{4}{y}$ ○ $xy=6$ ○ $x=xy$ ○ $x=\frac{7}{16}y$

2. Find the constant of variation, when $s\propto t^2$ and t = 10 when s = 5.
3. $y\propto\frac{1}{x^2}$. If $y=4$ when $x=3$, find the value of x when $y=9$.
4. Pressure of gas in a closed vessel varies directly to temperature. If pressure is 150 units then the temperature is 70 units. What will be the pressure if temperature rises to 140 units.
5. In an electric circuit, current varies inversely as the resistance. When current is 44 amp. The resistance is 30 ohm. How much current will flow if resistance becomes 22 ohm.

6. If a varies jointly as b and the square root of c, and a = 21 when b = 5 and c = 36, find a when b = 12 and c = 225.
7. What number should be added to each of the numbers 3,8,11 and 20 to make them in proportion?
8. What number must be subtracted from each of 6,8,7 and 11 so that the remaining numbers are in proportion?
9. The ratio between two numbers is 8 : 3 and their difference is 20. Find the numbers.
10. Find three numbers in continued proportion such that their sum is 14 and sum of their squares is 84.
11. The mean proportional between two numbers is 6 and their sum is 13. Find the numbers.
12. Find the angles of a triangle which are in the ratio 3 : 4 : 5.
13. If $\frac{a}{b}=\frac{c}{d}$, then prove that $\frac{ac(a+c)}{bd(b+d)}=\frac{(a+c)^3}{(b+d)^3}$
14. If a, b, c are in continued proportion then prove that $\frac{a}{c}=\frac{a^2+ab+b^2}{b^2+bc+c^2}=\frac{a^2-b^2}{b^2-c^2}$
15. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then prove that $\frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{ace}{bdf}$

Summary

- A rate is a special type of ratio.
- A rate is a comparison of two quantities of different kinds.
- A ratio compares two quantities of the same type.
- A proportion is a statement expressing the equivalence of two rates or two ratios.
- Two quantities are in direct proportion when one quantity is doubled, the other quantity is also doubled; when one quantity increases x times, the other quantity also increases x times.
- Two quantities are in inverse proportion when one quantity is doubled, the other quantity is halved; when one quantity increases y times, the other quantity becomes 1/y of the original.
- If y varies directly to x, then y = kx, where k is a constant and k = 0 is direct variation.
- If y varies inversely to x, then xy = k, where k is a constant and k = 0 is inverse variation.
  If y varies directly to two or more quantities x and z, then y = kxz, where k is a constant and k = 0 is joint variation.
- Let: a : b : c : d be a proportion then [redacted] = [redacted] = k (say)

  Thus a = bk, c = dk

  These equation are used to evaluate certain expressions more easily. This method is called K-method.

Unit 4
PARTIAL FRACTIONS

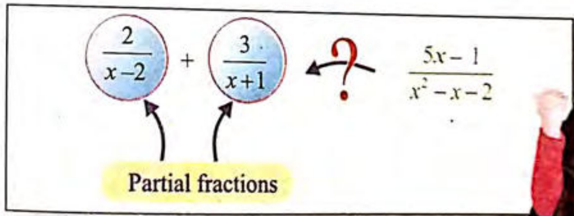
In this unit the students will be able to

- Define proper, improper and rational fraction.
- Resolve an algebraic fraction into partial fractions when its denominator consists of
  - non – repeated linear factors,
  - repeated linear factors,
  - non-repeated quadratic factors,
  - repeated quadratic factors.

$$\frac{4x-1}{(x+2)(x-1)}$$

Why it's important

The partial fractions is simpler, which can help to solve more complicated fraction. For example, it is very useful in calculus, which is a branch of mathematics.



We can do this directly: $\frac{2}{x+1}+\frac{3}{x-2}\rightarrow\frac{5x-1}{x^2-x-2}$

Like this:

$$\frac{2}{x+1}+\frac{3}{x-2}=\frac{2(x-2)+3(x+1)}{(x+1)(x-2)}=\frac{2x-4+3x+3}{(x+1)(x-2)}=\frac{5x-1}{x^2-x-2}$$

... but how do we go in the opposite direction?

$$\frac{2}{x+1}+\frac{3}{x-2}\leftarrow\frac{5x-1}{x^2-x-2}$$

How to find the "parts" that make the single fraction (This is **"partial fractions"**).

Partial fractions

A procedure which does splitting up a fraction into two or more fractions with only one factor in the denominator is called partial fraction. In other words a set of fractions whose algebraic sum is a given fraction is called partial fraction.

4.1 Proper and improper rational fractions

(a) Rational fraction A rational function can be written in the form:

$$f(x)=\frac{P(x)}{Q(x)}$$

Wher $P(x)$ and $Q(x)$ are polynomials where $Q(x)\neq 0$

(b) Proper rational fraction A rational fraction $\frac{P(x)}{Q(x)}$, $Q(x)\neq 0$ is proper fraction, if the degree of numerator $P(x)$ is less than the degree of denominator $Q(x)$.

For example,

$$\frac{1}{x+1},\ \frac{2x}{x^2+2},\ \frac{x^2+x-3}{x^3+x^2-x+1}$$

are proper rational fractions.

(c) Improper rational fractions

A rational fraction $\frac{P(x)}{Q(x)}$, $Q(x)\neq 0$ is as an improper fraction, if the degree of numerator $P(x)$ is greater than or equal to the degree of denominator $Q(x)$.

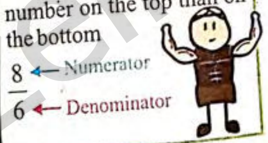
For example,

$$\frac{x^3+4}{(x+1)(x+2)},\ \frac{x}{2x+1},\ \frac{x^2+3x+2}{x^2+2x+3},\ \frac{x^3-x^2+x+1}{x^2+x-1}$$

Any improper rational fraction can be reduced into sum of polynomials and proper rational fraction by large division.

Tidbit

a fraction that has a larger number on the top than on the bottom



For example, consider $\frac{2x^2+1}{x-1}$, an improper rational fraction. Divide numerator by denominator as

$$\begin{array}{r} 2x+2 \\ x-1\overline{)\,2x^2+1} \\ \underline{\pm 2x^2 \mp 2x} \\ 2x+1 \\ \underline{\pm 2x \mp 2} \\ 3 \end{array}$$

Therefore we have $\frac{2x^2+1}{x-1}=2x+2+\frac{3}{x-1}$

4.2 Resolution of fraction into partial fractions

Resolution of rational fraction $\frac{P(x)}{Q(x)}$, where $Q(x)\neq 0$ into partial fractions depends upon the factors of denominator $Q(x)$. Thus there are four cases involved in resolution of fraction.

(i) When denominator $Q(x)$ consists of non-repeated linear factors.

(ii) When denominator $Q(x)$ consists of repeated linear factors

(iii) When denominator $Q(x)$ consists of non-repeated quadratic factors.

(iv) When denominator $Q(x)$ consists of repeated quadratic factors.

Case I When denominator consists of non-repeated linear factors

Let proper fraction $\frac{P(x)}{Q(x)}$ is given, factorize the polynomial Q(x) in the denominator if it is not already factorized. To every non-repeated linear factor $ax+b$ in the denominator corresponds a partial fraction of the form $\frac{A}{ax+b}$ where A is a constant. Thus if

$Q(x)=(a_1x+b_1)(a_2x+b_2)$ then

$$\frac{P(x)}{Q(x)}=\frac{A}{a_1x+b_1}+\frac{B}{a_2x+b_2}$$ where A and B are constants. Similarly if

$Q(x)=(a_1x+b_1)(a_2x+b_2)(a_3x+b_3)$ then

$$\frac{P(x)}{Q(x)}=\frac{A}{a_1x+b_1}+\frac{B}{a_2x+b_2}+\frac{C}{a_3x+b_3}$$

Where A,B and C are constants to be determined.

Did You Know?

$$(\sqrt{-1})(\sqrt{-1})=\sqrt{(-1)(-1)}$$
$$(\sqrt{-1})^2=\sqrt{1}$$
$$-1=1$$

Example 1 Resolve $\frac{1}{(x+1)(x+2)}$ into partial fractions.

Solution

Let $\quad \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \qquad$ (i)

$\Rightarrow \quad \frac{1}{(x+1)(x+2)} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$

Multiplying both sides by $(x+1)(x+2)$ we get

$1 = A(x+2)+B(x+1) \qquad$ (ii)

Putting $\quad x=-1$ in Eq. (2), we get

$1 = A(-1+2)+B(-1+1)$

$\Rightarrow \quad 1 = A(1)+B(0)$

$1 = A$

$\boxed{A=1}$

or

putting $x=-2$ in Eq. (ii), we get

$1 = A(-2+2)+B(-2+1)$

$\Rightarrow \quad 1 = A(0)+B(-1)$

$\Rightarrow \quad 1 = 0-B$

or $\quad \boxed{B=-1}$

Now putting these values of A and B in equation (i) we have

$\frac{1}{(x+1)(x+2)} = \frac{1}{x+2} + \frac{-1}{x+2}$

$\Rightarrow \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$

These are required partial fractions.

Math Fun

$111{,}111{,}111 \times 111{,}111{,}111$

$= 12{,}345{,}678{,}987{,}654{,}321.$

Example 2 Find partial fractions of $\frac{3x+2}{x^2-x-2}$

Solution

Since $\quad x^2-x-2 = (x+1)(x-2)$

Therefore, $\quad \frac{3x+2}{x^2-x-2} = \frac{3x+2}{(x+1)(x-2)}$

Let $\quad \frac{3x+2}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \qquad$ (i)

$\Rightarrow \quad \frac{3x+2}{(x+1)(x-2)} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$

Multiply both sides by $(x+1)(x-2)$ we have.

$3x+2 = A(x-2)+B(x+1) \qquad$ (ii)

Putting $x=-1$ in Eq. (ii) we get.

$3(-1)+2 = A(-1-2)+B(-1+1)$

$\Rightarrow \quad -3+2 = A(-3)+B(0)$

$\Rightarrow \quad -1 = -3A$

$\Rightarrow \quad \boxed{A = \frac{1}{3}}$

Putting $x=2$ in Eq. (ii), we get

$3(2)+2 = A(2-2)+B(2+1)$

$\Rightarrow \quad 6+2 = A(0)+B(3)$

$\Rightarrow \quad 8 = -3B$

$\Rightarrow \quad \boxed{A = \frac{8}{3}}$

Putting these values of A and B in Eq. (i), we have

$$\frac{3x+2}{(x+1)(x-2)} = \frac{\frac{1}{3}}{x+1} + \frac{\frac{8}{3}}{x-2}$$

$$= \frac{1}{3(x+1)} + \frac{8}{3(x-2)}$$

or $\quad \frac{3x+2}{x^2-x-2} = \frac{1}{3(x+1)} + \frac{8}{3(x-2)}$ These are required partial fractions.

Case II When denominator consists of repeated linear factors

...et, in the proper fraction $\frac{P(x)}{Q(x)}$ the denominator $Q(x)$ contains a repeated linear factor ...$(ax+b)^2$, corresponds to two partial fractions of the form $\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$, where A ...nd B are constant to be determined.

...the denominator $Q(x)$ contains the repeated factor $(ax+b)^3$, then the corresponding three ...rtial fraction are $\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$ where A,B and C are constants to be ...etermined.

Example 3 Find partial fractions of $\frac{x}{(x+1)^2}$

Solution

Let $\frac{x}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}$ (i)

$\Rightarrow \frac{x}{(x+1)^2}=\frac{A(x+1)+B}{(x+1)^2}$

Multiplying both sides by $(x+1)^2$, we get

$x=A(x+1)+B$ (ii)

$\Rightarrow x=Ax+A+B$ (iii)

Putting $x=-1$ in Eq. (ii), we get

$-1=A(-1-1)+B$

$\Rightarrow -1=A(0)+B$

$\Rightarrow -1=B$

$\Rightarrow \boxed{B=-1}$

To find, the value of A compare the co-efficient of x in Eq. (iii), we get

$\boxed{1=A}$

putting these values of A and B in Eq. (i), we get

$$\frac{x}{(x+1)^2}=\frac{1}{x+1}+\frac{-1}{(x+1)^2}$$

$$=\frac{1}{x+1}-\frac{1}{(x+1)^2}$$

These are required partial fractions.

Example 4 Find partial fractions of $\frac{2x^2+1}{(x-2)^2(x+3)}$.

Solution

Let $\frac{2x^2+1}{(x-2)^2(x+3)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+3}$ (i)

$\Rightarrow \frac{2x^2+1}{(x-2)^2(x+3)}=\frac{A(x-2)(x+3)+B(x+3)+C(x-2)^2}{(x-2)^2(x+3)}$

Multiplying both sides by $(x-2)^2(x+3)$, we get

$2x^2+1=A(x-2)(x+3)+B(x+3)+C(x-2)^2$ (ii)

$2x^2+1=A(x^2+x-6)+B(x+3)+C(x^2-4x+4)$

$2x^2+1=(A+C)x^2+(A+B-4C)x+(-6A+3B+4C)$ (iii)

Putting $x=2$ in Eq. (ii), we get

$2(2)^2+1=A(2-2)(2+3)+B(2+3)+C(2-2)^2$

$\Rightarrow 8+1=A(0)(5)+B(5)+C(0)^2$

$\Rightarrow 9=0+5B+0 \Rightarrow 9=5B \Rightarrow \boxed{B=\frac{9}{5}}$

Putting $x=-3$ in Eq. (ii) we get

$2(-3)^2+1=A(-3-2)(-3+3)+B(-3+3)+C(-3-2)^2$

$\Rightarrow 2\times 9+1=A(-5)(0)+B(0)+C(-5)^2$

$\Rightarrow 19=0+0+C(25) \Rightarrow \boxed{C=\frac{19}{25}}$

To find value of A, compare co-efficient of x^2 from Eq. (iii), we get

$2=A+C$

Putting value of C, we get

$2=A+\frac{19}{25} \Rightarrow 2-\frac{19}{25}=A \Rightarrow \boxed{A=\frac{31}{25}}$

Putting values of A, B and C in Eq. (i), we get

$$\frac{2x^2+1}{(x-2)^2(x+2)}=\frac{\frac{31}{25}}{x-2}+\frac{\frac{9}{5}}{(x-2)^2}+\frac{\frac{19}{25}}{x+3}$$

$$\frac{2x^2+1}{(x-2)^2(x+2)}=\frac{31}{25(x-2)}+\frac{9}{5(x-2)^2}+\frac{19}{25(x+3)}$$

These are required partial fractions.

Exercise 4.1

Resolve the following fractions into partial fractions.

(1). $\dfrac{3x-2}{2x^2-x}$ (2). $\dfrac{x-1}{x^2+6x+5}$ (3). $\dfrac{1}{x^2-1}$

(4). $\dfrac{x}{x^2+4x-5}$ (5). $\dfrac{4x+2}{(x+2)(2x-1)}$ (6). $\dfrac{x^2+5x+3}{(x^2-1)(x+1)}$

(7). $\dfrac{x^2+2}{(x+2)(x^2+5x+6)}$ (8). $\dfrac{2x-1}{x(x-3)^2}$ (9). $\dfrac{x^2}{x^2+2x+1}$

(10). $\dfrac{x^2}{(x-1)^2(x+1)}$

Case III When denominator consists of non-repeated quadratic factors

Let the proper fraction $\dfrac{P(x)}{Q(x)}$ is given then to every non-repeated quadratic factor ax^2+bx+c in Q(x) which is not factorizable corresponds the partial fraction of the form $\dfrac{P(x)}{Q(x)}=\dfrac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be determined.

Example 5 Find partial fractions of $\dfrac{1}{(x+1)(x^2+2)}$.

Solution Let $\dfrac{1}{(x+1)(x^2+2)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+2}$ (i)

$$\Rightarrow \quad \frac{1}{(x+1)(x^2+2)}=\frac{A(x^2+2)+(Bx+C)(x+1)}{(x+1)(x^2+2)}$$

Multiplying both sides by $(x+1)(x^2+2)$, we get

$$1=A(x^2+2)+(Bx+C)(x+1) \qquad \text{(ii)}$$
$$1=A(x^2+2)+Bx^2+Bx+Cx+C$$
$$1=(A+B)x^2+(B+C)x+2A+C \qquad \text{(iii)}$$

Putting $x=-1$ in Eq. (ii), we get

$$1=A\left[(-1)^2+2\right]+\left[B(-1)+C\right](-1+1)$$
$$\Rightarrow \quad 1=A(1+2)+(-B+C)(0)$$
$$\Rightarrow \quad 1=A(3)+0$$

$$1=3A\Rightarrow \boxed{A=\frac{1}{3}}$$

To find values of B and C comparing co-efficient of x^2 and x in Eq. (iii)

We get, $0=A+B$ (iv)

and $0=B+C$ (v)

using value of A in Eq. (iv), we get

$$0=\frac{1}{3}+B$$

$$\Rightarrow \quad \boxed{B=\frac{-1}{3}}$$

putting $B=-\dfrac{1}{3}$ in Eq. (v), $-\dfrac{1}{3}+C=0\Rightarrow C=\dfrac{1}{3}$

Putting these values of A, B and C in Eq. (i), we get

$$\frac{1}{(x+1)(x^2+2)}=\frac{\frac{1}{3}}{x+1}+\frac{\frac{-1}{3}x+\frac{1}{3}}{x^2+2}$$

$$\frac{1}{(x+1)(x^2+2)}=\frac{1}{3(x+1)}+\frac{-x+1}{3(x^2+2)}$$

$$\frac{1}{(x+1)(x^2+2)}=\frac{1}{3(x+1)}+\frac{1-x}{3(x^2+2)}$$

These are required partial fractions.

Example 6 $\dfrac{4x^2-28}{x^4-x^2-6}=\dfrac{4x^2-28}{(x^2+3)(x^2-2)}=\dfrac{Ax+B}{x^2+3}+\dfrac{Cx+D}{x^2-2}$

Solution $4x^2-28=(Ax+B)(x^2-2)+(Cx+D)(x^2+3)$

$$=(Ax^3+Bx^2-2Ax-2B)+(Cx^3+Dx^2+3Cx+3D)$$
$$=(A+C)x^3+(B+D)x^2+(3C-2A)x-2B+3D$$

Equating coefficients of like powers of x,

$$A+C=0, B+D=4, 3C-2A=0, -2B+3D=-28$$

Solving simultaneously, $A=0, B=8, C=0, D=-4$.

Hence, $\dfrac{4x^2-28}{x^4-x^2-6}=\dfrac{4x^2-28}{x^4+x^2-6}=\dfrac{8}{x^2+3}-\dfrac{4}{x^2-2}$

Case IV When denominator consists of repeated quadratic factors

Example 7 Resolve $\dfrac{1}{(x-1)(x^2+1)^2}$ into partial fractions.

Solution Let $\dfrac{1}{(x-1)(x^2+1)^2}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}$ (i)

$$\Rightarrow \frac{1}{(x-1)(x^2+1)^2}=\frac{A(x^2+1)^2+(Bx+C)(x-1)(x^2+1)+(Dx+E)(x-1)}{(x-1)(x^2+1)^2}$$

Multiplying both sides by $(x-1)(x^2+1)^2$ we get

$$1=A(x^2+1)^2+(Bx+C)(x-1)(x^2+1)+(Dx+E)(x-1) \quad \text{(ii)}$$
$$1=A(x^4+2x^2+1)+(Bx+C)(x^3-x^2+x-1)+Dx^2-Dx+Ex-E$$
$$1=A(x^4+2x^2+1)+Bx^4-Bx^3+Bx^2-Bx+Cx^3-Cx^2$$
$$+Cx-C+Dx^2-Dx+Ex-E$$
$$1=(A+B)x^4+(-B+C)x^3+(2A+B-C+D)x^2+(-B+C-D+E)x$$
$$+(A-C-E)$$

Putting $x=1$ in Eq. (ii), we get

$$1=A\left[(1)^2+1\right]^2+\left[B(1)+C\right](1-1)\left[(1)^2+1\right]+\left[D(1)+E\right](1-1)$$
$$1=A[1+1]^2+(B+C)(0)(1+1)+(D+E)(0)$$
$$1=A(2)^2+0+0$$
$$1=A(4)$$
$$\Rightarrow \boxed{A=\frac{1}{4}}$$

Comparing co-efficients of x^4, x^3, x^2, x and constant respectively.

$$0=A+B \quad \text{(iii)}$$
$$0=-B+C \quad \text{(iv)}$$
$$0=2A+B-C+D \quad \text{(v)}$$
$$0=-B+C-D+E \quad \text{(vi)}$$
$$1=A-C-E \quad \text{(vii)}$$

Using value of A in (iii) we get

$$0=\frac{1}{4}+B$$
$$\Rightarrow \boxed{B=-\frac{1}{4}}$$

Using value of B in (4) we get

$$0=-\left(-\frac{1}{4}\right)+C$$
$$0=\frac{1}{4}+C$$
$$\Rightarrow \boxed{C=-\frac{1}{4}}$$

Using values of A, B and C in Eq. (v), we get

$$0=2\left(\frac{1}{4}\right)+\left(-\frac{1}{4}\right)-\left(-\frac{1}{4}\right)+D$$
$$0=\frac{1}{2}-\frac{1}{4}+\frac{1}{4}+D$$
$$0=\frac{1}{2}+D$$
$$\boxed{D=\frac{-1}{2}}$$

Using values of A and C in Eq. (vii), we get

$$1=\frac{1}{4}-\left(-\frac{1}{4}\right)-E$$
$$1=\frac{1}{4}+\frac{1}{4}-E$$
$$1=\frac{1}{2}-E$$
$$E=\frac{1}{2}-1=-\frac{1}{2}$$
$$\boxed{E=-\frac{1}{2}}$$

Putting values of A, B, C, D, E in Eq. (i), we get

$$\frac{1}{(x-1)(x^2+1)^2}=\frac{\frac{1}{4}}{x-1}+\frac{\frac{-1}{4}x-\frac{1}{4}}{x^2+1}+\frac{\frac{-1}{2}x-\frac{1}{2}}{(x^2+1)^2}$$
$$=\frac{1}{4(x-1)}+\frac{-x-1}{4(x^2+1)}+\frac{-x-1}{2(x^2+1)^2}$$
$$\frac{1}{(x-1)(x^2+1)^2}=\frac{1}{4(x-1)}-\frac{x+1}{4(x^2+1)}-\frac{x+1}{2(x^2+1)^2}$$

Exercise 4.2

Resolve the following fractions into partial fractions.

(1). $\frac{1}{x(x^2+1)}$ (2). $\frac{x^2+3x+1}{(x-1)(x^2+3)}$ (3). $\frac{2x+1}{(x^2+1)(x-1)}$

(4). $\frac{-3}{x^2(x^2+5)}$ (5). $\frac{3x-2}{(x+4)(3x^2+1)}$ (6). $\frac{5x}{(x+1)(x^2-2)^2}$

(7). $\frac{5x^2-4x+8}{(x^2+1)^2(x-2)}$ (8). $\frac{4x-5}{(x^2+4)^2}$ (9). $\frac{8x^2}{(x^2+1)(1-x^4)}$

(10) $\frac{2x^2+4}{(x^2+1)^2(x-1)}$

Review Exercise 4

1. At the end of each question, four circles are given. Fill in the correct circle only.

(i). $\frac{1}{x^2-1}=$

○ $\frac{1}{x+1}-\frac{1}{x-1}$ ○ $\frac{1}{2(x+1)}-\frac{1}{2(x-1)}$

○ $\frac{1}{2(x-1)}-\frac{1}{2(x+1)}$ ○ $\frac{2}{x-1}-\frac{1}{2(x+1)}$

(ii). If $P(x)$ and $Q(x)$ are two polynomials then $\frac{P(x)}{Q(x)}$, $Q(x)\neq 0$ is

○ Rational fraction ○ Irrational fraction

○ Proper fraction ○ Improper fraction

(iii). $\frac{x^2+2}{x^2+2x+2}$ is

○ Proper fraction. ○ Improper fraction

○ Irrational fraction ○ None of these

(iv). What is the quotient when $x^3-8x^2+16x-5$ is divided by $x-5$?

○ x^2-x+5 ○ x^2-3x+2

○ x^2-3x+1 ○ $x^2+13x-49+\frac{240}{(x+5)}$

2. Resolve the following fractions into partial fractions.

(i). $\frac{2x^2}{(x+1)(x-1)}$ (ii). $\frac{2x^3-3x^2+9x+8}{x^2-3x+2}$ (iii). $\frac{3x-1}{x^3-2x^2+x}$

(iv). $\frac{x+1}{(x-1)^2}$ (v). $\frac{2x^2}{x^4-4}$ (vi). $\frac{3x^2+3x+2}{x^4-1}$

(vii). $\frac{x^3+3x^2+1}{(x^2+1)^2}$ (viii). $\frac{2x^3-1}{x^3+x^2}$ (ix). $\frac{4x^2+3x+14}{x^3-8}$

Challenge !

3. Resolve the following fraction into partial fractions. $\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}$

Summary

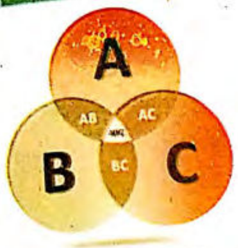
- If $P(x)$ and $Q(x)$ are two polynomials and $Q(x)$ is non zero polynomial than the fraction $\frac{P(x)}{Q(x)}$ is called a **rational fraction**.
- A rational fraction, $Q(x)\neq 0$ is a **proper rational fraction**, if the degree of numerator $P(x)$ is less than the degree of denominator $Q(x)$.
- A rational fraction $\frac{P(x)}{Q(x)}$, $Q(x)\neq 0$ is a **improper rational fraction**, if the degree of numerator $P(x)$ is equal to or greater than the degree of denominator $Q(x)$.
- Splitting up a single ration al fraction into two or more rational fraction with single factor in denominator, such a procedure is called **partial fractions**.

Math Fun

1 x 1 = 1
11 x 11 = 121
111 x 111 = 12321
1111 x 1111 = 1234321
11111 x 11111 = 123454321
111111 x 111111 = 12345654321
1111111 x 1111111 = 1234567654321
11111111 x 11111111 = 123456787654321
111111111 x 111111111 = 12345678987654321

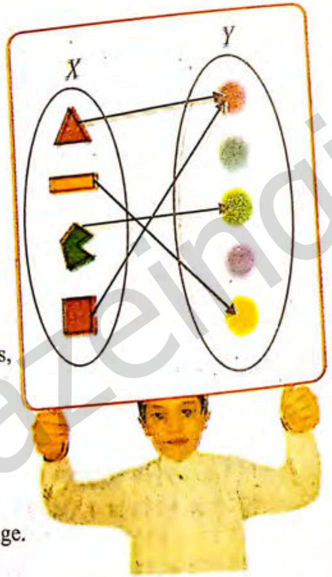
Unit 5

SETS AND FUNCTIONS



In this unit the students will be able to

- Recall the sets denoted by N, Z, W, E, O, P and Q.
- Recognize operation on sets (∪, ∩, \, ...)
- Perform operation on sets
  - union,
  - intersection,
  - difference,
  - complement.
- Give formal proofs of the following fundamental properties of union and intersection of two or three sets.
  - Commutative property of union,
  - Commutative property of intersection,
  - Associative property of union,
  - Associative property of intersection,
  - Distributive property of union over intersection,
  - Distributive property of intersection over union,
  - De Morgan's laws.
- Verify the fundamental properties for given sets.
- Use Venn diagram to represent
  - union and intersection of sets,
  - complement of a set.
- Use Venn diagram to verify
  - commutative law for union and intersection of sets,
  - De Morgan's Law,
  - associative laws,
  - distributive laws.
- Recognize ordered pairs and Cartesian product.
- Define binary relation and identify its domain and range.
- Define function and identify its domain, co-domain and range.
- Demonstrate the following:
  - into function,
  - one-one function,
  - into and one-one function (injective function),



  - onto function (surjective function),
  - one-one and onto functin (bijective function).
- Examine whether a given relation is a function or not.
- Differentiate between one-one correspondence and one-one function.
- Include sufficient exercise to clarify/differentiate between the above concepts.

Why it's important

Perhaps the single most important concept in mathematics is that of a function. At the heart of the function concept is the idea of a correspondence between two sets of objects. Correspondences between two sets of objects occur frequently in every day life. For example to each item in supermarket there corresponds a price (item → price).

5.1 Some important sets

N = Set of natural numbers = $\{1, 2, 3, \ldots\}$

W = Set of whole numbers = $\{0, 1, 2, 3, \ldots\}$

Z = Set of integers = $\{0, \pm 1, \pm 2, \ldots\}$

E = Set of even integers = $\{0, \pm 2, \pm 4, \ldots\}$

O = Set of odd integers = $\{\pm 1, \pm 3, \pm 5, \ldots\}$

P = Set of prime numbers = $\{2, 3, 5, 7, 11, \ldots\}$

Q = Set of rational numbers = $\left\{x \mid x = \dfrac{p}{q}, q \neq 0 \wedge p, q \in Z\right\}$

5.1.1 Operation on sets

(a) Union of two sets

If A and B two sets then union of set A and set B consists of all elements in set A or in set B or in both A and B, and it is denoted by A∪B. In set builder form,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Example 1 If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6\}$

then $A \cup B = \{1, 2, 3\} \cup \{3, 4, 5, 6\}$

$= \{1, 2, 3, 4, 5, 6\}$

(b) Intersection of two sets

If A and B are two sets then intersection of set A and set B consists of all those elements which are common to both A and B and it is denoted by A∩B symbolically.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

(c) Disjoint sets

If A and B are disjoint sets then

$A \cap B = \varphi$

Example 2 Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$

$C = \{5, 11, 12\}$, $D = \{8, 9, 10\}$

Then

$A \cap B = \{1, 2, 3, 4, 5\} \cap \{3, 4, 5, 6, 7\} = \{3, 4, 5\}$

and $B \cap C = \{3, 4, 5, 6, 7\} \cap \{5, 11, 12\} = \{5\}$

$A \cap D = \{1, 2, 3, 4, 5\} \cap \{8, 9, 10\} = \{\ \}$ or φ

Clearly here A and D are disjoint sets.

(d) Difference of two sets

If A and B are two sets then their difference consists of all those elements of A which are not in B and it is denoted by A\B or A−B. In set builder form

$$A \setminus B = \{x \mid x \in A \wedge x \notin B\}$$

Example 3

If $A = \{5, 6, 7, 8\}$, $B = \{7, 8, 9, 10\}$

Then $A \setminus B = \{5, 6, 7, 8\} \setminus \{7, 8, 9, 10\}$

$= \{5, 6\}$

and $B \setminus A = \{7, 8, 9, 10\} \setminus \{5, 6, 7, 8\}$

$= \{9, 10\}$

(e) Complement of a set

If U is a universal set and A is subset of U then U \ A is called complement of the set A and is denoted by A' or A^c i.e. $A' = U \setminus A$

For your information

George Boole (1815-1864) introduce a symbolic approach to the study of logic.
This allowed him to clarify difficult logical problems in symbolic forms based on sets.
The algebra of sets having union and intersection as its basic operations is known as Boolean Algebra.
Today Boolean Algebra is used widely as a tool to aid sound reasoning.

Example 4 If $U = \{1, 2, 3, 4, 5, 6\}$

$A = \{3, 4, 5\}$ and $B = \phi$, then find

(i) A' (ii) B'

Solution

$A' = U \setminus A = \{1, 2, 3, 4, 5, 6\} \setminus \{3, 4, 5\}$

$= \{1, 2, 6\}$

$B' = U \setminus B = \{1, 2, 3, 4, 5, 6\} \setminus \varphi$

$= \{1, 2, 3, 4, 5, 6\}$

Exercise 5.1

1. If $A = \{1, 2, 3\}$, $B = \{0, 1\}$ and $C = \{1, 3, 4\}$ then find
   (i). $A \cup B$ (ii). $A \cap B$ (iii). $A \cup C$
   (iv). $A \cap C$ (v). $B \cup C$ (vi). $A \cap A$
2. Find A \ B and B \ A when
   (i). $A = \{1, 3, 5, 7\}$, $B = \{3, 4, 5, 6, 7, 8\}$
   (ii). $A = \{0, \pm1, \pm2, \pm3\}$, $B = \{-1, -2, -3\}$
   (iii). $A = \{1, 2, 3, 4, \ldots\}$, $B = \{1, 3, 5, 7, \ldots\}$
3. If $U = \{1, 2, 3, \ldots 20\}$, $A = \{2, 4, 6, \ldots 20\}$
   $B = \{1, 3, 5, \ldots 19\}$ and $C = \varphi$ then find
   (i). A' (ii). B' (iii). C' (iv). $A' \cup B'$
   (v). $A' \cap B'$ (vi). $A' \cap B$ (vii). $A' \cup C'$ (viii). $A \cap C'$
   (ix). $C' \cap C$ (x). $B' \cup C'$
4. If U = set of natural numbers up to 15
   and A = set of even numbers up to 15
   and B = set of odd numbers up to 15
   then find.
   (i). $A' \cup B'$ (ii). $A' \cap B'$ (iii). U' (iv). φ'
   (v). $B \cap A'$ (vi). $B \cup B'$ (vii). $A \cap A'$ (viii). $A \cup B'$

5.1.2 Properties of union and intersection

(a) Commutative property of union

If A and B are any two sets then

$$A \cup B = B \cup A$$

Proof Let $x \in A \cup B$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in B \cup A$

Hence $A \cup B \subseteq B \cup A$ (i)

(b) Commutative property of intersection

Conversely

Let $x \in B \cup A$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in A \cup B$

Hence $B \cup A \subseteq A \cup B$ (ii)

From(i) and (ii), we have

$A \cup B = B \cup A$

(c) Associative property of union

If A, B and C are any three sets then

$A \cup (B \cup C) = (A \cup B) \cup C$

Proof

Let $x \in A \cup (B \cup C)$

$\Rightarrow x \in A$ or $x \in B \cup C$

$\Rightarrow x \in A$ or $x \in B$ or $x \in C$

$\Rightarrow x \in A \cup B$ or $x \in C$

$\Rightarrow x \in (A \cup B) \cup C$

Hence $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ (i)

Conversely,

Let $x \in (A \cup B) \cup C$

$\Rightarrow x \in A \cup B$ or $x \in C$

$\Rightarrow x \in A$ or $x \in B$ or $x \in C$

$\Rightarrow x \in A$ or $x \in B \cup C$

$\Rightarrow x \in A \cup (B \cup C)$

Hence $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ (ii)

From (i) and (ii), it follows that

$A \cup (B \cup C) = (A \cup B) \cup C$

(d) Associative property of intersection

If A, B and C are any three sets then

$A \cap (B \cap C) = (A \cap B) \cap C$

Proof

Let $x \in A \cap (B \cap C)$

$\Rightarrow x \in A$ and $x \in B \cap C$

$\Rightarrow x \in A$ and $x \in B$ and $x \in C$

$\Rightarrow x \in A \cap B$ and $x \in C$

$\Rightarrow x \in (A \cap B) \cap C$

Hence $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ (i)

Conversely,

Let $x \in (A \cap B) \cap C$

$\Rightarrow x \in A \cap B$ and $x \in C$

$\Rightarrow x \in A$ and $x \in B$ and $x \in C$

$\Rightarrow x \in A$ and $x \in B \cap C$

$\Rightarrow x \in A \cap (B \cap C)$

Hence $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ (ii)

From (i) and (ii),it follows

$A \cap (B \cap C) = (A \cap B) \cap C$

(e) Distributive property of union over intersection

If A, B and C are any three sets then

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof

Let $x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ or $x \in B \cap C$

$\Rightarrow x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (i)

Conversely,

Let $x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Rightarrow x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow x \in A$ or $x \in B \cap C$

$\Rightarrow x \in A \cup (B \cap C)$

Hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ (ii)

From (i) and (ii), it follows

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(f) Distributive property of intersection over union

If A, B and C are any three sets then

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof

Let $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in B \cup C$

$\Rightarrow x \in A$ and $x \in B$ or $x \in C$

$\Rightarrow\ x\in A$ and $x\in B$ or $x\in A$ and $x\in C$
$\Rightarrow\ x\in A\cap B$ or $x\in A\cap C$
$\Rightarrow\ x\in (A\cap B)\cup(A\cap C)$ (i)

Hence $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$

Conversely,

Let $x\in(A\cap B)\cup(A\cap C)$
$\Rightarrow\ x\in A\cap B$ or $x\in A\cap C$
$\Rightarrow\ x\in A$ and $x\in B$ or $x\in A$ and $x\in C$
$\Rightarrow\ x\in A$ and $x\in B$ or $x\in C$
$\Rightarrow\ x\in A$ and $x\in B\cup C$
$\Rightarrow\ x\in A\cap(B\cup C)$ (ii)

Hence $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$

From (i) and (ii),

$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$$

De Morgan's Laws

(g) If U is universal set and A and B are any two subsets of U then;

(i) $(A\cup B)'=A'\cap B'$ (ii) $(A\cap B)'=A'\cup B'$

Proof

(i) Let $x\in(A\cup B)'$
$\Rightarrow\ x\notin A\cup B$
$\Rightarrow\ x\notin A$ and $x\notin B$
$\Rightarrow\ x\in A'$ and $x\in B'$
$\Rightarrow\ x\in A'\cap B'$ (i)

Hence $(A\cup B)'\subseteq A'\cap B'$

Conversely,

Let $x\in A'\cap B'$
$\Rightarrow\ x\in A'$ and $x\in B'$
$\Rightarrow\ x\notin A$ and $x\notin B$
$\Rightarrow\ x\notin A\cup B$
$\Rightarrow\ x\in(A\cup B)'$ (ii)

Hence $A'\cap B'\subseteq(A\cup B)'$

From (i) and (ii), it follows

$$(A\cup B)'=A'\cap B'$$

Proof (ii) Let $x\in(A\cap B)'$
$\Rightarrow\ x\notin A\cap B$
$\Rightarrow\ x\notin A$ and $x\notin B$
$\Rightarrow\ x\in A'$ or $x\in B'$
$\Rightarrow\ x\in A'\cup B'$

Hence $(A\cap B)'\subseteq A'\cup B'$ (i)

Conversely

Let $x\in A'\cup B'$
$\Rightarrow\ x\in A'$ or $x\in B'$
$\Rightarrow\ x\notin A$ and $x\notin B$
$\Rightarrow\ x\notin A\cap B$
$\Rightarrow\ x\in(A\cap B)'$

Hence $A'\cup B'\subseteq(A\cap B)'$ (ii)

From (i) and (ii), it follows $(A\cap B)'=A'\cup B'$

5.1.3 Verification of fundamental properties of union and intersection

(a) Commutative property of union

For any two sets A and B, $A\cup B=B\cup A$

Example 5

Solution If $A=\{1,2,3\}$

$B=\{4,5,6\}$, then

$A\cup B=\{1,2,3\}\cup\{4,5,6\}=\{1,2,3,4,5,6\}$

and $B\cup A=\{4,5,6\}\cup\{1,2,3\}=\{1,2,3,4,5,6\}$

Hence $A\cup B=B\cup A$

(b) Commutative property of intersection

For any two sets A and B, $A\cap B=B\cap A$

Example 6

Solution If $A=\{a,b,c\}$

$B=\{b,c,d,e\}$, then

$A\cap B=\{a,b,c\}\cap\{b,c,d,e\}$

$=\{b,c\}$

and $B\cap A=\{b,c,d,e\}\cap\{a,b,c\}$

$=\{b,c\}$

Hence $A\cap B=B\cap A$

(c) Associative property of union

For any three sets A, B and C, $A\cup(B\cup C)=(A\cup B)\cup C$

Example 7

If $A=\{3,4,5\}$

$B=\{5,6,7\}$

and $C=\{8,9,10\}$, then prove that $A\cup(B\cup C)=(A\cup B)\cup C$

Solution

$$A\cup(B\cup C)=\{3,4,5\}\cup[\{5,6,7\}\cup\{8,9,10\}]$$
$$=\{3,4,5\}\cup\{5,6,7,8,9,10\}$$
$$=\{3,4,5,6,7,8,9,10\}$$

and
$$(A\cup B)\cup C=[\{3,4,5\}\cup\{5,6,7\}]\cup\{8,9,10\}$$
$$=\{3,4,5,6,7\}\cup\{8,9,10\}$$
$$=\{3,4,5,6,7,8,9,10\}$$

Hence,
$$A\cup(B\cup C)=(A\cup B)\cup C$$

(d) Associative property of intersection

For any three sets A, B and C, $A\cap(B\cap C)=(A\cap B)\cap C$

Example 8

If $A=\{1,2,3\}$

$B=\{2,3,4\}$

and $C=\{3,4,5\}$, then prove that $A\cap(B\cap C)=(A\cap B)\cap C$

Solution

$$A\cap(B\cap C)=\{1,2,3\}\cap[\{2,3,4\}\cap\{3,4,5\}]$$
$$=\{1,2,3\}\cap\{3,4\}$$
$$=\{3\}$$

and
$$(A\cap B)\cap C=[\{1,2,3\}\cap\{2,3,4\}]\cap\{3,4,5\}$$
$$=\{2,3\}\cap\{3,4,5\}$$
$$=\{3\}$$

Hence,
$$A\cap(B\cap C)=(A\cap B)\cap C.$$

(e) Distributive property of union over intersection

For any three sets A, B and C, $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

Example 9 Let $A=\{1,2,3,4\}$

$B=\{5,6,7\}$

and $C=\{7,8,9\}$,

then prove that $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

Solution

$$A\cup(B\cap C)=\{1,2,3,4\}\cup[\{5,6,7\}\cap\{7,8,9\}]$$
$$=\{1,2,3,4\}\cup\{7\}$$
$$=\{1,2,3,4,7\}$$

and
$$(A\cup B)\cap(A\cup C)=[\{1,2,3,4\}\cup\{5,6,7\}]\cap[\{1,2,3,4\}\cup\{7,8,9\}]$$
$$=\{1,2,3,4,5,6,7\}\cap\{1,2,3,4,7,8,9\}$$
$$=\{1,2,3,4,7\}$$

Hence,
$$A\cup(B\cap C)=(A\cup B)\cap(A\cup C).$$

(f) **Distributive property of intersection over union**

For any three sets A, B and C, $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

Example 10 Let $A=\{a,b,c\}$

$B=\{c,d,e\}$

and $C=\{e,f,g\}$,

then prove that $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

Solution

$$A\cap(B\cup C)=\{a,b,c\}\cap[\{c,d,e\}\cup\{e,f,g\}]$$
$$=\{a,b,c\}\cap\{c,d,e,f,g\}$$
$$=\{c\}$$

and
$$(A\cap B)\cup(A\cap C)=[\{a,b,c\}\cap\{c,d,e\}]\cup[\{a,b,c\}\cap\{e,f,g\}]$$
$$=\{c\}\cup\{\ \}$$
$$=\{c\}$$

Hence,
$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C).$$

(g) De. Morgan's laws

For any two sets A and B which are subsets of U

(i) $(A\cup B)' = A'\cap B'$ and (ii) $(A\cap B)' = A'\cup B'$

Example 11

Solution If $U = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 3\}$

$B = \{3, 4, 5\}$

(i) $A\cup B = \{2, 3\}\cup\{3, 4, 5\}$

$= \{2, 3, 4, 5\}$

$(A\cup B)' = U\setminus(A\cup B)$

$(A\cup B)' = \{1, 2, 3, 4, 5, 6\}\setminus\{2, 3, 4, 5\}$

$(A\cup B)' = \{1, 6\}$ (i)

also $A' = U - A = \{1, 2, 3, 4, 5, 6\}\setminus\{2, 3\}$

$= \{1, 4, 5, 6\}$

and $B' = U - B = \{1, 2, 3, 4, 5, 6\}\setminus\{3, 4, 5\}$

$= \{1, 2, 6\}$

$\therefore$ $A'\cap B' = \{1, 4, 5, 6\}\cap\{1, 2, 6\}$

$A'\cap B' = \{1, 6\}$ (ii)

From (i) and (ii) it follows;

$(A\cup B)' = A'\cap B'$

(ii) Now

$A\cap B = \{2, 3\}\cap\{3, 4, 5\}$

$= \{3\}$

$(A\cap B)' = U - (A\cap B)$

$(A\cap B)' = \{1, 2, 3, 4, 5, 6\}\setminus\{3\}$

$= \{1, 2, 4, 5, 6\}$ (i)

also $A'\cup B' = \{1, 4, 5, 6\}\cup\{1, 2, 6\}$

$= \{1, 2, 4, 5, 6\}$ (ii)

From (i) and (ii), it follows; $(A\cup B)' = A'\cap B'$

Exercise 5.2

1. Verify commutative property of union and intersection for the following sets.
   (i). $A = \{1, 2, 3, ...12\}$, $B = \{2, 4, 5, 8, 10, 12\}$
   (ii). $A = N$, $B = \{x \mid x \in N \wedge x$ is an even integer$\}$
   (iii). A = Set of first ten prime numbers.
   B = Set of first ten composite numbers.
2. Verify associative properties of union and intersection for the following sets.
   (i). $A = \{a, b, c, ...z\}$
   $B = \{a, e, i, o, u\}$
   $C = \{a, d, i, l, m, n, o\}$
   (ii). $A = \{1, 2, ... 100\}$
   $B = \{2, 4, 6, ... 100\}$
   $C = \{1, 3, 5, ... 99\}$
3. Verify distributive properties of union over intersection and intersection over u
   (i). $A = \{0, 1, 2\}$
   $B = \{0\}$
   $C = \varphi$
   (ii). $A = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$
   $B = \{-1, -2, -3, -4, -5\}$
   $C = \{-1, -2, +3, +4\}$
4. Verify De. Morgan's laws for the following sets.
   (i). $U = \{x \mid x \in N \wedge 1 \le x \le 20\}$
   $A = \{2, 3, 5, 7, 11, 12, 13, 17\}$
   $B = \{1, 4, 6, 8, 10, 14, 17, 18\}$
   (ii). If $U = \{1, 2, 3, ... 10\}$
   $A = \{2, 4, 6, 8, 10\}$
   $B = \{1, 3, 5, 7, 9\}$

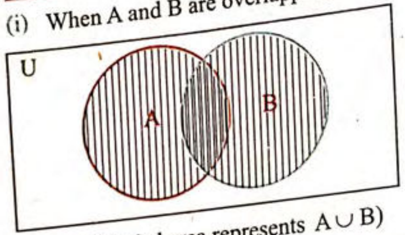
5.1.4 Venn diagrams

A Venn diagram is a visual way to show the relationships among or between sets th something in common. Usually, the Venn diagram consists of two or more ove circles, with each circle representing a set of elements, or members and univers represented by a rectangle. If two circles overlap, the members in the overlap belon sets; if three circles overlap, the members in the overlap belong to all three sets.

Example 12 Draw Venn diagrams for A ∪ B , A ∩ B A/B and B/A when;
(i) A and B are overlapping sets. (ii) A and B are disjoint sets. (iii) A ⊆ B.

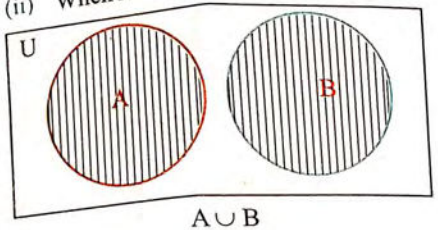
Solution Venn diagrams of union.

(i) When A and B are overlapping sets.



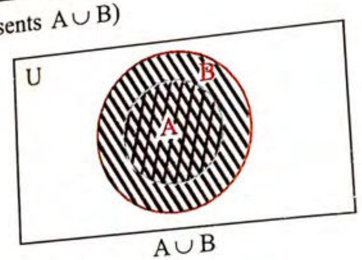
A ∪ B (Shaded area represents A ∪ B)

(ii) When A and B are disjoint sets.



A ∪ B

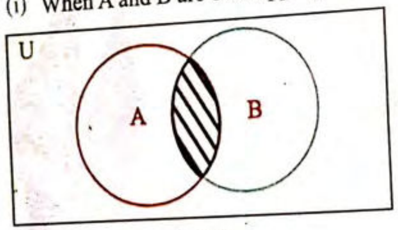
(iii) When A ⊆ B



A ∪ B

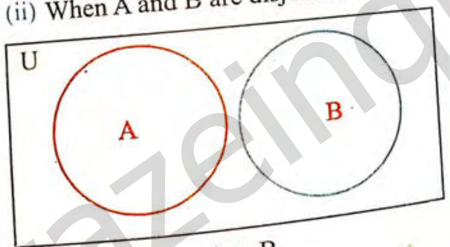
(a). **Venn diagrams of intersection**

(i) When A and B are overlapping sets.



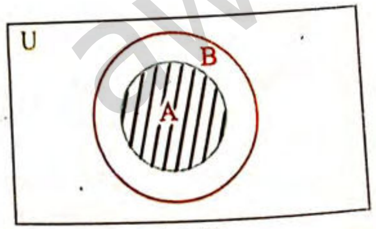
A∩B

(ii) When A and B are disjoint sets.



A ∩ B

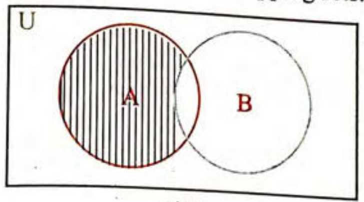
(iii) When A ⊆ B



A ∩ B

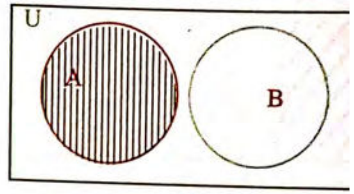
(b). **Venn diagrams of A\B**

When A and B are overlapping sets.



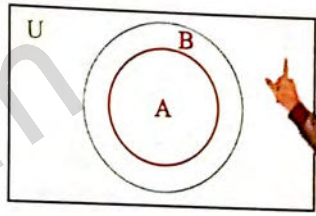
A\B

When A and B are disjoinfs sets.



A\B

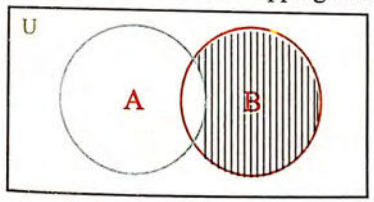
When A ⊆ B.



A\B

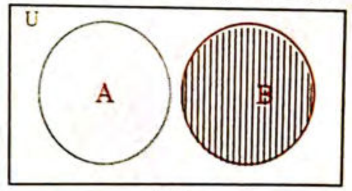
(c). **Venn diagrams of B\A**

When A and B are overlapping sets.



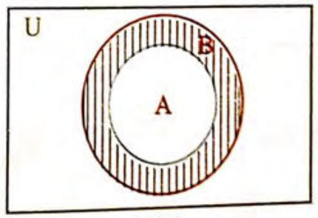
B\A

When A and B are disjoints sets.



B\A

When A ⊆ B.

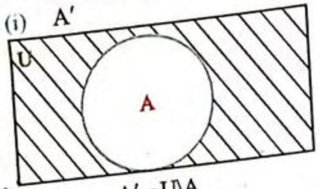


B\A

(d). Venn diagrams of complement sets A', B', $(A\cup B)'$ and $(A\cap B)'$

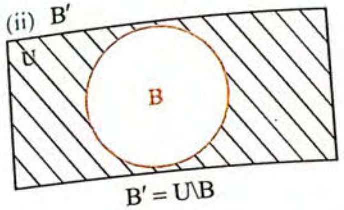
When A and B are overlapping sets

(i) A'



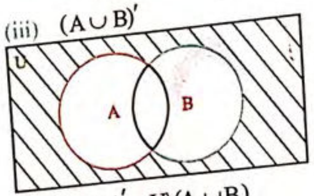
$A' = U\backslash A$

(ii) B'



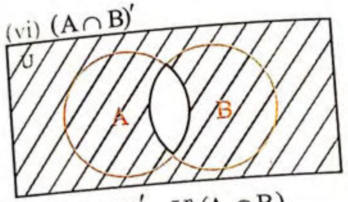
$B' = U\backslash B$

(iii) $(A\cup B)'$



$(A\cup B)' = U\backslash(A\cup B)$

(vi) $(A\cap B)'$



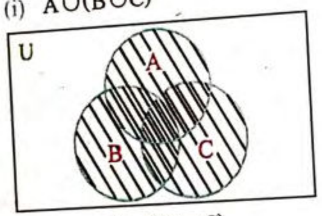
$(A\cap B)' = U\backslash(A\cap B)$

(e). Venn diagrams of union and intersection of three sets

Example 13 If A, B and C are any three sets then draw Venn diagrams of $A\cup(B\cup C)$, $(A\cap B)\cap C$, $A\cup(B\cap C)$ and $A\cap(B\cup C)$

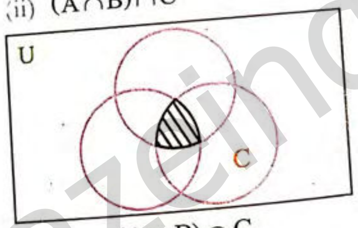
Solution

(i) $A\cup(B\cup C)$



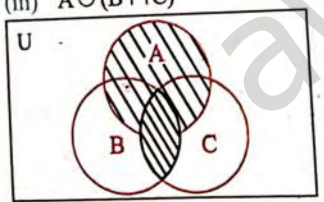
$A\cup(B\cup C)$

(ii) $(A\cap B)\cap C$



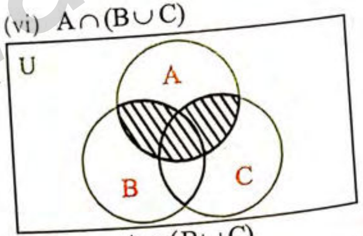
$(A\cap B)\cap C$

(iii) $A\cup(B\cap C)$



$A\cup(B\cap C)$

(vi) $A\cap(B\cup C)$



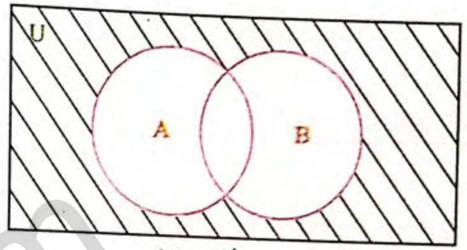
$A\cap(B\cup C)$

(f). Venn diagrams of De. Morgan's Laws

If A and B are any two sets then

(i) $(A\cup B)' = A'\cap B'$, (ii) $(A\cap B)' = A'\cup B'$

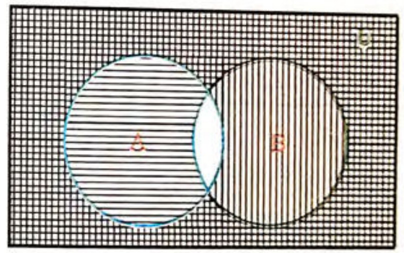
Venn diagrams of $(A\cup B)$



$(A\cup B)'$ (Fig. 1)

Where U is universal set and A and B are overlapping sets of U. Here $A\cup B$ is white part and $(A\cup B)'$ is shaded part.

Venn diagrams of $A'\cap B'$



$A'\cap B'$ (Fig. 2)

In Fig. 2,

$A' = U\backslash A$ is represent by vertical lines.

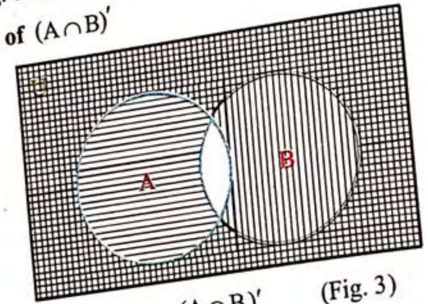
$B' = U\backslash B$ is represent by horizontal lines.

Therefore, $A'\cap B'$ is represented by ▦ lines i.e. horizontal and vertical lines cross each other as shown in figure.

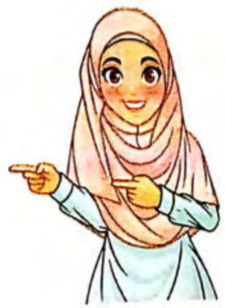
$(A\cup B)' = A'\cap B'$

Hence from Fig. 1 and Fig. 2, we see that

Venn diagram of $(A\cap B)'$

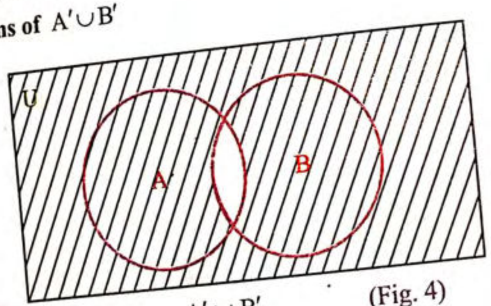


$(A\cap B)'$ (Fig. 3)



· In this figure shaded area represents $(A\cap B)'$

Venn diagrams of $A'\cup B'$



$A'\cup B'$ (Fig. 4)

In Fig. 4,

A' is represented by vertical lines

B' is represented by horizontal lines

While $A'\cup B'$ is represented by either lines.

Hence From fig 3 and fig 4, we see that

$$(A\cap B)' = A'\cup B'$$

Verification of fundamental properties with the help of Venn diagrams

Example 14 If $A=\{1,2,3,4\}$
$B=\{3,4,5,6\}$
$C=\{3,4,7,8\}$

Then verify the following with the help of Venn diagrams.

(i) $A\cup B=B\cup A$

(ii) $A\cap B=B\cap A$

(iii) $A\cup(B\cup C)=(A\cup B)\cup C$

(iv) $A\cap(B\cap C)=(A\cap B)\cap C$

(v) $A\cup(B\cap C)=(A\cap B)\cap(A\cup C)$

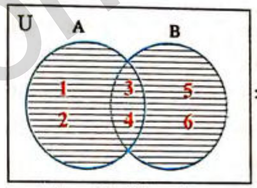
(vi) $A\cap(B\cup C)=(A\cap B)\cap(A\cap C)$

Solution

(i) $A\cup B=\{1,2,3,4\}\cup\{3,4,5,6\}$
$=\{1,2,3,4,5,6\}$

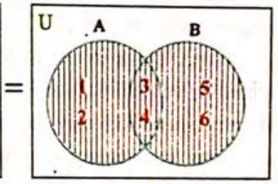
and $B\cup A=\{3,4,5,6\}\cup\{1,2,3,4\}$
$=\{1,2,3,4,5,6\}$

Venn diagrams are



$A\cup B$ (Fig. a)

=



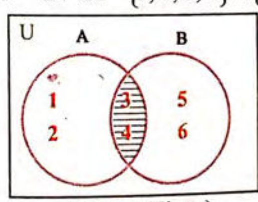
$B\cup A$ (Fig. b)

It is clear from Fig. (*a*) and Fig. (b), that both regions representing $A\cup B$ and $B\cup A$ are are identical therefore

$$A\cup B=B\cup A$$

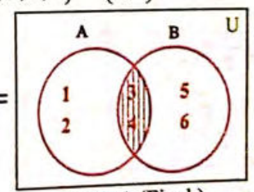
(ii) $A\cap B=\{1,2,3,4\}\cup\{3,4,5,6\}=\{3,4\}$

and $B\cap A=\{3,4,5,6\}\cap\{1,2,3,4\}=\{3,4\}$



$A\cap B$ (Fig. a)

=



$B\cap A$ (Fig. b)

It is clear from Fig. (*a*) and Fig. (b) that both regions representing $A\cap B$ and $B\cap A$ are are identical. Therefore,

$$A\cap B=B\cap A$$

(iii) $A\cup(B\cup C)=\{1,2,3,4\}\cup[\{3,4,5,6\}\cup\{3,4,7,8\}]$

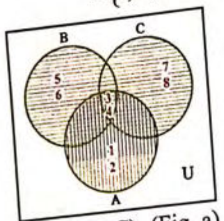
$=\{1,2,3,4\}\cup\{3,4,5,6,7,8\}$

$=\{1,2,3,4,5,6,7,8\}$

and $(A\cup B)\cup C=[\{1,2,3,4\}\cup\{3,4,5,6\}]\cup\{3,4,7,8\}$

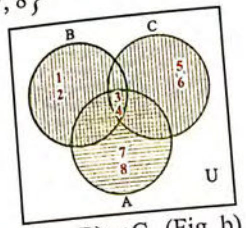
$=\{1,2,3,4,5,6\}\cup\{3,4,7,8\}$

$=\{1,2,3,4,5,6,7,8\}$



$A\cup(B\cup C)$ (Fig. a)

=



$(A\cup B)\cup C$ (Fig. b)

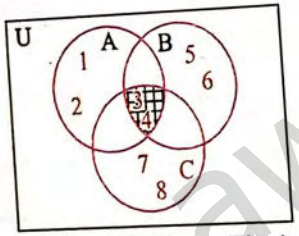
It Fig. (*a*) A is represented by vertical lines and $B\cup C$ represented by horizontal lines. The region which is lined either in one or both ways represented $A\cup(B\cup C)$. In Fig. (b) $A\cup B$ is represented by vertical lines and C is represented by horizontal lines. The region which is lined either in one or both ways represents $A\cup(B\cup C)$. It is clear from Fig. (a) and Fig. (b) both the regions representing $A\cup(B\cup C)$ and $(A\cup B)\cup C$ are identical. Therefore,

$A\cup(B\cup C)=(A\cup B)\cup C$

(iv) $A\cap(B\cap C)=\{1,2,3,4\}\cap[\{3,4,5,6\}\cap\{3,4,7,8\}]$

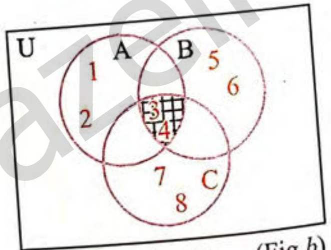
$=\{1,2,3,4\}\cap\{3,4\}=\{3,4\}$

Similarly, $(A\cap B)\cap C=\{3,4\}$



$A\cap(B\cap C)$ (Fig.*a*)

=



$(A\cap B)\cap C$ (Fig.*b*)

It is clear from fig.(*a*) and fig.(b) that both regions representing $A\cap(B\cap C)$ and $(A\cap B)\cap C$ are identical.

Therefore, $A\cap(B\cap C)=(A\cap B)\cap C$.

(v) $A\cup(B\cap C)=\{1,2,3,4\}\cup[\{3,4,5,6\}\cap\{3,4,7,8\}]$

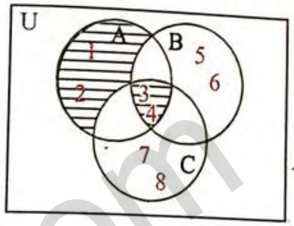
$=\{1,2,3,4\}\cup\{3,4\}$

$=\{1,2,3,4\}$

$(A\cup B)\cap(A\cup C)=[\{1,2,3,4\}\cup\{3,4,5,6\}][\{1,2,3,4\}\cup\{3,4,7,8\}]$

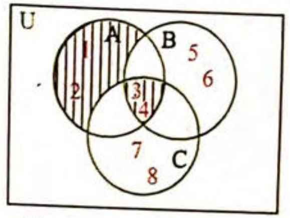
$=\{1,2,3,4,5,6\}\cap\{1,2,3,4,7,8\}$

$=\{1,2,3,4\}$



$A\cup(B\cap C)$ (Fig.*a*)

=



$(A\cup B)\cap(A\cup C)$ (Fig.*b*)

It is clear from Fig. (a) and Fig. (b) that both regions representing $A\cup(B\cap C)$ an $(A\cup B)\cap(A\cup C)$ are identical.

(vi) $A\cap(B\cup C)=\{1,2,3,4\}\cap[\{3,4,5,6\}\cup\{3,4,7,8\}]$

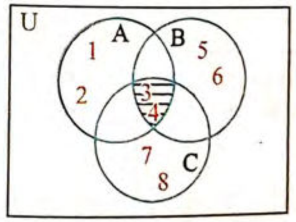
$=\{1,2,3,4\}\cap\{3,4,5,6,7,8\}$

$=\{3,4\}$

$(A\cap B)\cup(A\cap C)=[\{1,2,3,4\}\cap\{3,4,5,6\}]\cup[\{1,2,3,4\}\cap\{3,4,7,8\}]$

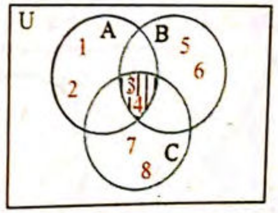
$=\{3,4\}\cup\{3,4\}$

$=\{3,4\}$



$A\cap(B\cup C)$ (Fig.*a*)

=



$(A\cap B)\cup(A\cap C)$ (Fig.*b*)

Example 15 If $U = \{1, 2, 3, 4, 5, 6, 7\}$
$A = \{2, 5, 6\}$ and $B = \{1, 2, 3\}$

then draw Venn diagrams for

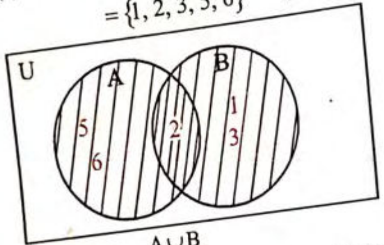
(i) $A \cup B$ (ii) $A \cap B$ (iii) A' (iv) B' (v) $(A \cup B)'$

(vi) $A' \cap B'$ (vii) $(A \cap B)'$ (viii) $A' \cup B'$

Hence verify De Morgan's laws.

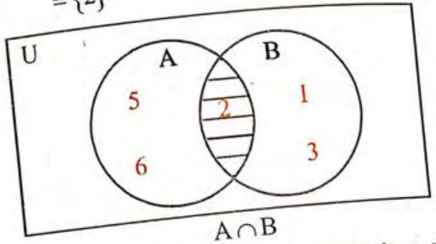
Solution

(i) $A \cup B = \{2, 5, 6\} \cup \{1, 2, 3\}$
$= \{1, 2, 3, 5, 6\}$



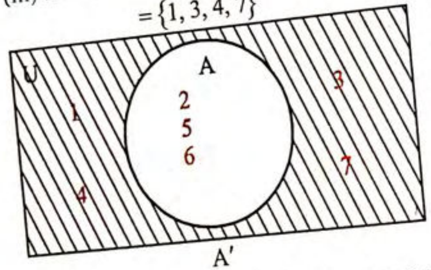
$A \cup B$

(ii) $A \cap B = \{2, 5, 6\} \cap \{1, 2, 3\}$
$= \{2\}$



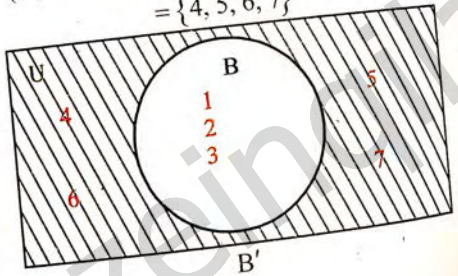
$A \cap B$

(iii) $A' = U \setminus A = \{1, 2, 3, 4, 5, 6, 7\} \setminus \{2, 5, 6\}$
$= \{1, 3, 4, 7\}$



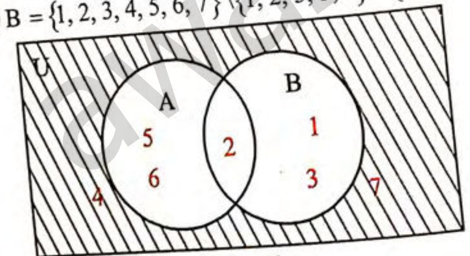
A'

(iv) $B' = U \setminus B = \{1, 2, 3, 4, 5, 6, 7\} \setminus \{1, 2, 3\}$
$= \{4, 5, 6, 7\}$



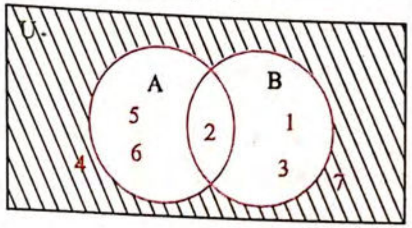
B'

(v) $(A \cup B)' = U \setminus A \cup B = \{1, 2, 3, 4, 5, 6, 7\} \setminus \{1, 2, 3, 5, 6\} = \{4, 7\}$



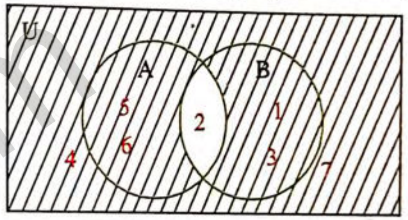
$(A \cup B)'$

(vi) $A' \cap B' = \{1, 2, 3, 4, 7\} \cap \{4, 5, 6, 7\} = \{4, 7\}$



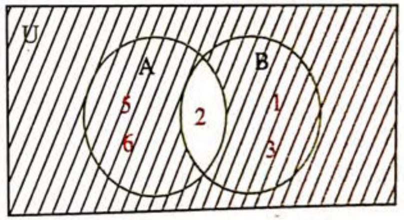
$A' \cap B'$

(vii) $(A \cap B)' = U \setminus A \cap B = \{1, 2, 3, 4, 5, 6, 7\} \setminus \{2\} = \{1, 3, 4, 5, 6, 7\}$



$(A \cap B)'$

(viii) $A' \cup B' = \{1, 3, 4, 7\} \cup \{4, 5, 6, 7\} = \{1, 3, 4, 5, 6, 7\}$



$A' \cup B'$

From (v) and (vi), it is proved that

$(A \cup B)' = A' \cap B'$

and from (vi) and (viii), it is proved that

$(A \cap B)' = A' \cup B'$

Exercise 5.3

1. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 6, 7\}$, then draw Venn diagrams for the following.
   (i). $A \cup B$ (ii). $A \cap B$
2. If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{3, 4, 5, 6, 7, 8\}$ and $C = \{5, 6, 9, 10\}$, then verify with the help of Venn diagrams.
   (i). $A \cup (B \cup C) = (A \cup B) \cup C$ (ii). $A \cap (B \cap C) = (A \cap B) \cap C$
   (iii). $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (iv). $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
3. If $U = \{1, 2, 3, 4, 5, 6, 7,\}$, $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$
   Draw Venn diagrams for A', B', $A' \cup B'$, $A' \cap B'$ and also verify that
   (i). $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
4. If $U = \{a, b, c, 1, 2, 3, 4\}$, $A = \{c, 3\}$ and $B = \{a, 3, 4\}$ then draw Venn diagrams for A', B', $A \backslash B$ and $B \backslash A$.
5. If $U = \{a, b, c, d, e, f, g\}$, $A = \{a, b, c\}$ and $B = \{c, d, e\}$ then verify De Morgan's laws with the help of Venn diagrams.

5.1.5 Ordered pairs and Cartesian product

Ordered Pairs

An ordered pair (a, b) consists of two elements "a" and "b" in which "a" is the first element and "b" is the second element.
The ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$.
Note that (a, b) and (b, a) are not equal unless $a = b$.

Example 16 Find x and y given $(2x, x + y) = (6, 2)$

Solution Two ordered pairs are equal if and only if the corresponding components are equal. Hence, we obtain the equations:

$$2x = 6 \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$
$$\text{and } x + y = 2 \quad \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Solving equation (i) we get $x = 3$ and when substituted in (ii) we get $y = -1$.

Cartesian Product Of Two Sets

The Cartesian product of A and B, denoted $A \times B$ (read "A cross B") is the set of all ordered pairs (a, b), where a is in A and b is in B.

Symbolically: $A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$

Remarks:
(i) $A \times B \neq B \times A$ for non-empty and unequal sets A and B (ii) $A \times \phi = \phi \times A = \phi$

5.2 Binary Relation:

Let A and B be sets. A (binary) relation R from A to B is a subset of $A \times B$.
When $(a, b) \in R$, we say a is related to b by R, written a R b.
Otherwise if $(a, b) \notin R$, we write $a \not{R} b$.

Real Life Application

Relationships between elements of sets occur in many contexts. Every day we deal with relationships such as those between a business and its telephone number, an employee and his or her salary, a person and a relative, and so on. In mathematics we study relationships such as those between a positive integer and one that it divides, a real number and one that is larger than it, a real number x and the value $f(x)$ where f is a function, and so on.

Example 17 If $A = \{a, b\}$ and $B = \{1, 2\}$ then
$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$

Solution Number of elements in $A \times B = 2 \times 2 = 4$
Number of all possible subsets of
$A \times B = 2^4 = 16$

Since every subset of $A \times B$ is a binary relation from A to B. Therefore, here we have 16 binary relations which are given below.

$R_1 = \varphi$ $R_2 = \{(a, 1)\}$
$R_3 = \{(a, 2)\}$ $R_4 = \{(b, 1)\}$
$R_5 = \{(b, 2)\}$ $R_6 = \{(a, 1), (a, 2)\}$
$R_7 = \{(a, 1), (b, 1)\}$ $R_8 = \{(a, 1)(b, 2)\}$
$R_9 = \{(a, 2)(b, 1)\}$ $R_{10} = \{(a, 2), (b, 2)\}$
$R_{11} = \{(b, 1), (b, 2)\}$ $R_{12} = \{(a, 1), (a, 2), (b, 1)\}$
$R_{13} = \{(a, 1), (b, 1), (b, 2)\}$ $R_{14} = \{(a, 1), (a, 2), (b, 2)\}$
$R_{15} = \{(a, 2), (b, 1), (b, 2)\}$ $R_{16} = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$

Similarly, total number of binary relation in $B \times A = 2^4 = 16$

Note
If set A has m elements and set B has n elements then A B has m n elements.

Example 18 Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$
Then $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$
Write any five relations from A to B.

Solution

Let

$R_1 = \{(1,1), (1, 3), (2, 2)\}$
$R_2 = \{(1, 2), (2, 1), (2, 2), (2, 3)\}$
$R_3 = \{(1, 1)\}$
$R_4 = A \times B$
$R_5 = \varnothing$

All being subsets of $A \times B$ are relations from A to B.

Domain Of A Relation

The domain of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R denoted Dom(R).

Symbolically: $\text{Dom}(R) = \{a \in A \mid (a,b) \in R\}$

Range Of A Relation

The range of a relation R from A to B is the set of all second elements of the ordered pairs which belong to R denoted Ran(R).

Symbolically: $\text{Range}(R) = \{b \in B \mid (a,b) \in R\}$

Example 19 Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$.
Define a binary relation R from A to B as follows:
$R = \{(a, b) \in A \times B \mid a < b\}$

Then
(i) Find the ordered pairs in R.
(ii) Find the Domain and Range of R.
(iii) Is 1R3, 2R2?

Solution Given $A = \{1, 2\}$, $B = \{1, 2, 3\}$.
$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

(i) $R = \{(a, b) \in A \times B \mid a < b\}$
$R = \{(1,2), (1,3), (2,3)\}$

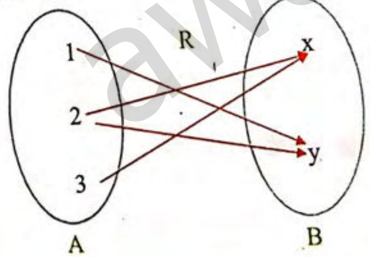
(ii) $\text{Dom}(R) = \{1,2\}$ and $\text{Range}(R) = \{2, 3\}$

(iii) Since $(1,3) \in R$ so 1R3
But $(2, 2) \notin R$ so 2 is not related with 3.

Arrow Diagram Of A Relation

Let
$A = \{1, 2, 3\}$, $B = \{x, y\}$ and
$R = \{(1,y), (2,x), (2,y), (3,x)\}$
be a relation from A to B.

The arrow diagram of R is:



Exercise 5.4

1. If $A = \{1, 2, 3\}$, $B = \{4, 5\}$ then
   (i). Write three binary relations from A to B.
   (ii). Write four binary relations from B to A.
   (iii). Write four binary relations on A.
   (iv). Write two binary relations on B.
2. If $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5\}$ and $R = \{(x,y) \mid y < x\}$ is a binary relation from A to B, then write it in tabular form.
3. Domain of a binary relation $R = \{(x,y) \mid y = 2x\}$ is the set $\{0, 4, 8\}$, find Range of R.
4. Domain of a binary relation $R = \{(x, y) \mid y + 1 = 2x^2\}$ is set N. Find its range.

5.3 Functions

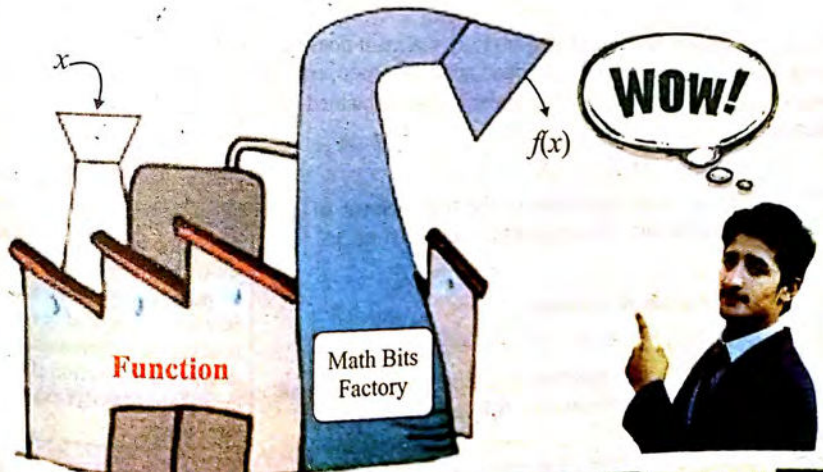
Let A and B are two non-empty sets, then a binary relation f is said to be a function from A to B if

- Dom $f = A$
- There should be no repetition in the first element of all ordered pairs contained in f.

Symbolically, we write it as

$$f : A \longrightarrow B$$

and say f is function from A to B.



Example 20

If A = {1, 2, 3} and B = {a, b, c, d} then which of the following are functions?

$f_1 = \{(1, a), (2, b)\}$

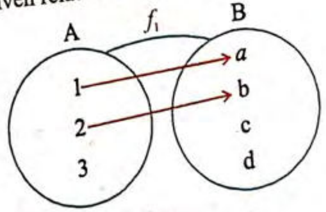
$f_2 = \{(1, a), (2, b), (3, c), (3, d)\}$

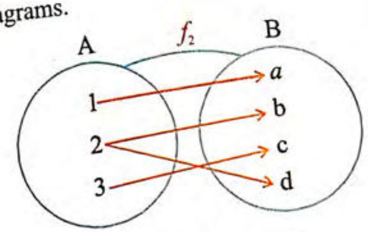
$f_3 = \{(1, a), (2, b), (3, c)\}$

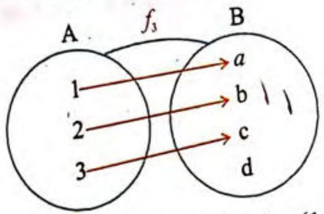
$f_4 = \{(1, a), (2, a), (3, d)\}$

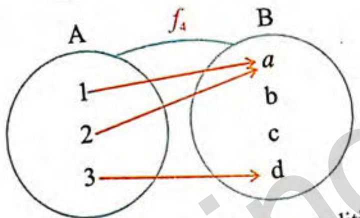
Solution

The given relations are shown in the following diagrams.









f_1 is not a function because Dom $f_1 = \{1, 2\} \neq A$ i.e. it does not satisfy the first condition of being a function. f_2 is also not a function because second condition of being a function is not satisfied i.e. first element $2 \in A$ of ordered pairs contained in f_2 is repeated. f_3 is a function because

(i) Dom $f_3 = \{1, 2, 3\} = A$

(ii) There is no repetition in the first element of all ordered pairs contained in f_3. It can be written as

$$f_3 : A \longrightarrow B$$

f_4 is a function because

(i) Dom $f_4 = \{1, 2, 3\} = A$

(ii) There is no repetition in the first element of all ordered pairs contained in f_4. It can be written as

$$f_4 : A \longrightarrow B$$

Note

A relation is any pairing of set of inputs with a set of outputs. Every function is relation, but not every relation is a function. A relation is a function if for every input there is exactly one output.

5.3.1 Domain, Co-domain and Range of a function

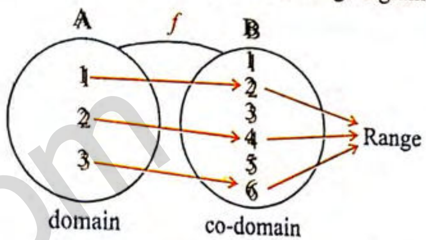
Let f: A⟶B be a function, then the set A is called domain of "f" (or set of first elements of the ordered pairs contained in f).

The set B is co-domain of f and the set of second elements of all ordered pairs contained in f is called range of the function. Range is always a subset of co-domain. i.e. Range $f \subseteq B$.

Example:

Let A = {1, 2, 3} and B = {1, 2, 3, 4, 5, 6}

f : A⟶B as shown in the following diagram.



In this diagram, the set A is domain, the set B is co-domain and that set {2, 4, 6} which is a subset of B is range of the function f.

5.3.2 Kinds of a function

(a) Into function

Let f be a function from A to B, then f is into function if Range $f \neq B$

It can be written as

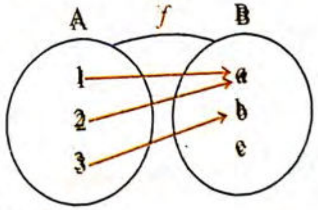
$$f : A \xrightarrow{\text{into}} B$$

Example:

Let A = {1, 2, 3} and B = {a, b, c} then a function f from A to B is defined by

$f = \{(1, a), (2, a), (3, b)\}$

The following figure illustrates the function f.



We see that f is into function because

Range $f = \{a, b\} \neq B$

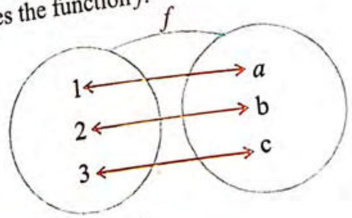
(b) One-one function

Let f be a function from A to B, then f is one-one function, if there is no repetition in the second element of all ordered pairs contained in f. This means that element of the domain set A is associated with one and only one element of the set B. It can be written as:

$$f: A \xrightarrow{\text{one–one}} B$$

Example:

If A = {1, 2, 3} and B = {a, b, c} then
f = {(1, a), (2, b), (3, c)}
The following figure illustrates the function f.



From figure it is clear that f is one – one function.

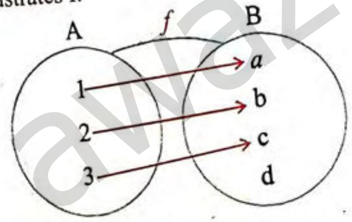
(c) Into and one-one function (Injective function)

Let f be a function from A to B, then f is into one-one or injective function, if
(i) Range $f \neq$ B
(ii) There is no repetition in the second element of all ordered pairs contained in f.
It can be written as

$$f: A \xrightarrow{\text{injective}} B$$

Example:

If A = {1, 2, 3} and B = {a, b, d} and f is a function from A to B defined by,
f = {(1, a), (2, b), (3, c)}
The following figure illustrates f.



Here we see that f is injective function because
(i) Range f = {a, b, c} $\neq$ B
(ii) There is no repetition in the second element of all ordered pairs contained in f.

(d) Onto function (surjective function)

Let f be a function from A to B, then f is onto function if
Range f = B
It can be written as

$$f: A \xrightarrow{\text{onto}} B$$

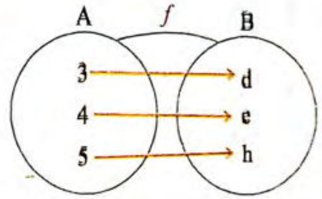
(e) One-one and onto function (bijective function)

Let f be a function from A to B, then f is one-one and onto (bijective) function if it is both one-one and onto.

i.e. $f: A \xrightarrow{\text{bijective}} B$

Example:

Let A = {3, 4, 5} and B = {d, e, h} then a function f from A to B is define by
f = {(3, d), (4, e), (5, h)}
The following figure illustrates f.



Here we see that f is onto function
Range f = {d, e, h} = B

f is one – one function because for each $x \in A$ there is unique $y \in B$.
For $3 \in A$ there is unique $d \in B$
For $4 \in A$ there is unique $e \in B$
For $5 \in A$ there is unique $h \in B$
Hence f is bijective function.

Difference between one – one correspondence and one – one function

If A and B are two non-empty sets then a rule for which each element of A is paired with one and only one element of B and each element of B is paired with one and only one element of A is called one-one correspondence.

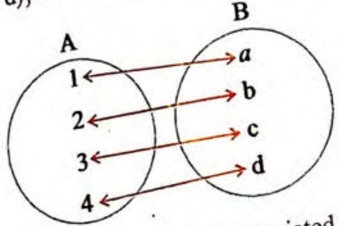
Remember

If the two sets are finite and if there exists a one-one correspondence between them, they have the same number of elements.

The difference is clear from the following two examples.

Example

Let A = {1, 2, 3, 4} and B = {a, b, c, d} then a one – one correspondence is given by {(1, a), (2, b), (3, c), (4, d)} as shown in the following figure.

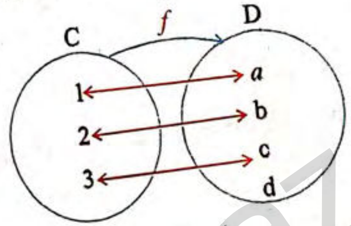


In one – one function every element of the set A is associated with one and only one element of set B. This means that Range f may not be equal to set B.

Example

If C = {1, 2, 3} and D = {a, b, c, d}

then f = {(1, a), (2, b), (3, c)} is one – one function, as shown in the following figure.



From figure, it is clear that there does not exist one – one correspondence between sets C and D because $d \in D$ is unpaired.

WARNING

(a) Note that $f(a + 3) \neq f(a) + f(3)$. Function notation is not to be confused with the distributive law.

(b) Note that $f(x^2) \neq f \cdot x^2$. The use of parentheses in function notation does not imply multiplication.

Exercise 5.5

1. A = {1, 2, 3, 4} and B = {6, 7} and the following are the relations from A to B, then state whether these are functions or not?

   If these are functions then state which kind of functions are these?

   R_1 = {(1, 6), (2, 7), (3, 6)}

   R_2 = {(1, 6), (2, 6), (3, 7), (4, 7)}

   R_3 = {(1, 6), (2, 6), (3, 6), (4, 6)}

2. Which of the following relations on set {a, b, c, d} are functions? State the kind of functions as well

   (i). {(a, b), (c, d), (b, d), (d, b)} (ii). {(b, a), (c, b), (a, b), (d, d)}

   (iii). {(d, c), (c, b), (a, b), (d, d)} (iv). {(a, b), (b, c), (c, b), (d, a)}

3. If A = {0, 1, 2, 3} and B = {x, y, z, p} then state whether following relations show that there exist one – one correspondence between the elements of set A and B. If not, give reasons.

   (i). {(0, x), (2, z), (3, y), (1, p)} (ii). {(0, x), (1, z), (2, y), (3, z)}

4. If A = {a, b, c} and B = {2, 3, 4, 5} then state, whether the following relations show that there exists one – one correspondence between the elements of sets A and B, if not what kind of the relations they are?

   (i). {(a, 2), (b, 3), (c, 4)} (ii). {(a, 3), (b, 4), (c, 3)}

5. If X = {1, 2, 3, 4} and Y = {5, 6, 7, 8} then write

   (i). a function from X to Y.

   (ii). a one- one function from X to Y.

   (iii). a relation which shows that there exist one – one correspondence between X and Y.

   (iv). a function which is onto from Y to X.

   (v). bijective function from Y to X.

   (vi). a function from X to Y which is neither one- one nor onto.

6. Let $A = \{1,2,3,4,5\}$. Check whether the following sets are functions on A. In case these are functions, indicate their ranges. Which function is onto.

   (i). {(1, 5), (2, 3), (3, 3), (4, 2), (5, 1)} (ii). {(1, 1), (2, 4), (3, 2), (4, 1), (5, 3)}

   (iii). {(1, 2), (2, 1), (3, 1), (4, 4), (5, 5)}

Review Exercise 5

1. At the end of each question, four circles are given. Fill in the correct circle only.

(i). If A = {1, 2, 3}, B = {4, 5} and R = {(1, 4), (2, 5), (3, 4)} then R is ________

○ A one-one function from A to B ○ A function from A to B
○ Not a function ○ An onto function from A to B

(ii). If A has two elements and B has three elements, then number of binary relations in A × B is ________.

○ 2×3 ○ 2^3 ○ 2^6 ○ 2^2

(iii). Which one of the following is an example of disjoint sets?

○ {0,1,2,3} and {3,2,1,0} ○ {0,2,4,6} and {2,4,6,8}
○ {0,3,6,9} and {9,16,25,36} ○ {0,4,8,12} and {6,10,14,18}

(iv). If the universal set U = {x|x is a positive odd integer less than 30}, R = {1, 5, 7}, and S = {1, 3, 7, 11, 13}, how many elements are in $(R \cap S)'$?

○ 15 ○ 13 ○ 7 ○ 2

(v). If f is a function from A to B, then f is onto function if

○ Range f = B ○ Range f ≠ A ○ Dom f = A
○ second element of all ordered pairs contained in f is not repeated.

(vi). If R = {(0, 0), (8, 2), (10, 3), (14, 12)}, then Dom R = ________

○ {0,8,10,14} ○ {0,2,3,12} ○ {8,10,4} ○ {0,10}

2. If U = {Natural numbers up to 100}
A = {positive even numbers up to 100}
B = {positive odd numbers up to 100}

Then find.

(i). $A' \cup B'$ (ii). $A' \cap B'$ (iii). $A \cap B'$ (iv). $A' \cap B$

3. If A = {1, 2, 3, 5, 7}, B = {2, 4, 6}, C = {2, 5, 9} then verify.

(i). Associative property of union
(ii). Associative property of intersection
(iii). Distributive property of union over intersection.
(iv). Distributive property of union over union.

4. If $U = \{x \mid x \in N \wedge 1 \le x \le 40\}$
A = {1, 6, 11, 16, 21, 26, 31}
B = {2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32}
then verify De. Morgan's laws.

5. If U = {1, 2, 3, 5, 6, 7}
A = {2, 5, 6}
B = {1, 2, 3}

Then verify De-Morgan's laws with the help of Venn diagrams.

6. If U = {1, 2, ... 10}
A = {1, 2, 3, 4}
B = {3, 4, 5, 6}
C = {3, 4, 7, 8}

Then verify distributive laws with the help of Venn diagrams.

7. Let A = {−2, −1, 0, 1, 2} B = {a, b, c, d, e}. Determine which sets of ordered pair represent a function. In case of a function, mention one-one function, onto function and bijective function.

(i). {(−2, a), (−1, a), (0, b), (1, c), (2, d)} (ii). {(−1, a), (1, e), (−2, d), (0, c), (2, b)}
(iii). {(2, d), (0, a), (−2, b), (−1, c), (1, e)} (iv). {(−2, b), (−1, b), (0, a), (1, d), (−2, e)}

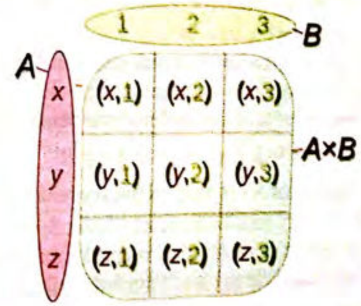
8. Let A = {1, 2, 3, 4, 5}, check whether the following sets are functions on A. In case these are functions, indicate their ranges. Which function are onto.

(i). {(1, 5), (2, 3), (3, 3), (4, 2), (5, 1)} (ii). {(1, 1), (2, 4), (3, 2), (4, 1), (5, 3)}
(iii). {(1, 2), (2, 1), (3, 1), (4, 4), (5, 5)} (iv). {(1, 2), (2, 3), (1, 4), (3, 5)}

9. If X = {−6, −5, −4, −3}
Y = {1, 2, 3, 4} then write

(i). A one – one function from X to Y. (ii). Onto function from X to Y.
(iii). A function which is one – one and onto from X to Y.
(iv). A function from X to Y which is neither one – one nor onto.

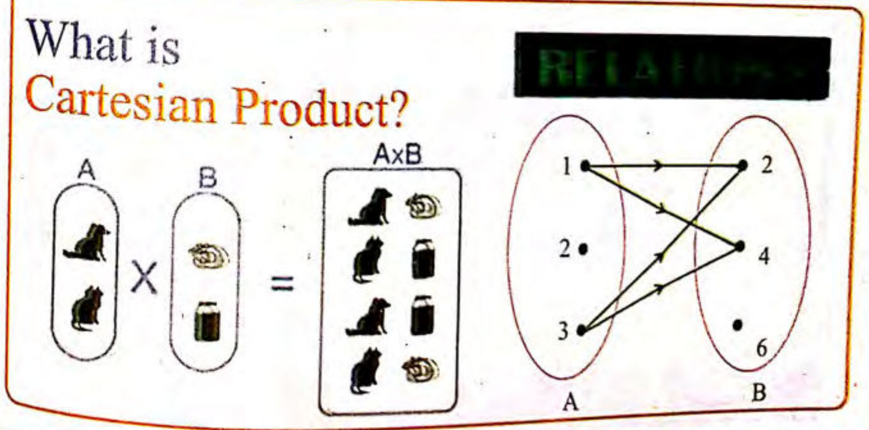
PROJECT (−2, 5), (−1, 0), (0, 1), and (2, 5) is a function. If the x-values and y-values of this function are switched, the relation that results is not a function. Provide an example of a function that will still be a function if the x-values and y-values are switched.



Summary

- A set is a "collection of well-defined distinct objects, sets are represented by capital English alphabets, A, B, C, …. , Z and elements of sets are represented by small English alphabets, a, b, c, … z.
- If A and B are two sets, then **union** of set A and set B consists of all elements in set A or in set B or in both A and B, and it is denoted by $A\cup B$. Symbolically.
  $A\cup B=\{x \mid x\in A \text{ or } x\in B\}$
- If A and B are two sets, then **intersection** of set A and set B consists of all those elements which are common to both A and B, and it is denoted by $A\cap B$, Symbolically
  $A\cap B=\{x \mid x\in A \text{ or } x\in B\}$
- Two sets A and B are disjoint if $A\cap B=\varphi$.
- If U is universal set and A is subset of U, then U \ A is called **complement** of set A and is denoted by A' or A^c.
- If A and B are two sets, then their **difference** consists of all those elements of A which are not in B, and it is denoted by A \ B or A – B symbolically,
  $A\setminus B=\{x \mid x\in A \text{ or } x\notin B\}$
- For any two sets A and B,
  - $A\cup B=B\cup A$
  - $A\cup(B\cup C)=(A\cup B)\cup C$
  - $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$
  - $(A\cup B)'=A'\cap B'$ and $(A\cap B)'=A'\cup B'$
  - $A\cap B=B\cap A$
  - $A\cap(B\cap C)=(A\cap B)\cap C$
  - $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
- Concept of sets i.e. union, intersection, complement, difference of sets, can be explained easily with the help of **Venn diagrams**. In these diagrams, a set is usually represented by a circle and universal set is represented by a rectangle.
- (a, b) is called **ordered pair** of two elements a and b of a set or of different sets, where a is the first element and b is the second element.
  $(a, b)\neq(b, a)$
- If A and B are two non- empty sets then A × B is called **Cartesian product**, which is set of all ordered pairs such that the first element of each ordered pair belongs to set A and second element of each ordered pair belongs to set B symbolically.
  - $A\times B=\{(x, y) \mid x\in A \text{ and } y\in B\}$
  - If $A\neq B$, then $A\times B\neq B\times A$
- Any subset of A × B is called **binary relation**, where A and B are two non-empty sets. If R is binary relation from set A to set B, i.e. $R=\{(x, y) \mid x\in A\wedge y\in B\}$, then domain of R is set of first elements of all ordered pairs in R and is denoted by "Dom R". Range of R is set of second elements of all ordered pairs in R, and is denoted by "Range R".

- If A and B are two non-empty sets, then a binary relation f is said to be a function from A to B, if
  - Dom f = A
  - There should be no repetition in the first elements of all ordered pairs contained in f. Symbolically $f: A\longrightarrow B$.
- Let $f: A\longrightarrow B$ be a function, then set A is called **domain** of f, set B is called **co-domain** of f and the set of second elements of all ordered pairs contained in f is called **range** of f. The range is subset of co-domain i.e. Range $f\subseteq B$.
- Let f be a function from A to B, then f is **into function**, if Range $f\neq B$.
- Let f be a function from A to B, then f is **one-one function**, if for each $x\in A$ there exist unique $y\in B$, i.e. there is no repetition in the second element of all ordered pairs contained in f.
- Let f be a function from A to B, then f is **injective** (into and non- one) function if
  (i). Range $f\neq B$
  (ii). There is no repetition in the second elements of all ordered pairs contained in f.
- Let f be a function from A to B, then f is **onto function** if Range f = B.
- Let f be function from A to B, then f is **one-one and onto (bijective) function** if it is both one-one and onto.
- If A and B are two non-empty sets then **one-one correspondence** between A and B is a rule for which each element of set A is paired with one and only one element of B and each element of B is paired with one and only one element of A, and none of the members of any set remains unpaired. It is also known as **one-to-one function**. In one-one correspondence both sets A and B have same number of elements.



Unit 6

BASIC STATISTICS

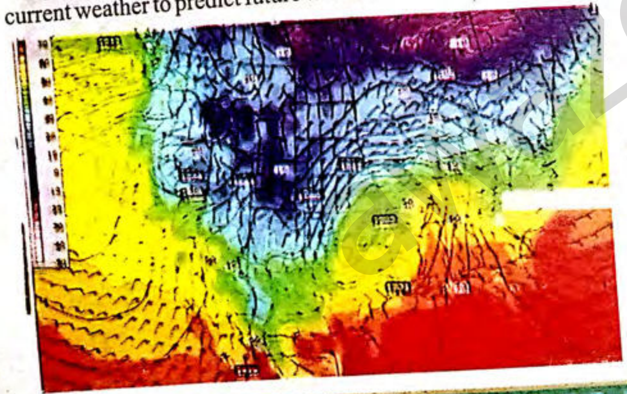
In this unit the students will be able to

- Construct a grouped frequency table.
- Construct histograms with equal and unequal class intervals;
- Construct a frequency polygon;
- Construct a cumulative frequency table.
- Draw a cumulative frequency polygon;
- Calculate (for ungrouped and grouped data)
  - arithmetic mean by definition and using deviation from assumed mean,
  - median, mode, geometric mean, harmonic mean.
- Recognize the properties of arithmetic mean;
- Calculate weighted mean and moving averages.
- Estimate median, quartiles and mode graphically
- Measure range, variance and standard deviation.

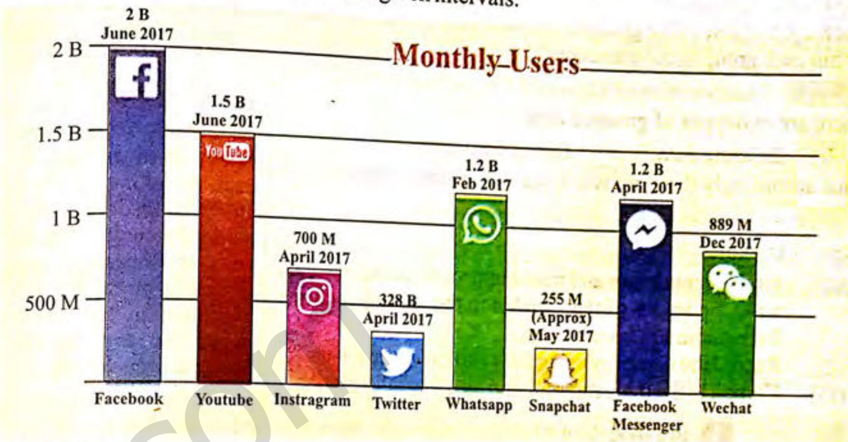
Why it's important

Weather Forecasts

Do you watch the weather forecast sometime during the day? How do you use that information? Have you ever heard the forecaster talk about weather models? These computer models are built using statistics that compare prior weather conditions with current weather to predict future weather.



Statistics is the science of collecting, arranging, analyzing, and interpreting data to draw conclusions or answer questions. Data are pieces of information, which are often numerical. Large amounts of data can be organized in a frequency table. A frequency table shows the number of pieces of data that fall within given intervals.



Activity

Scooters Riding is a fun to get exercise. The table shows prices for 35 types of nonmotorized scooters found at an online store.

(i). What is the cost of the least expensive scooter? the most expensive scooter?

(ii). How many scooters cost \$51 to \$75?

(iii). How could you reorganize the prices so that they are easier to find and read?

Prices of Scooters(\$.)

60	25	29	25	40
35	50	80	30	70
80	90	80	55	70
40	100	52	45	60
99	60	100	89	11
99	100	49	35	90
92	20	49	80	50

The scale allows you to record all of the data. It includes the least value, 11, and the greatest value, 100. The scale is 1 to 100.

Price(\$)	Tally Marks	Frequency
1 — 25	||||	4
26 — 50	~~||||~~ ~~||||~~ |	11
51 — 75	~~||||~~ ||	7
76 — 100	~~||||~~ ~~||||~~ |||	13

The interval separates the scale into equal parts. The interval is 25.

Tally marks are counters used to record items in a group.

From the frequency table, you can see that just over half the scooters cost between \$51 and \$100.

6.1 Frequency Distribution

The grouped frequency table is a statistical method to organize and simplify a large set of data into smaller "groups."
The main purpose of the grouped frequency table is to find out how often each value occurred within each group of the entire data.

6.1.1 Construction of Grouped Frequency Table

There are two types of grouped data.
(i) Discrete data (ii) Continuous data
Thus accordingly there are two types of frequency table.

For your information
The study of statistics began when an English man, John Graunt (1620-1674) collected and studied the death records in various cities of Britain and was fascinated by the patterns he found in the whole population even though people died randomly.

(a) Construction of Discrete Frequency Table

Steps of construction:

(i). Find the minimum and maximum value in the data and write the values of the variable in the variable column from minimum to maximum.
(ii). Record the values by using tally marks (vertical bars '|').
(iii). Complete the frequency column by using tally marks.

Example 1 In a shoe store 40 customers bought shoes with following shoe size.
6, 6, 7, 6, 8, 7, 7, 8, 6, 10, 6, 8, 8, 10 ,7, 9, 7, 10, 6 ,10 ,10 ,9, 7 ,9, 6, 10, 10, 7, 11, 8, 8, 7, 6, 6 ,8 ,9, 7, 8, 7 ,9. Construct a frequency table of the data.

Solution Let x = shoe size.

x	Tally Marks	Frequencies (f)
6	卌 IIII	9
7	卌 卌	10
8	卌 III	8
9	IIII	4
10	卌 III	8
11	I	1

Construction of Continuous Frequency Table

Steps of construction:

(i). Find Range: Deduct lowest value from the highest value.
(ii). Determine the number of groups (k). Most of the data has between 5 to 15 groups. It is your decision to choose the number of groups for your data. Number of classes must be balanced that is not too large or not too small.
(iii). Determine the width (number of values per group) of group interval by dividing Range by k.
(iv). Create three columns titled "Groups." "Tally Marks." And "Frequency."
(v). Insert the data in the table.
(vi). Determine the frequencies for all the groupsby tallying the data.

Related concepts

Class Limits

Every class has two limits, the **upper-class limit** and the **lower class limit**. For each, class the two limits may be fixed such that the mid-point of each class falls on an integer rather than a fraction.

Note Choose the first class interval such that the smallest observation is included.

The mid-point of each class is calculated by the formula.

$$\text{Mid-point} = \frac{\text{Lower limit} + \text{Upper limit}}{2}$$

For example if in a data the lowest value is 3 and the highest value is 20, then the better way is to make the classes as 3 — 7, 8 — 12, 13 —17, 18 — 22, because the mid-point of each class is an integer. The mid-points usually denoted by (x_i) are called the **class marks**.

Class Width

The class width is the difference between consecutive lower-class limits. It is found by dividing the range by the number of groups formed.

Class Boundaries

For computing classboundaries, the upper limit of the preceding class is subtracted from the lower limit of the following class and the difference is then divided by '2'. The quantity which is obtained is then subtracted from the lower limit and added to the upper limit of each class.

For example, consider the following table.

Classes-Limits	Class-Boundaries
1 — 4	0.5 — 4.5
5 — 8	4.5 — 8.5
9 — 12	8.5 — 12.5
13 — 16	12.5 — 16.5
17 — 20	16.5 — 20.5

Formula for calculating class-boundaries (CB)

$$\frac{5-4}{2} = \frac{1}{2} = 0.5$$

So 0.5 is subtracted from the lower limit and added to the upper limit of each class.

Example 2 The heights of 30 students of 10th class in cm are as follows. Construct group frequency table.

Heights in cm
162, 165, 170, 170, 162, 159, 162, 163, 175, 166, 171, 174, 155, 160, 173,
140, 145, 140, 146, 150, 172, 158, 155, 163, 165, 171, 153, 158, 149, 153

Solution

The width of each group is 5 because $\frac{175-140}{7} = 5$. The width of class boundary is 5.

Groups	Class boundaries	Heights in cm	Frequencies (f)
139 – 144	138.5 – 144.5	140, 140	2
145 – 150	144.5 – 150.5	146, 150, 149, 145	4
151 – 156	150.5 – 156.5	155, 155, 153, 153,	4
157 – 162	156.5 – 162.5	162, 162, 159, 162, 160, 158,158	7
163 – 168	162.5 – 168.5	165, 163,165, 163, 166	5
169 – 174	168.5 – 174.5	170, 170, 171, 174,173, 172, 171	7
175 – 180	174.5 – 180.5	175	1

Example 3 Construct a frequency table of the weights (kg) of 30 students from the following data by using 5 as a class-interval. Find the class-boundaries and class-marks also.
25, 30, 40, 21, 24, 25, 36, 30, 45, 50, 22, 25, 36, 46, 35, 38, 40, 28, 34, 45, 42, 46, 38, 48, 28, 29, 31, 33, 30, 26.

Solution

Class-Limits	Tally Marks	Frequency	Class-Boundaries	Class Marks
21 – 25	𝍸 I	6	20.5 — 25.5	23
26 – 30	𝍸 II	7	25.5 — 30.5	28
31 – 35	IIII	4	30.5 — 35.5	33
36 – 40	𝍸 I	6	35.5 — 40.5	38
41 – 45	III	3	40.5 — 45.5	43
46 – 50	IIII	4	45.5 — 50.5	48

6.1.2 Histogram

A histogram is a vertical bar graph with no space between the bars. The area of each bar is proportional to the frequency it represents.
A histogram has advantages over the other methods that it can be used to represent data with both equal and unequal class intervals.
Whenever we construct a histogram representing a set of figures which have been rounded off from the original measurements, we have to make the class intervals continuous by using class boundaries. This is to avoid having gaps in between the bars.

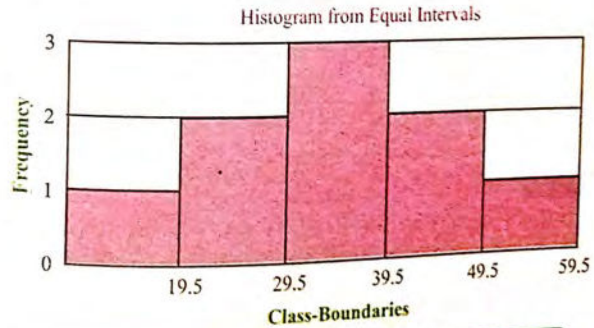
Example 4 Construct a histogram from the following frequency distribution.

Class-limits	20 — 29	30 — 39	40 — 49	50 — 59	60 — 69
Frequency	1	2	3	2	1

Solution

To draw a histogram class-boundaries are marked along x-axis and frequencies of each class are marked along Y-axis as shown in the figure.

Class-Limits	Frequency	Class-Boundaries
20 — 29	1	19.5 — 29.5
30 — 39	2	29.5 — 39.5
40 — 49	3	39.5 — 49.5
50 — 59	2	49.5 — 59.5
60 — 69	1	59.5 — 69.9



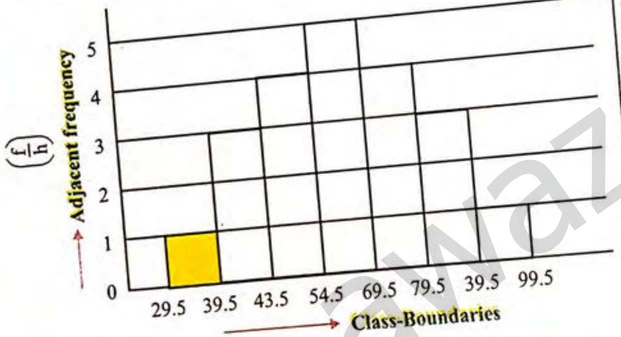
Example 5 Draw a histogram for the following data.

Class-Limits	30 — 39	40 — 43	44 — 54	55 — 69	70 — 79	80 — 89	90 — 99
Frequency	10	12	44	75	40	30	10

Solution Histogram with unequal class intervals.

The class intervals are not equal. In constructing the histogram, we must ensure that the areas of the rectangle are proportional to the class frequencies, as the frequency in a histogram is represented by the area of each rectangle.

Class-Limits	Class-Boundaries	Class-Intervals (h)	Frequency (f)	Adjusted Frequency $\frac{f}{h}$
30 — 39	29.5 — 39.5	10	10	1
40 — 43	39.5 — 43.5	4	12	3
44 — 54	43.5 — 54.5	11	44	4
55 — 69	54.5 — 69.5	15	75	5
70 — 79	69.5 — 79.5	10	40	4
80 — 89	79.5 — 89.5	10	30	3
90 — 99	89.5 — 99.5	10	10	1



Note Since the areas of the rectangles in a histogram must be proportional to the class frequencies, height of rectangle x class width = class frequency.
OR
height of rectangle = class frequency/class width Thus, the height of a rectangle may be found by dividing the class frequency by the class width.

For your information
In a histogram, the frequency is represented by (height of bar) x (number of standard intervals)

6.1.3 Frequency polygon

A frequency polygon is drawn by joining all the midpoints at the top of each rectangle. The midpoints at both ends are joined to the horizontal axis to accommodate the end points of the polygon. This will make the graph nearer with the end points falling off to zero on the horizontal axis.

We can draw the frequency polygon of a distribution without first drawing the histogram. We plot each class frequency against the mid value of the class to obtain points which we join by straight lines to form the frequency polygon.

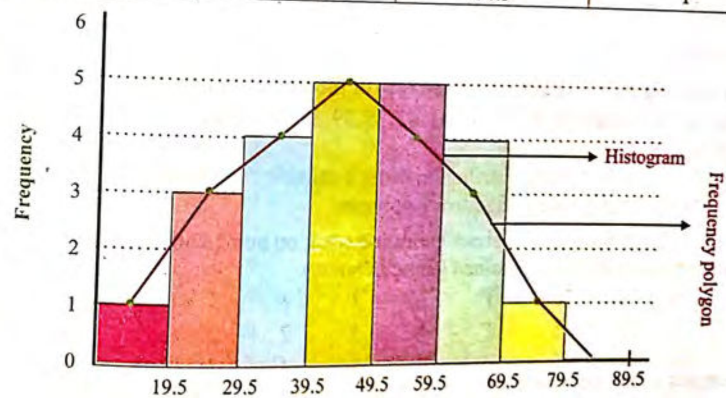
Frequency polygon are specially useful to compare two sets of data e.g. prices of commodities (vegetables, pulses, gold etc) over a given period of time. Now a days it is commonly used in cricket to show the progress of a batting team or comparision of scores of chasing a team.

Example 6 Construct a frequency polygon for the following frequency distribution.

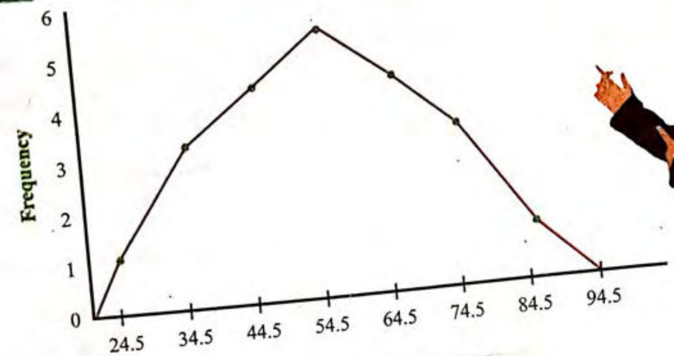
Class-Limits	20 — 29	30 — 39	40 — 49	50 — 59	60 — 69	70 — 79	80 — 89
Frequency	1	3	4	5	4	2	1

Solution

Classes	Class-Boundaries	Class Marks (Mid-Points)	Frequency
20 — 29	19.5 — 29.5	24.5	1
30 — 39	29.5 — 39.5	34.5	3
40 — 49	39.5 — 49.5	44.5	4
50 — 59	49.5 — 59.5	54.5	5
60 — 69	59.5 — 69.5	64.5	4
70 — 79	69.5 —79.5	74.5	2
80 — 89	79.5 — 89.5	84.5	1



Frequency polygon



Exercise 6.1

1. Construct a frequency distribution of the marks of 30 students during a quiz with 100 points by taking 10 as the class-interval. Indicate the class-boundaries and class-marks. 40, 60, 65, 70, 35, 50, 56, 74, 72, 49, 85, 76, 82, 83, 68, 90, 67, 66, 58, 46, 74, 88, 76, 69, 57, 63, 66, 47, 82, 90.
2. Following are the mistakes made by a group of students of class 10^{th} in a test of essay writing. Using an appropriate size of class interval, make a frequency distribution and also indicate the number of class intervals. 4, 7, 12, 9, 21, 16, 3, 19, 17, 24, 14, 15, 8, 13, 11, 16, 15, 6, 5, 8, 11, 20, 18, 22, 6.
3. Draw a Histogram for the following data.

Class-Limit	20 — 24	25 — 29	30 — 34	35 — 39	40 — 44	45 — 49	50 — 54
Frequency	1	3	4	5	4	2	1

4. The following data give the weights in (kg) of the students in the 10th class. 25, 30, 32, 29, 24, 40, 36, 37, 28, 27, 41, 42, 35, 39, 31, 32, 34, 42, 40, 43, 36, 26, 22, 23, 42, 39, 35, 41, 39, 29.
   (i). Prepare a frequency distribution using a suitable class interval.
   (ii). Draw histogram and frequency polygon.
5. A teacher asked the students about their time spent on homework completion time. The following set of data was obtained (Time in hours).

4	4	6	3	1	2	2	3	1	4
1	2	5	3	4	5	2	2	3	1
3	1	2	2	3	1	4	2	6	2

Construct a frequency table and draw a histogram showing the results.

Activity

"Getting Ready for School in the morning"

Gather your Data:

Ask your school-mates the question, how much time (minutes) do you take in getting ready for your school? Write the data in a notebook like this Ali Haidar 20 minutes. Collect responses from approximately 20 students in the form of minutes only. Like 30 min, 10 min, 12 min, 15 min. and so on. Make a grouped frequency table for your data.

6.2 Cumulative Frequency Distribution

A cumulative frequency table provides information about the sum of a variable against the other value. For example, how many workers of a factory have salary less than Rs.7000, the number of students scoring below 50% marks, number of maiz cob measuring more than 6 inches etc.

When the same data is presented on a graph paper the freehand curve formed is called an Ogive.

6.2.1 Construct a cumulative frequency table

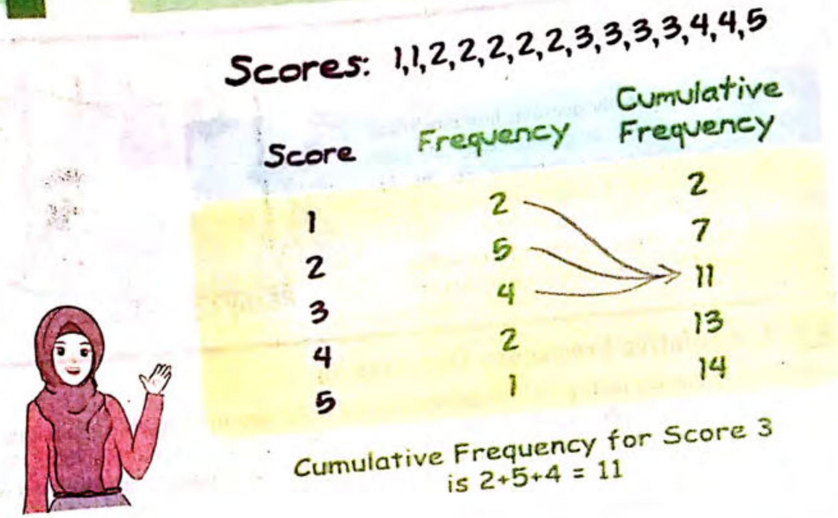
The frequency of the last class is the total frequency of a set of data.

Example 7 Find the cumulative frequency of the following data.

x	3	4	5	6	7	8	9	10	11	12
f	1	2	3	4	5	6	7	4	3	8

Solution

x	f	Method of finding (c.f)	c.f
3	1	1	1
4	2	$1+2=3$	3
5	3	$3+3=6$	6
6	4	$6+4=10$	10
7	5	$10+5=15$	15
8	6	$15+6=21$	21
9	7	$21+7=28$	28
10	4	$28+4=32$	32
11	3	$32+3=35$	35
12	8	$35+8=43$	43



Example 8 The consumption of petrol of 1000cc cars of a particular brand was surveyed and the following data was obtained. Construct a cumulative frequency distribution.

Distance in Km/ Litre (Milage)	10 — 12	13 — 15	16 — 18	19 — 21	22 — 24
Frquencey	16	20	36	21	7

Solution The cumulative frequency disturbution is consturcted below.

Milage (Km/Litre)	Upper Class Boundaries	Frequency	Cumulative Frequency
10 — 12	12.5	16	16
13 — 15	15.5	20	16 + 20 = 36
16 — 18	18.5	36	36 + 36 = 72
19 — 21	21.5	21	72 + 21 = 93
22 — 24	24.5	7	93 + 7 = 100

6.2.2 Cumulative Frequncy Polygon

A polygon in which cumulative frequencies are used for plotting the curve is called a cumulative frequency polygon. The curve is also called a Ogive. The construction of an Ogive is described in the following example.

Example 9 In the following data marks of students are given during first pre-board exam in the subject of maths.

25, 30, 27, 28, 35, 36, 40, 41, 42, 45, 50, 44, 29, 26, 36, 31, 43, 46, 52, 53, 51, 42, 37, 27, 3. 46, 44, 34, 51, 54

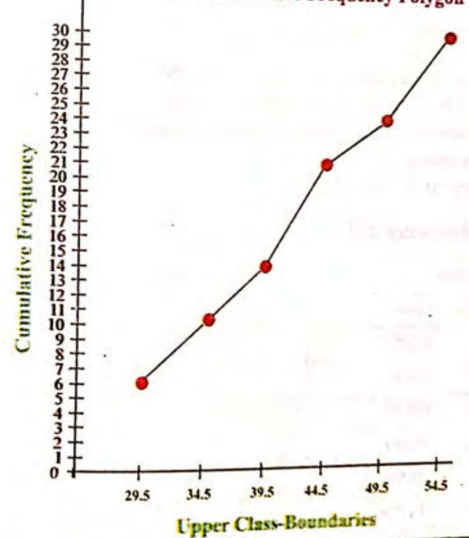
By taking a suitable class-interval, prepare a frequency distribution, find less than cumulativ frequency and draw a cumulative frequency polygon (Ogive).

Solution

To construct the cumulative frequency distribution, we take the class-interval as 5.

Class-limits	Frequency	Class-boundaries	Cumulative Frequency
25 — 29	6	24.5 — 29.5	6
30 — 34	4	29.5 — 34.5	10
35 — 39	4	34.5 — 39.5	14
40 — 44	7	39.5 — 44.5	21
45 — 49	3	44.5 — 49.5	24
50 — 54	6	49.5 — 54.5	30

Cumulative Frequency Polygon



Exercise 6.2

1. The following data give the wages (in Rs) of workers.
60, 75, 80, 85, 90, 84, 70, 73, 76, 84, 95, 100, 150, 66, 58, 90, 98, 120, 77, 90.
By taking 10 as the class-interval, prepare.
(i). Cumulative frequency distribution (ii). Cumulative frequency polygon

2. Make cumulative frequency table for the following data

Age in Years	20–24	25–29	30–39	40–44	45–49	50–54	55–59
Number of persons (f)	1	2	26	22	20	15	14

3. In a city during the first week of August rainfall recorded is as fallows. Construct a cumulative frequency graph.

Day	Sun	Mon	Tue	Wed	Thu	Fri	Sat
Rainfall in ml	70	40	30	35	50	55	80

4. Draw less than and more than cumulative frequency polygon for the data given.

Marks	Number of Students
40 — 49	1
50 — 59	2
60 — 69	3
70 — 79	4
80 — 89	5
90 — 99	6

5. Determine from the data of Q4, the following.
(i). Number of students who obtained more than 50 marks.
(ii). Number of students who obtained less than 70 marks.
(iii). Number of students who secured marks between 50 and 70.
(iv). Class interval of all classes.
(v). Lower-class boundary of 5^{th} class.

6. Construct an ogive for the following table

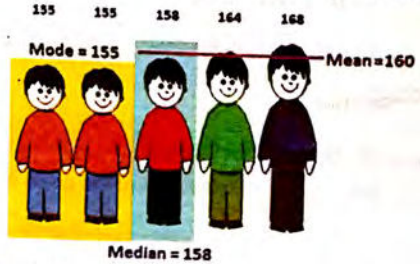
Salary	Workers
4000 — 5000	3
5001 — 6000	5
6001 — 7000	12
7001 — 8000	9
8001 — 9000	5
9001 — 10000	4
10001 — 11000	2

6.3 Measures of Central Tendency

Central Tendency of a data is the value that represents the whole data or the stage at which the largest number of item tends to concentrate and so it is called central tendency. Central Tendency or Averages are also sometimes called measures of location, because they locate the centre of a distribution.

Types of Central Tendency Averages

(i). Arithmetic Mean (A.M)
(ii). Median
(iii). Mode
(iv). Geometric Mean (G.M)
(v). Harmonic Mean (H.M)
(vi). Quartiles



6.3.1 Calculate for ungrouped and grouped data

(i) Arithmetic Mean

(a) Arithmetic Mean (for ungrouped data)

Arithmetic Mean or simply Mean is calculated by adding all values of the data divided by the number of items (values). If $x_1, x_2, x_3, \ldots, x_n$ are n values; then the Arithmetic Mean denoted by $(\bar{x})$ is given as

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \ldots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$\bar{x}$ = Arithmetic Mean, $\sum$ = sigma (Notation used for the summation)

$x_i = x_1, x_2, x_3, \ldots x_n (i = 1, 2, 3 \ldots n)$

n = Total number of items in the data.

By short-cut method

Formula for short-cut method is,

Mean = $\dfrac{\text{Sum of data}}{\text{Number of data}}$

$$\bar{x} = a + \frac{\sum D_x}{n}$$

Where

$\bar{x}$ = Arithmetic Mean

a = Provisional Mean (P.M)

$D_x = (x-a)$ (Deviation from Provisional Mean)

$\sum D_x$ = Sum of Deviations from P.M

n = Total number of values in the data.

Example 10

Find the A.M of the values 2, 3, 4, 5, 6, 7, 8, 9, 10.

(i) By direct method

(ii) By short cut method

Solution

(i) By direct method

Let $\bar{x}$ be the A.M of the given items. Then by using formula

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \ldots + x_n}{n}$$

$$= \frac{2+3+4+5+6+7+8+9+10}{9} = \frac{54}{9} = 6$$

$\therefore \bar{x} = 6$

(ii) By short cut method

x	D_x $(x-a)$, $a=6$
2	$2-6=-4$
3	$3-6=-3$
4	$4-6=-2$
5	$5-6=-1$
6	$6-6=0$
7	$7-6=1$
8	$8-6=2$
9	$9-6=3$
10	$10-6=4$
	$\sum D_x = 0$

By using formula,

$$\bar{x} = a + \frac{\sum D_x}{n} = 6 + \frac{0}{9} = 6 + 0$$

$\therefore \bar{x} = 6$

(b) Arithmetic Mean (for grouped data)

If $x_1, x_2, x_3, \ldots, x_n$ are n values with $f_1, f_2, f_3, \ldots f_n$ as frequencies of the respective values, hence the Arithmetic Mean $\bar{x}$ is given as

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots + f_nx_n}{f_1 + f_2 + f_2 + \ldots + f_n} = \frac{\sum fx}{\sum f}$$

Example 11

In a coaching class of 13 students, a test was conducted, and the following marks were obtained by the students.

10, 12, 12, 14, 9, 18, 9, 13, 16, 9, 17, 16, 14

Make frequency table and find Arithmetic mean.

Solution

Score	9	10	12	13	14	16	17	18
Frequency (f)	3	1	2	1	2	2	1	1

(x) Score	(f) Frequency	fx
9	3	27
10	1	10
12	2	24
13	1	13
14	2	28
16	2	32
17	1	17
18	1	18
	$\sum f = 13$	$\sum fx = 169$

$$\text{Arithmetic Mean} = \frac{\sum fx}{\sum f} = \frac{169}{13} = 13$$

Example 12

The prices of 2 KW generators are given below along with their frequencies. Find the mean price.

(i) By direct method

(ii) By short-cut method

Price (In 100s)	90–94	95–99	100–104	105–109	110–114	115–119	120–124
Frequency (f)	4	11	15	24	18	9	3

Solution

(i) By direct method

Class Interval	(x) Mid Value	(f) Frequency	fx
90 — 94	92	4	368
95 — 99	97	11	1067
100 — 104	102	15	1530
105 — 109	107	24	2568
110 — 114	112	19	2016
115 — 119	117	9	1053
120 — 124	122	3	366
		$\sum f = 85$	$\sum fx = 8968$

The mean price of the generators $= \bar{x}$

$$= \frac{\sum fx}{\sum f}$$

$$= \frac{8968}{85}$$

$= 105.5$ hundred rupees

$= 10550/-$

(ii) Short–cut method

Let the assumed mean a = 107.

Class Interval	(x) Mid -point	(f) Frequency	$D_x = x - a$	fD_x
90 — 94	92	4	$92 - 108 = -15$	-60
95 — 99	97	11	$97 - 107 = -10$	-110
100 — 104	102	15	$102 - 107 = -5$	-75
105 — 109	107	24	$107 - 107 = 0$	0
110 — 114	112	19	$112 - 107 = 5$	95
115 — 119	117	9	$117 - 107 = 10$	90
120 — 124	122	3	$122 - 107 = 15$	45
		$\sum f = 85$		$\sum fD_x = -15$

$$\bar{x} = a + \frac{\sum fD_x}{\sum f}$$

$$= 107 + \left(-\frac{15}{85}\right)$$

$$= \frac{9095 - 15}{85}$$

$$= \frac{9080}{85}$$

$= 106.83$

$= 10683/-$

(ii) Median

It is a value which is in the center of observations when all the observations are arranged in ascending or descending order i.e. Median divide the data in two equal parts. Median is calculated for the following data:

(a) ungrouped data (b) discrete data (c) continuous data

(a) Median for ungrouped data

If in a data there are n values, then after arranging the values in ascending or descending order Median is calculated by the following rules:

$$\text{Median} = \text{Size of } \left(\frac{n+1}{2}\right)\text{th item} \quad (\text{if } n \text{ is odd})$$

or

$$\text{Median} = \frac{1}{2}\left[\text{Size of } \frac{n}{2}\text{th} + \text{Size of } \left(\frac{n+2}{2}\right)\text{th item}\right] \quad (\text{if } n \text{ is even})$$

Example 13 Find the median for the following values.

2, 4, 5, 6, 3

Solution Writing the given data in increasing order

2, 3, 4, 5, 6

S.No	*x (value)*
1	2
2	3
3	4
4	5
5	6

What's the error?

0, 5, 7, 9, 10, 3, 5, 8

Median = 9.5

Note

The median for an odd number of data is the middle value when the data are arranged in ascending /descending order. The median for an even number of data is arranged is the mean of two middle values when the data are arranged in ascending/descending order.

By using formula

$$\text{Median} = \text{size of } \left(\frac{n+1}{2}\right)\text{th item}$$

$$= \text{size of } \left(\frac{5+1}{2}\right)\text{th item}$$

$$= \text{size of } 3^{rd} \text{ item}$$

$$= 4$$

So Median = 4

Example 14 The following is the daily pocket money in rupees for the children of a family 10, 20, 15, 30. Calculate the median for the data.

Solution

S.No	*x*
1	10
2	15
3	20
4	30

Since number of items is even, so

$$\text{Median} = \frac{1}{2}\left[\text{Size of } \frac{n}{2}\text{th} + \text{Size of } \left(\frac{n+2}{2}\right)\text{th}\right]\text{item}$$

$$= \frac{1}{2}\left[\text{Size of } \frac{4}{2}\text{th} + \text{Size of } \left(\frac{6}{2}\right)\text{th}\right]\text{item}$$

$$= \frac{1}{2}[\text{Size of 2nd} + \text{Size of 3rd}]\text{ item} = \frac{1}{2}(15+20) = \frac{35}{2} = 17.5$$

(b) Median for discrete data

In a frequency distribution if $\sum f$ is odd then the value of x opposite to $\left(\frac{\sum f+1}{2}\right)$th item in the cumulative frequency column is its median. But if $\sum f$ is even, then the value of x opposite to $\frac{\sum f}{2}$ th item in the cumulative frequency column is its median. This can be illustrated with the help of the following examples.

Example 15 The following are the marks obtained by 35 students in a test.

x	10	12	15	20	25	30
f	1	10	5	13	2	4

Find median of the data.

Solution

x	f	$c.f$
10	1	1
12	10	11
15	5	16
20	13	29
25	2	31
30	4	35

Since $n = \sum f = 35$

So $\text{Median} = \text{Size of } \left(\frac{n+1}{2}\right)\text{th item}$

$$= \text{Size of } \left(\frac{35+1}{2}\right)\text{th item}$$

$$= \text{Size of 18th item}$$

$$= 20$$

Example 16 Find the median marks from the following distribution:

(Marks) x	10	20	22	25
Number of students	0	2	4	6

Solution

Marks x	*Frequency*	*c.f*
10	0	0
20	2	2
22	4	6
25	6	12
	$\sum f = 12$	

$$\text{Median} = \text{Size of } \left(\frac{n}{2}\right) \text{th item}$$

$$= \text{Size of } \left(\frac{12}{2}\right) \text{th item}$$

$$= \text{Size of 6th item}$$

$$= 22$$

So, Median = 22

(c) Median from continuous data

For computing median in continuous data, it is important to make class boundaries and then median is calculated by the following formula:

$$\text{Median} = l + \frac{h}{f}\left(\frac{n}{2} - c\right) \text{ where,}$$

l = lower limit of the median class

h = width (class-interval of the median class)

f = frequency of the median class

$\frac{n}{2} = \frac{\sum f}{2}$

c = cumulative frequency of the class preceding the median class.

Example 17 Find the median of the following distribution.

Daily wages (in Rs.)	60—69	70—79	80—89	90—99	100—109
Labour	4	6	8	10	5

Solution

Daily wages	f	(C.B) Class-Boundaries	(C.F) Cumulative Frequency
60 — 69	4	59.5 — 69.5	4
70 — 79	6	69.5 — 79.5	10
80 — 89	8	79.5 — 89.5	18
90 — 99	10	89.5 — 99.5	28
100 — 109	5	99.5 — 109.5	33
	$\sum f = 33$		

$$\text{Median} = \frac{n}{2}\text{th item} = \frac{33}{2}\text{th item}$$

$$= 16.5\text{th item}$$

Median lies in the group 79.5 — 89.5

$$\text{Median} = l + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

$$= 79.5 + \frac{10}{8}(16.5 - 10)$$

$$= 79.5 + \frac{10}{8}(6.5)$$

$$= 79.5 + 8.125$$

$$= 87.625$$

(iii) Mode

The value that appears more times in a data, is called **mode** for the given data or the most frequent value in a data is called **mode**.

Example 18 From the following sizes of kids trousers, find the model size.

25, 30, 31, 25, 35, 25

Solution In the above data, 25 is the most frequent value, so the model size is 25.

Example 19 The following data shows the weights of the students. Find the model weight.

Weights (in KG)	40	42	50	51	55
Number of students	10	8	3	2	1

Solution Mode = 40, because 40 is the most frequent value in the data.

Example 20 Calculate the mode from the following frequency distribution.

Marks (C.B)	0 — 4	4 — 8	8 — 12	12 — 16	16 — 20
No. of students	3	5	4	6	2

Solution

Marks (C.B)	*No. of students f*
0 — 4	3
4 — 8	5
8 — 12	4
12 — 16	6
16 — 20	2

→ Modal group (12 — 16)

The mode lies in the group 12 — 16.

So by using the formula

$$\text{Mode} = l + \left(\frac{f_m - f_0}{2f_m - f_0 - f_1}\right) \times h$$

Where
l = lower limit of the modal group
f_m = Frequency of the modal group
f_0 = Frequency of the group preceding the modal group
f_1 = Frequency of the group following the modal group
h = class-interval

So, $\text{Mode} = 12 + \left(\frac{6-4}{2(6)-4-2}\right) \times 4$

$= 12 + \left(\frac{2}{12-6}\right) \times 4$

$= 12 + \frac{8}{6}$

$= 12 + 1.33 = 13.33$

(iv) Geometric Mean (G.M)

Geometric Mean is the nth positive root of the product of n values.

(a) Geometric Mean from ungrouped data

If $x_1, x_2, x_3, \ldots x_n$ are n values in a data, then the Geometric Mean is given as

$$G.M = (x_1 \times x_2 \times x_3 \times \ldots \times x_n)^{\frac{1}{n}}$$

or $$G.M = \text{Anti-log}\left(\frac{\sum \log x}{n}\right)$$

Example 21 Find the Geometric Mean (G.M) of the marks obtained by the 9th class students.

60, 65, 70, 80, 85, 90, 75.

Solution

x	Log x
60	log 60 = 1.7781
65	log 65 = 1.8129
70	log 70 = 1.8450
75	log 75 = 1.8750
80	log 80 = 1.9030
85	log 85 = 1.9294
90	log 90 = 1.9542
	$\sum \log x = 13.0976$

$G.M = \text{Anti-log} \frac{\sum \log x}{n}$

$= \text{Anti-log} \frac{13.0976}{7}$

$= \text{Anti-log } 1.8710$

$= 74.31$

(b) Geometric Mean of Grouped data

Let $x_1, x_2, x_3, \ldots, x_n$ be the mid-points in a frequency distribution and $f_1, f_2, f_3, \ldots f_n$ are their respective frequencies. Then the G.M is calculated by the following formula:

$$G.M = \text{Anti-log}\left(\frac{\sum f \log x}{\sum f}\right)$$

Example 22

Calculate the Geometric Mean (G.M) of the following data.

Marks	0 — 20	20 — 40	40 — 60	60 — 80
Number of students	3	4	10	11

Solution

Marks	x	f	$\log x$	$f \log x$
0 — 20	10	3	1.0000	3.0000
20 — 40	30	4	1.4771	5.9084
40 — 60	50	10	1.6989	16.989
60 — 80	70	11	1.8450	20.295
		$\sum f = 28$		$\sum f \log x = 46.1924$

By formula,

$$\text{G.M} = \text{Anti-log}\,\frac{\sum f \log x}{\sum f}$$

$$= \text{Anti-log}\,\frac{46.1924}{28}$$

$$= \text{Anti-log } 1.6497$$

$$= 44.64$$

Did You Know?

When you multiply 1089 by 9, the answer you get is 9801, which is the exact reverse of the number 1089

(v) Harmonic Mean (H.M)

It is the reciprocal of the Arithmetic Mean of the reciprocal values.

(a) Harmonic Mean from ungrouped data

Let there be 'n' values i.e., $x_1, x_2, x_3, \ldots, x_n$ in a data, then the Harmonic Mean (H.M) is calculated by the following formula:

$$\text{H.M} = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+,\ldots+\frac{1}{x_n}} = \frac{n}{\sum\left(\frac{1}{x}\right)}$$

Example 23

Find the Harmonic Mean of the following values. 5, 6, 8, 9 and 10.

Solution

x	$\log \frac{1}{x}$
5	$\frac{1}{5} = 0.2$
6	$\frac{1}{6} = 0.16$
8	$\frac{1}{8} = 0.125$
9	$\frac{1}{9} = 0.11$
10	$\frac{1}{10} = 0.1$
	$\sum \frac{1}{x} = 0.695$

By using formula:

$$\text{H.M} = \frac{n}{\sum\left(\frac{1}{x}\right)} = \frac{5}{0.695} = 7.194$$

(b) Harmonic Mean of grouped data

The Harmonic Mean of grouped data is computed by the following formula:

$$\text{H.M} = \frac{\sum f}{\sum\left(\frac{f}{x}\right)}$$

Example 24

Find the Harmonic Mean from the following data.

Classes	0 — 6	6 — 12	12 — 18	18 — 24	24 — 30
Frequency	1	2	5	4	6

Solution

Classes	Frequency (f)	(Mid-point)	$\frac{f}{x}$
0 — 6	1	3	$\frac{1}{3} = 0.33$
6 — 12	2	9	$\frac{2}{9} = 0.22$
12 — 18	5	15	$\frac{5}{15} = 0.33$
18 — 24	4	21	$\frac{4}{21} = 0.19$
24 — 30	6	27	$\frac{6}{27} = 0.22$
	$\sum f = 18$		$\sum\left(\frac{f}{x}\right) = 1.29$

$$\text{H.M} = \frac{\sum f}{\sum\left(\frac{f}{x}\right)} = \frac{18}{1.29} = 13.95$$

6.3.2 Properties of Arithmetic Mean

- The sum of deviations of values measured from their A.M is always equal to zero.
  i.e. $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$
- The sum of squared deviations of values measured from their A.M is least (minimum).
  $\sum_{i=1}^{n}(x_i - \bar{x})^2 < \sum_{i=1}^{n}(x_i - a)^2$, where $\bar{x}$ = A.M and a = any constant
- If $y_i = ax_i + b$ where a and b are some non-zero constants, than $\bar{y} = a\bar{x} + b$.
- If K-subgroups of a data having their respective means $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_k$, with respective frequencies $n_1, n_2, n_3, \ldots n_k$ then the mean of the combined data is given as:
  $$\bar{x}_c = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \ldots + n_k\bar{x}_k}{n_1 + n_2 + n_3 + \ldots + n_k} = \frac{\sum_{i=1}^{n} n_i\bar{x}_i}{\sum_{i=1}^{n} n_i}$$

6.3.3(a) Weighted mean

- The numerical values which show the relative importance of different items are called weights and the average of different items having different weights is called weighted mean. Let $x_1, x_2, x_3, \ldots x_n$ are different values of items having weights $w_1, w_2, w_3 \ldots w_n$ then weighted mean:
  $$\bar{x}_w = \frac{x_1 w_1 + x_2 w_2 + \ldots + x_n w_n}{w_1 + w_2 + \ldots + w_n} = \frac{\sum x_i w_i}{\sum w_i}$$

Example 25 The marks obtained by a student in Maths, English, Urdu and Statistics were 70, 60, 80, 65 respectively. Find the average if weights of 2, 1, 3, 1 are assigned to the marks.

Solution

x	w	wx
70	2	140
60	1	60
80	3	240
65	1	65
	$\sum w = 7$	505

Now $\bar{x}_w = \frac{\sum wx}{\sum w} = \frac{505}{7} = 72.14$

6.3.3(b) Moving Averages

- It is succession of averages (Arithmetic means) derived from the successive segments of a series of values. It is continuously recomputed as new data becomes available. It progresses by dropping the earliest value and adding the latest value.

Example 26

During first week of May, daily temperatures were recorded as given in the table. Calculate 3-day moving average temperature.

Days	Temperature
Saturday	40
Sunday	37
Monday	36
Tuesday	38
Wednesday	37
Thursday	41
Friday	39

Solution

Days	Temperature	3 day moving average means
Saturday	40	...
Sunday	37	$\frac{40+37+36}{3}=37.67$
Monday	36	$\frac{37+36+38}{3}=37$
Tuesday	38	$\frac{36+38+37}{3}=37$
Wednesday	37	$\frac{38+37+41}{3}=38.67$
Thursday	41	$\frac{37+41+39}{3}=39$
Friday	39	...

6.3.4 Estimation of Median, Mode and Quartiles graphically

(i) Median

For median the class-boundaries are plotted on the horizontal axis and their cumulative frequencies on the vertical axis. Then points are located above the boundary points against their respective cumulative frequencies. All the points are then joined by straight line to get a curve called 'Ogive'.

Horizontal and vertical lines are drawn from the point where median is located, so the value on the horizontal axis at which the vertical lines intersect the Ogive, determines the median.

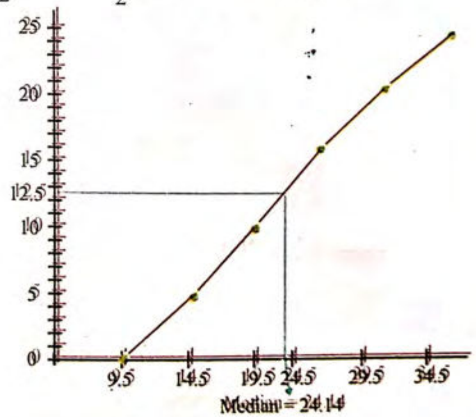
Example 27 Find median graphically from the following frequency distribution:

Classes	10 — 14	15 — 19	20 — 24	25 — 29	30 — 34	35 — 39
Frequency	1	5	7	2	6	4

Solution

Classes	f	Cumulative Frequency $c.f$	Class-Boundaries
10 — 14	1	1	9.5 — 14.5
15 — 19	5	6	14.5 — 19.5
20 — 24	7	13	19.5 — 24.5
25 — 29	2	15	24.5 — 29.5
30 — 34	6	21	29.5 — 34.5
35 — 39	4	25	34.5 — 39.5

Median $= \frac{n}{2}th$ item $= \frac{25}{2}th$ item $= 12.5th$ item



ERROR ANALYSIS Describe and correct the error made in finding the median of the data set.

X median
10, 11, 24, 45, 41, 15, 45, 24, 50

(ii) Mode

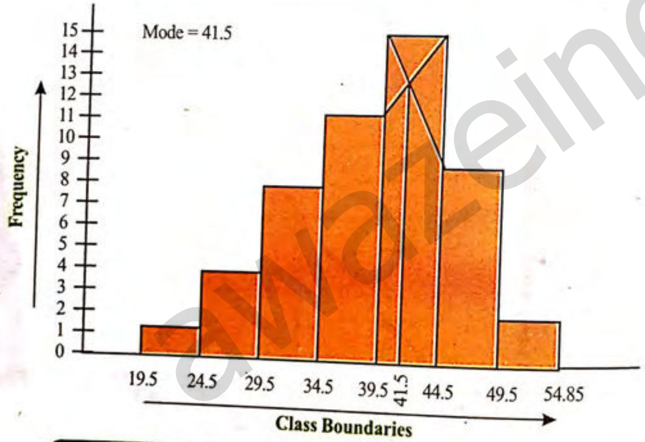
We need histogram to estimate mode graphically which is explained with the help of the following example.

Example 28 Find mode graphically from the following frequency distribution.

Classes	20 — 24	25 — 29	30 — 34	35 — 39	40 — 44	45 — 49	50—54
Frequency	1	4	8	11	15	9	2

Solution

Classes	f	Class-Boundaries
20 — 24	1	19.5 — 24.5
25 — 29	4	24.5 — 29.5
30 — 34	8	29.5 — 34.5
35 — 39	11	34.5 — 39.5
40 — 44	15	39.5 — 44.5
45 — 49	9	44.5 — 49.5
50 – 54	2	49.5 — 54.5



(iii) Quartiles

To find the values of Q_1 and Q_3 from graph the cumulative frequency polygon (ogive) is drawn.

Example 29 Find Q_1 and Q_3 from the following distribution.

Marks	0 — 10	10 — 20	20 — 30	30 — 40	40 — 50
Frequency	3	5	9	3	2

Solution

Marks	f	$c.f$
0 — 10	3	3
10 — 20	5	8
20 — 30	9	17
30 — 40	3	20
40 — 50	2	22

Location of $Q_1 = \frac{n}{4}$ th item

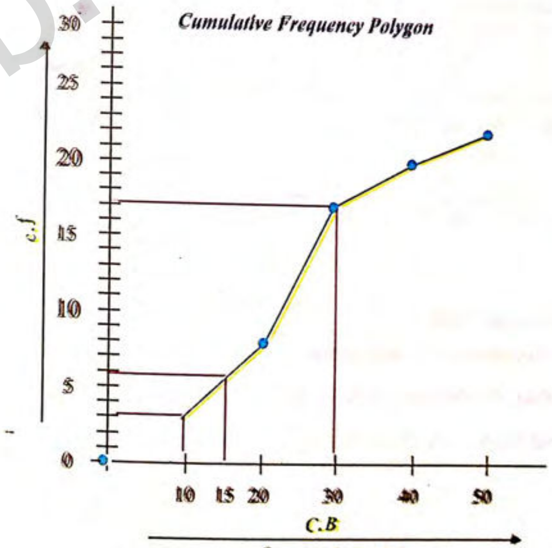
$= \frac{22}{4}$ th item

$= 5.5$th item

Location of $Q_3 = \frac{3n}{4}$ th item

$= 3\left(\frac{22}{4}\right)$th item

$= 16.5$th item



$Q_1 = 15,\ Q_3 = 29.4$

Exercise 6.3

1. The following are the weights (in kg) of students of the 10^{th} grade.
   45, 30, 25, 36, 42, 27, 31, 43, 49 and 50.
   Calculate mean of the weights.
2. Find the mean of the weights given in Question 1 by using short-cut method.
3. Using assumed mean find the mean of the following numbers.
   1242, 1248, 1252, 1244, 1249
4. Find the mean of the following frequency distribution.
   Marks obtained by students of 10^{th} class in mathematics.

Score (out of 75)	0 – 15	16 – 31	32 – 47	48 – 63	64 – 75
Frequency (f)	0	10	40	70	45

5. Find the median of the following data.
   (i). Heights of boys in inches.
   64, 65, 65, 66, 66, 67
   (ii). Salaries of 9 workers of a factory
   7000, 6600, 8000, 4500, 7500, 11000, 9000, 7500

6. Find the Arithmetic mean, Geometric mean, Median and Mode of the following data
   58 59 60 62 64 64 65 67 67 68
   70 71 71 71 73
7. A set of data contains the values as 148, 145, 160, 157, 156 and 160. Show that Mode > Median > Mean.

8. From the following distribution

Daily wages (in Rs)	112—116	117—121	122—126	127—131	132—136
No. of workers	3	20	11	4	5

   (i). Construct a frequency table.
   (ii). Find the class-boundaries for each group.
   (iii). Calculate Median, Mode, Harmonic Mean, and Geometric Mean for the table.
9. Find Median, Q_1, Q_3 and Mode from the following distribution graphically.

Classes	10 — 14	15 — 19	20 — 24	25 — 29	30 — 34
Frequency	1	3	7	12	2

6.4 Measures of Dispersion

Dispersion is the scatterdness of values from its central value (Average).
Types of measure of dispersion are:

(i) Range
(ii) Standard deviation (S.D)
(iii) Variance

(i) Range

The RANGE is the difference between the smallest observation and the largest observation.

Range = Largest value – Smallest value

Example 30 What is the range of the data 209, 260, 270, 311, 311?

Solution

The largest value = 311
The smallest value = 209 m
Range = largest value–smallest value
Range = 311 – 209 = 102.

So the range is 102.

Tidbit
The range tells you how different the observations are. A large range means that the observations are very different. A smaller range means that they are more similar.

Example 31 Following are the names and heights of mountains in Karakoram. Find the range of heights.

K – 2	8611m
Ghasherbrum I	8068m
Broad	8047m
Ghasherbrum II	8035m
Ghasherbrum III	7952m
Ghasherbrum IV	7925m
Rakaposhi	7788m

Solution

The maximum height = 8611 m
The minimum height = 7788 m
Range = maximum – minimum
= 8611 – 7788 = 823

So the range is 823 m.

Note
Range is very rarely used as it does not tell us about the observations in-between the largest and smallest values.

Example 32 Calculate the range from the given data.

Classes	5 — 9	10 — 14	15 — 19	20 — 24	25 — 29
Frequency	10	15	12	21	3

Solution

Classes	f	Class-Boundaries
5 — 9	10	4.5 — 9.5
10 — 14	15	9.5 — 14.5
15 — 19	12	14.5 — 19.5
20 — 24	21	19.5 — 24.5
25 — 29	3	24.5 — 29.5

In the given data the lower limit of the first group is 4.5 and the upper limit of the last group is 29.5. So,

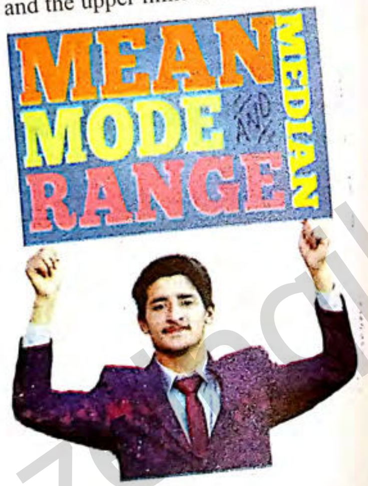
Range = 29.5 – 4.5 = 25

Example 33

The numbers of grams in various candy bars are listed below.

Find the mean, median, mode, and range. Round to the nearest tenth if necessary. Then select the appropriate measure of central tendency or range to describe the data. Justify your answer.

9, 8, 9, 8, 9, 13, 24



Solution

mean: $\dfrac{8+8+9+9+9+13+24}{7} = 11.6$ g

median: 8, 8, 9, 9, 9, 13, 24

mode: 9 g occurs most frequently.

range: 24 – 8 = 16 g

The appropriate measure of central tendency or range to describe the data is the median or the mode. The mean is affected by the highest value, 24 grams.

Did You Know?

π = 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164 06286 20899 86280 34825 34211 70679 82148 08651 32823 ...

(ii) Standard Deviation (S.D)

It is the positive square root of the average of squared deviations measured from A.M (arithmetic mean).

i.e. $\text{S.D} = \sqrt{\dfrac{\sum(x-\bar{x})^2}{n}}$ for ungrouped data

and $\text{S.D} = \sqrt{\dfrac{\sum f(x-\bar{x})^2}{\sum f}}$ for frequency distribution

(iii) Variance

Variance is the square of standard deviation. Variance is usually denoted by the symbol 'S'. Mathematically it is expressed as:

$$S^2 = \frac{\sum(x-\bar{x})^2}{n} \text{ or } \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 \text{ for ungrouped data}$$

$$\text{and } S^2 = \frac{\sum f(x-\bar{x})^2}{\sum f} \text{ or } \frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2 \text{ for discrete and continuous data}$$

Example 34

Find the variance and standard deviation for the following data:

6, 8, 10, 12, 14

Solution

$$\bar{x} = \frac{6+8+10+12+14}{5} = \frac{50}{5} = 10$$

x	$x-\bar{x}$	$(x-\bar{x})^2$
6	6 – 10 = –4	16
8	8 – 10 = –2	4
10	10 – 10 = 0	0
12	12 – 10 = 2	4
14	14 – 10 = 4	16
		40

$$\text{S.D} = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{40}{5}} = \sqrt{8}$$

Variance $(S^2) = (\sqrt{8})^2 = 8$

Example 35 The following is the distribution for the number of rotten eggs found in 60 crates. Find the standard deviation and variance of the rotten eggs.

No. of rotten eggs	0 — 4	4 — 8	8 — 12	12 — 16	16 — 20	20 — 24
No. of crates	5	10	15	20	6	4

Solution

C.B	f	x	fx	x^2	fx^2
0 — 4	5	2	10	4	20
4 — 8	10	6	60	36	360
8 — 12	15	10	150	100	1500
12 — 16	20	14	280	196	3920
16 — 20	6	18	108	324	1944
20 — 24	4	22	88	484	1936
			696		9680

$$\text{Variance} = \frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2$$

$$= \frac{9680}{60} - \left(\frac{696}{60}\right)^2$$

$$= 161.33 - 134.56 = 26.77$$

$$\text{S. D} = \sqrt{Variance} = \sqrt{26.77} = 5.18 \text{ (Approximately)}$$

Activity

Substances with pH values less than 7 are acids, those with pH values greater than 7 are bases, and substances with pH values equal to 7 are neutral. Test several solutions and list the pH values. Make the *frequency* table of pH values by completing the table at the right.

pH number	Tally	Frequency
Less than 7		
Equal to 7		
Greater than 7		

Exercise 6.4

1. Find the range for the following items:
11, 13, 15, 21, 19, 23.
2. A bank branch manager was interested in the waiting times of customers. For this he carried out a survey. A random sample of 12 customers was selected and yielded the following data.
5.90 9.66 5.79 8.02 8.73 8.01 10.49 8.35 6.68 5.64 5.47 9.91

Calculate the following
(i). Average (ii). Median (iii). Standard Deviation
3. Calculate the Range, Variance and Standard Deviation for the following discrete data:

x	5	10	11	13	15
f	2	3	4	1	5

4. The Following table shows the marks obtained by 10 students of two sections of 10[th] class.

Section A	7	9	6	9	4	7	5	8	8	7
Section B	6	10	6	4	2	8	10	6	9	9

Find their
(a) Arithmetic Mean (b) Variance
5. Following are the marks (out of 75) of the eight students in two subjects.

Student	A	B	C	D	E	F	G	H
Marks in Mathematics	54	63	59	45	52	35	61	68
Physics	52	55	57	51	56	58	50	59

Compare the standard deviation of the marks and tell that in which subject students are more consistent.
6. The following is the distribution for the number of defective bulbs in 30 cartons (Packs). Find Variance and Standard Deviation of defective bulbs.

No. of defective bulbs	0 — 2	2 — 4	4 — 6	6 — 8	8 — 10
No. of packs (f)	1	3	15	10	2

Review Exercise 6

At the end of each question, four circles are given. Fill in the correct circle only.

(i). The difference between upper limit of two consecutive classes in a frequency table is called.
- ○ Class limit ○ Class interval ○ Class mark ○ Range

(ii). A cumulative frequency histogram is also called
- ○ Histogram ○ Pie chart
- ○ Ogive ○ frequency polygon

(iii). The number of times a value appears on a set of data is called
- ○ frequency ○ average ○ mode ○ median

(iv). The data below represents the number of televisions that 11 students have in their homes. Find the mode of the data.

3, 2, 1, 1, 1, 5, 3, 1, 2, 1, 2
- ○ 1 ○ 2 ○ 3 ○ 4

(v). In which data set are the mean, median, mode, and range all the same number?
- ○ 1, 2, 3, 3, 2, 1, 2 ○ 1, 2, 3, 1, 2, 3, 1
- ○ 1, 3, 3, 3, 2, 3, 1 ○ 2, 2, 1, 2, 3, 2, 3

(vi). The n^{th} root of product of 'n' number of values is called.
- ○ arithmetic ○ geometric mean
- ○ harmonic ○ standard deviation

(vii). In a set of data.

63, 65, 66, 67, 69, median is
- ○ 63 ○ 66 ○ 67 ○ 69

(viii). In a set of data

41, 43, 47, 51, 57, 52, 59 median is
- ○ 51 ○ 47 ○ 47 ○ none of these

(ix). In the given set of data 5, 7, 7, 5, 3, 7, 2, 8, 2 mode is
- ○ 9 ○ 5 ○ 2 ○ 7

(x). In the given set of data 5, 5, 5, 5, 5, 5, 5 the standard deviation is
- ○ 5 ○ 0 ○ 7 ○ none of these

(xi). The average pocket money of 30 student is Rs. 20/-. The total amount in the class is
- ○ Rs. 20/- ○ Rs. 30/- ○ Rs. 300/- ○ Rs. 600/-

(xii). The sum of 30 observations is 1500. Its average will be
- ○ 1500 ○ 150 ○ 15 ○ none of these

(xiii). The difference of the largest and smallest value in the data is called
- ○ Mean ○ Mode ○ Range ○ Standard deviation

(xiv). The formula $\frac{\sum x}{n}$ determines
- ○ Arithmetic Mean ○ Median
- ○ Mode ○ G.M

(xv). What is the difference between the mean of Set B and the median of Set A?

Set A: {2, −1, 7, −4, 11, 3}

Set B: {12, 5, −3, 4, 7, −7}
- ○ −0.5 ○ 0 ○ 0.5 ○ 1

(xvi). $\frac{\sum f(x-\bar{x})^2}{\sum f}$ is called ________
- ○ Range ○ Median ○ S.D ○ Variance

(xvii). The most frequent value in the data is called its ________
- ○ Mean ○ Median ○ Mode ○ G.M

2. The ages (in years) of 27 students of 10^{th} class are given. Prepare a frequency distribution of suitable class interval.

17,	17,	16,	16,	17,	16,	16,	17,	18,	18,
15,	17,	19,	18,	18,	17,	16,	15,	16,	17,
15,	19,	19,	15,	15,	16,	18			

3. Prepare a histogram of the following table;

Brand of car	A	B	C	D	E
Sale in 1 month	100	120	110	72	169

4. Prepare a frequency polygon of the following frequency distribution.

Score in test	Frequency
0 – 10	2
11 – 21	7
22 – 32	25
33 – 43	11
44 – 50	5

...ulative frequency polygon. ...in kilograms of 250 boys. Represent these data by means

Weight (kg)	Number of boys
44.0 – 47.9	3
48.0 – 51.9	17
52.0 – 55.9	50
56.0 – 59.9	81
60.0 – 63.9	57
64.0 – 67.9	23
68.0 – 71.9	9

PROJECT

The following scores were made on a 60-item test:

25	30	34	37	41	42	46	49	53
26	31	34	37	41	42	46	50	53
28	31	35	37	41	43	47	51	54
29	33	36	38	41	44	48	52	54
30	33	36	39	41	44	48	52	55
30	33	37	40	42	45	48	52	

(a) Group the data into class intervals of size 2, beginning with the interval 24.5–25.5.
(b) Set up a frequency table for the above data.
(c) Build the histogram for the data.
(d) Draw the frequency polygon for this histogram.
(e) Find the cumulative frequencies.
(f) Draw the ogive.
(g) Find the range of the data.
(h) Find the mean of the data.
(i) Find the standard deviation of the data.
(j) Find the variance of the data.

Summary

- A set of raw data can be organized and then arranged in an orderly way in the form of a frequency table.
- A frequency table can be represented graphically by a Histogram.
- A histogram is a vertical bar graph with no space in between the bars.
- The area of each bar is proportional to the frequency it represents.
- If the midpoints of the tops of the consecutive bars in a histogram are joined by line segments and mid points at both ends are joined to the horizontal axis, a frequency polygon is obtained.
- The cumulative frequency table provide another way of representing a set of data. A cumulative frequency table can be represented by a cumulative frequency curve known as an Ogive.
- The median is the middle value when a set of data is arranged in order of increasing magnitude. the median divides a set of data into two equal halves. The middle value of the lower half is the lower quartile and the middle value of the upper half is the upper quartile. The lower quartile, the median and the upper quartile which divides the data into four equal parts are called quartiles.
- The inter quartile range of a set of numbers is the difference between the upper quartile and the lower quartile.
- A set of data can be described by numerical quantities called averages.
- The three common averages are the mean, the median and the mode.
- The mode is the number that occurs most frequently.
- The mean is the sum of values divided by the number of values in a set of data.
- The mean of a set of grouped data is $\overline{x} = \frac{\sum fx}{\sum f}$, where x is the middle value of the class interval and f is the frequency of the class interval.
- The median for an odd number of data is the middle value when the data are arranged in ascending/descending order.
- The median for an even number of data is the mean of the two middle values when the data are arranged in ascending/descending order.

Unit 7

INTRODUCTION TO TRIGNOMETRY

In this unit the students will be able to

- Measure an angle in sexagesimal system (degree, minute and second).
- Convert an angle given in D°M′S″ form into a decimal form and vice versa.
- Define a radian (measure of an angle in circular system) and prove the relationship between radians and degrees.
- Establish the rule $\ell = r\theta$, where r is the radius of the circle, ℓ the length of circular arc and θ the central angle measured in radians.
- Prove that the area of a sector of a circle is $\frac{1}{2}r^2\theta$.
- Define and identify:
  - general angle (coterminal angles)
  - angle in standard position.
- Recognize quadrants and quadrantal angles.
- Define trigonometric ratios and their reciprocals with the help of a unit circle.
- Recall the values of trigonometric ratios for 45°,30°,60°.
- Recognize signs of trigonometric ratios in different quadrants.
- Find the values of remaining trigonometric ratios if one trigonometric ratio is given.
- Calculate the values of trigonometric ratios for 0°, 90°, 180°, 270°, 360°.
- Prove the trigonometric identities and apply them to show different trigonometric relations.
- Find angle of elevation and depression.
- Solve real life problems involving angle of elevation and depression.

Sine

Sine of angle A = $\frac{\text{Opp}}{\text{Hyp}}$

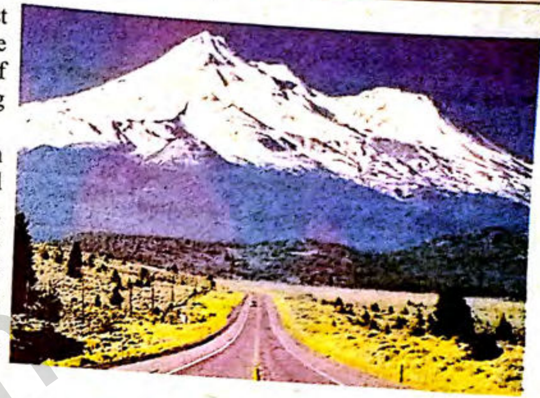
Cosine

Cosine of angle A = $\frac{\text{Adj}}{\text{Hyp}}$

Tangent

Tangent of angle A = $\frac{\text{Opp}}{\text{Adj}}$

Why it's important



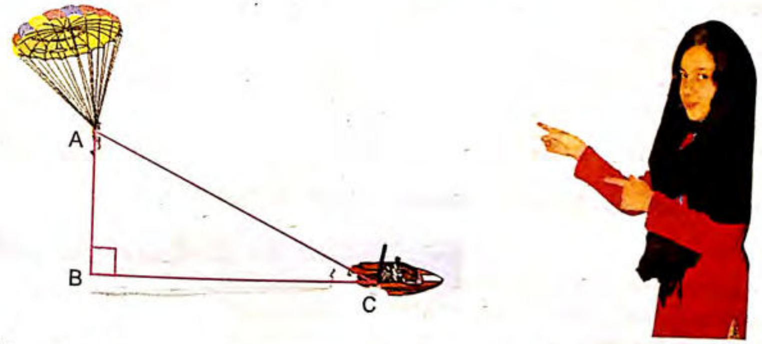
Mount Everest is Earth's highest mountain, peaking at an incredible 29,035 feet. The heights of mountains can be found using trigonometry.

Trigonometry is used in navigation, building, and engineering. For centuries, Muslims used trigonometry and the stars to navigate across the Arabian desert to Mecca, the birthplace of the prophet Muhammad ﷺ, the founder of Islam. The ancient Greeks used trigonometry to record the locations of thousands of stars and worked out the motion of the Moon relative to Earth. Today, trigonometry is used to study the structure of DNA, the master molecule that determines how we grow from a single cell to a complex, fully developed adult and much more.

Activity

How ratios in right triangles used in the real world?

In parasailing, a towrope is used to attach the parachute to the boat.



(a). What type of triangle do the towrope, water, and height of the person above the water form?

(b). What is the hypotenuse of the triangle.

(c). What type of angle do the towrope and the water form?

(d). Which side is opposite this angle?

(e). Other than the hypotenuse, what is the side adjacent to this angle.

7.1

7.1.1 Sexagesimal system (Degree, minute and second)

Sexagesimal (base 60) is a numeral system with sixty as its base. It originated with the ancient Sumerians in the 3rd millennium BC, was passed down to the ancient Babylonians, and is still used—in a modified form—for measuring time, angles and geographic coordinates.

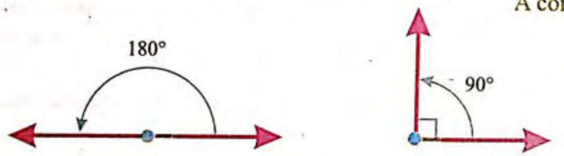
It is a system of measurement of angles, in which angles are measured in degrees, minutes and seconds. If the initial ray $\overrightarrow{OA}$ completes one rotation in anticlock wise direction, then the angle formed is said to be 360 degree or 360°.

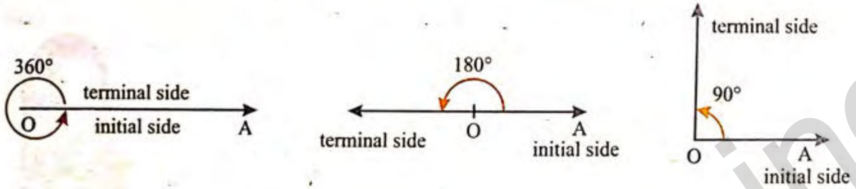
A complete 360° rotation

One complete rotation $= 360°$.

$\frac{1}{2}$ of complete rotation $= 180°$ is called straight angle.

$\frac{1}{4}$ of complete rotation $= 90°$ is called right angle.





Thus one degree is defined as the measure $\frac{1}{360}$th of a complete rotation and it is denoted by '1°'.

A degree is further divided into 60 equal parts called minutes and each minute is further subdivided into 60 equal parts called seconds. We denote minute by (′) and second by (″).

Try This

(i). Convert 5°42'30" to decimal degree notation.

(ii). Convert 72.18° D°M'S" to decimal degree notation.

$1° = 60 \text{ minutes} = 60'$

$1' = 60 \text{ seconds} = 60''$

$1° = 3600^{th} \text{ seconds} = 3600''$

49 degree 25 minutes and 15 seconds are written in symbolic form as 49°25′15″

7.1.2 Conversion of D°M'S" form into decimal form and vice versa

Example 1 Convert 15°30′25″ to decimal form.

Solution since $1' = \left(\frac{1}{60}\right)^\circ$ and $1'' = \left(\frac{1}{3600}\right)^\circ$

therefore, $15°30'25'' = \left[15 + 30 \times \frac{1}{60} + 25 \times \frac{1}{3600}\right]^\circ$

$= [15 + 0.5 + 0.00694]°$

$= 15.50694°$

Activity

Use coolconversion.com to covert 142°34'12" to decimal degrees. And also 142.57° to D°M′S″ form.

Example 2 Convert 38.39° to D°M′S″ form.

Solution Since $1° = 60'$ and $1' = 60''$

therefore, $38.39° = 38° + (0.39)°$

$38.39° = 38° + (0.39 \times 60)'$

$= 38° + (23.4)'$

$= 38° + 23' + (0.4)'$

$= 38°23' + (0.4 \times 60)''$

$= 38°23'24''$

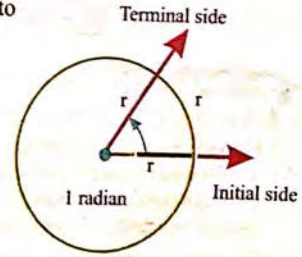
7.1.3 Circular system (Radians)

So far we have been using the measurement of 360 to denote the angle for one complete rotation. However, this value is arbitrary and in some branches of mathematics, angular measurement cannot be conveniently done in degrees. Thus a new unit called the radian, is introduced to describe the magnitude of an angle. This system of angular measurement, known as circular measure, is applied specially to those branches of mathematics which involve the differentiation and integration of trigonometric ratios.

A radian is the measure of the angle subtended at the centre of a circle by an arc ℓ equal in length to its radius r of the circle.

In general radians is the ratio of the length of an arc to the radius of the circle.

i.e. $\theta = \frac{\ell}{r}$



In the following figure, the radius of circle is r and length of an arc AB is also r then angle subtended at the centre is one radian i.e.

$m\angle AOB = \frac{r}{r} = 1$ radian.

or one radian is an angle subtended at the centre by an arc of length equal to radius of circle.

7.1.4 Relation between radians and degrees

If r is radius of circle then number of radians in one complete rotation is given by

$$\frac{\text{circumference of circle}}{\text{radius of circle}}$$

Since circumference of circle $= 2\pi r$

Therefore number of radians in one complete rotation $= \frac{2\pi r}{r} = 2\pi$ radians but 2π radian $= 360°$ because the circumference of a circle subtends an angle of measure 360°.

Hence we establish a relation between angles measure in radians and the angles measure in degrees as follows.

$$2\pi \text{ radians} = 360°$$

$$\pi \text{ radians} = 180° \text{ .... (i)}$$

$$\frac{\pi}{2} \text{ radians} = 90°$$

$$\frac{\pi}{3} \text{ radians} = 60°$$

$$\frac{\pi}{6} \text{ radians} = 30° \quad \text{and so on.}$$

Hence, $1 \text{ radian} = \frac{180°}{\pi} \approx \frac{180°}{3.14159} \approx 57.296°$

$$\boxed{1 \text{radian} = 57.3°}$$

Target

To find equivalents for radian and degree measure.

Similarly from (i) we have

$$180° = \pi \text{ radians}$$

$$\Rightarrow \quad 1° = \frac{\pi}{180} = \frac{3.14159}{180} \text{ radians}$$

$$1° \approx 0.0175 \text{radians}$$

$$\boxed{1° = 0.0175}$$

Did You Know?

WHAT IS A 'GOOGOL' ?

Googol is one of the largest numbers that has ever been named. It has been defined as 1 followed by one hundred zeros;
10000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000
However, a googol can be expressed using exponents, as 10^{100}

Example 3 Convert $\frac{4\pi}{7}$ radians to degrees.

Solution Since π radians $= 180°$

Therefore $\frac{4\pi}{7} = \frac{4}{7}(180°) = 102.85714°$

$= 102°51'26''$

Example 4 Convert 31° 45′ to radians.

Solution Since $180° = \pi$ radians

$$\Rightarrow \quad 1° = \frac{\pi}{180} \text{ radians}$$

As $31°45' = 31.75°$

$$\because \quad 31°45' = 31.75 \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{31.75 \times 3.14159}{180} \text{ radians}$$

$$= 0.5541 \text{ radians}$$

Exercise 7.1

1. Convert the following angles from D°M′S″ forms to decimal forms.
   - (i). 8°15′35″ (ii). 39°48′55″
   - (iii). 84°19′10″ (iv). 18°6′21″
2. Convert the following angles from decimal forms to D°M′S″.
   - (i). 4225° (ii). 57.325°
   - (iii). 12.9956° (iv). 32.625°
3. Convert the following radian measures of angles into the measures of degrees.
   - (i). 2 radians (ii). $\frac{5\pi}{3}$ radians
   - (iii). $\frac{\pi}{6}$ radians (iv). $\frac{-3\pi}{4}$ radians
4. Convert the following angles in terms of radians.
   - (i). 45° (ii). 120°
   - (iii). −210° (iv). 60°35′48″

7.2 Sector of a circle

7.2.1 Length of an arc of circle

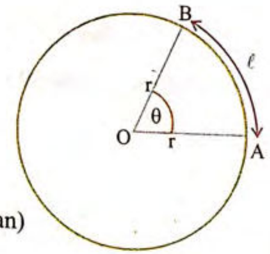
Consider a circle with centre 'O' and radius r, which subtends an angle θ radian at the centre O. Let $\widehat{AB}$ is the minor arc of the circle whose length is equal to ℓ as shown in figure.

By the definition of radian, we have

$$\text{Radian} = \frac{\text{length of an arc } \widehat{AB}}{\text{radius of cirlce}}$$

$$\theta = \frac{\ell}{r}$$

or $\quad \ell = r\,\theta$



Which gives length of an arc in terms of angle (in radian) subtended at the center of a circle.

Example 5 Find the length of an arc of a circle of radius 5 cm which subtends an angle of $\frac{3\pi}{4}$ radians at the centre.

Solution

$r = 5\text{cm}$

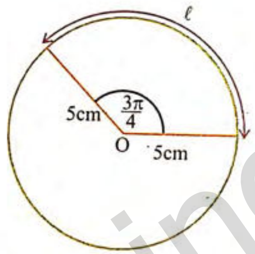
$\theta = \frac{3\pi}{4}$ radians

$\ell = ?$

we have

$\ell = r\,\theta$

$\ell = 5 \times \frac{3\pi}{4} = \frac{15\pi}{4} = 11.78$ cm



Hence length of an arc = 11.78 cm.

Example 6

Find the distance travelled by a cyclist moving on a circle of radius 15m, if he makes 3.5 resolutions.

Solution

l revolution = 2π radians

3.5 revolution = $2\pi \times 3.5$ m

Distance travelled = $l = r\theta$

$l = 15 \times 2\pi \times 3.5 = 105\pi$ m

Example 7 An arc of length 2.5 cm of a circle subtends an angle θ at the centre O of diameter 6 cm. Find the value of θ.

Solution Here

$m\overline{AB} = 6\text{cm}$

$\therefore \quad \overline{OB} = \overline{OC} = r = 3\text{cm}$

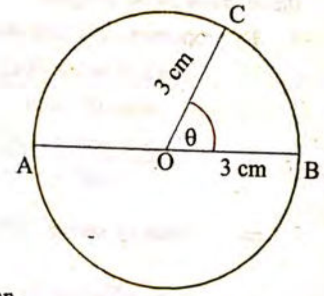
$m\widehat{BC} = 2.5\text{cm}$

$\theta = ?$

We know that $\quad \ell = r\,\theta$

$\Rightarrow \quad \theta = \frac{\ell}{r}$

$\theta = \frac{2.5}{3} \Rightarrow \theta = 0.833$ radian.



Example 8 If length of an arc of a circle is 5 cm which subtends an angle of measure 60°, find the radius of the circle.

Solution $\ell = 5\text{cm}$

$\theta = 60° = \frac{60 \times \pi}{180}$ radians

$= 1.047$radians

$r = ?$

We have formula $\quad \ell = r\theta$

$\Rightarrow \quad r = \frac{\ell}{\theta} = \frac{5}{1.047}$

$r = 4.78\text{cm}$

Hence radius of circle is 4.78cm.

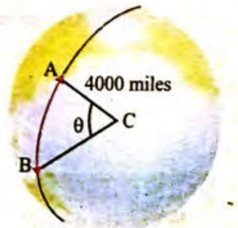
Tidbit

In general,
The length of an arc
= x/360 × circumference
The length of an arc
= x/360 × 2πr
The area of a sector
= x/360 × area of the circle
= $x/360 \times \pi r^2$

Activity

How do we measure the distance between two points, A and B, on Earth?

We measure along a circle with a center C, at the center of Earth. The radius of the circle is equal to the distance from C to the surface. Use the fact that Earth is a sphere of radius equal to approximately 4000 miles to solve the following questions.



(a). If two points A and B, are 8000 miles apart, express angle θ in radians and in degrees.

(b). If two points A and B, are 10,000 miles apart, express angle θ in radians and in degrees.

(c). If θ = 30°, find the distance between A and B to the nearest mile.

(d). If θ = 10°, find the distance between A and B to the nearest mile.

7.2.2 Area of sector

Consider a circle of radius r with centre O, $\overset{\frown}{PQ}$ is an arc which subtends an angle θ radians at the centre as shown in figure.

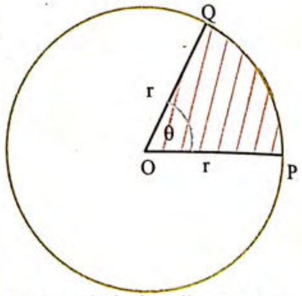
By proportion, it is clear that

$$\frac{\text{Area of sector POQ}}{\text{Area of circle}} = \frac{\text{Central angle of sector}}{2\pi}$$

$$\Rightarrow \quad \frac{\text{Area of sector POQ}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\Rightarrow \quad \text{Area of sector POQ} = \frac{\theta}{2\pi} \times \pi r^2$$

$$\Rightarrow \quad \text{Area of sector POQ} = \frac{1}{2} r^2 \theta$$



Hence, Area of sector of a circle with radius r, whose central angle is θ radian is given by

$$\boxed{A = \frac{1}{2} r^2 \theta}$$

Example 9

Find the area of sector with central angle of 60° in a circular region whose radius is 5 cm.

Solution

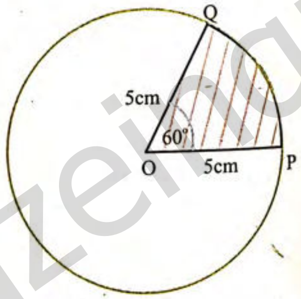
$r = 5\text{cm}$

$\theta = 60° = 60 \times \frac{\pi}{180}$ radian $= 1.047$ radian

Area of sector $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} (5)^2 (1.047)$

$= 13.08\text{cm}^2$

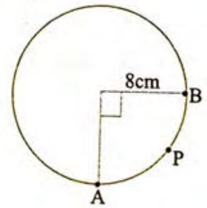


Did You Know?

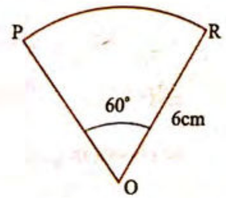
What are digits?
Digits are actually the alphabets of numbers. Just as we use the twenty-six letters of the alphabet to build words, we use the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 to build numerals.

Exercise 7.2

1. Find ℓ when
   (i). $\theta = \frac{\pi}{6}$ radian, $r = 2\text{cm}$ (ii). $\theta = 30°$, $r = 6\text{cm}$
   (iii). $\theta = \frac{4\pi}{6}$ radian, $r = 6\text{cm}$
2. Find θ when
   (i). $\ell = 5$ cm, $r = 2$ cm (ii). $\ell = 30$ cm, $r = 6$ cm
   (iii). $\ell = 6$ cm, $r = 2.87$ cm
3. Fined *r* when
   (i). $\theta = \frac{\pi}{6}$ radians, $\ell = 2\text{cm}$ (ii). $\theta = 3\frac{1}{2}$ radians, $\ell = \frac{4}{7}\text{m}$
   (iii). $\theta = \frac{3\pi}{4}$ radians, $\ell = 15\text{cm}$
4. Find the area of sector of a circle whose radius is 4 m, with central angle 12 radian.
5. The arc of a circle subtends an angle of 30° at the centre. The radius of circle is 5 cm. find;
   (i). Length of the arc (ii). Area of sector formed.
6. An arc of a circle subtends an angle of 2 radian at the centre. If the area of sector formed is 64 cm^2, find the radius of the circle.
7. In a circle of radius 10m, find the distance travelled by a point moving on this circle if the point makes 3.5 revolutions. (3.5 revolutions $= 7\pi$).
8. What is the circular measure of the angle between the hands of the watch at 3 o'clock.
9. What is the length of the arc *APB*?



10. Find the area of the sector *OPR*.



7.3 Trigonometric Ratios

Trigonometry is a powerful tool for making indirect measurements of distance or height. It plays an important role in the field of surveying, navigation, engineering and many other branches of physical science.

7.3.1 (a) The general angle (Coterminal angles)

Angles having the same initial and terminal sides are called coterminal angles, and they differ by a multiple of 2π radians or $360°$. They are also called general angles.

If let $\theta = 30°$ is a terminal angle, then it coterminal angles are $\theta + 360°$ and $\theta - 360°$ i.e. $390°$ and $-330°$ as shown in the following figures.

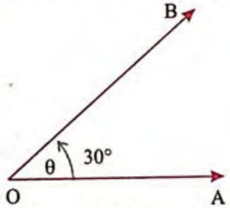


Fig.(a)

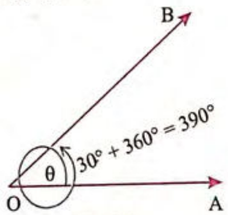


Fig.(b)

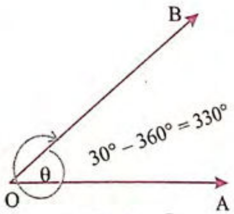
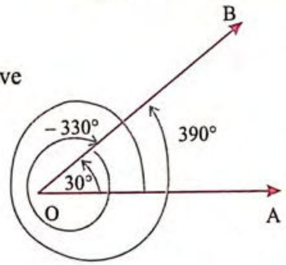


Fig.(b)

Combining figures a, b, c we have



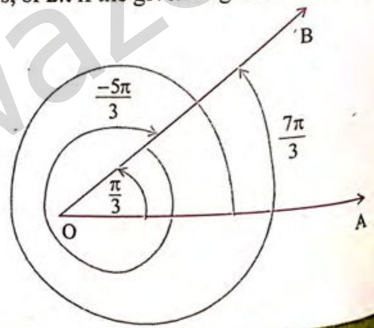
To find a positive and negative general angles (coterminal) with a given angle, add and subtract $360°$, if the given angle is measured in degrees, or 2π if the given angle is measured in radians. For example

coterminal angles of $\frac{\pi}{3}$ are

(i) $\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$

and (ii) $\frac{\pi}{3} - 2\pi = \frac{-5\pi}{3}$

as shown in figure.



Example 10 Find coterminal angle of $60°$ and $-60°$

Solution Since $60° + 360° = 420°$

and $60° - 360° = -300°$

therefore $420°$ and $-300°$ are coterminal angles of $60°$.

Now $-60° + 360° = 300°$

and $-60° - 360° = -420°$

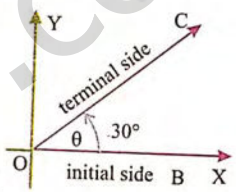
therefore $300°$ and $-420°$ are coterminal angles of $-60°$.

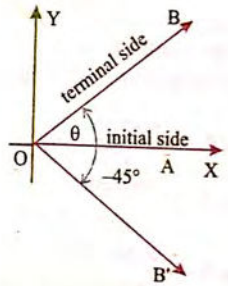
(b) Angle in standard position

In XY-plane, an angle is in standard position if

- its vertex is at the origin of a rectangular coordinate system and
- its initial side lies along the positive x-axis.

An angle is said to be positive if it is measured in anticlock wise direction from positive x-axis and is negative, if it is measured clockwise from +ve x-axis. In the following figure terminal side is in anti-clock wise direction which makes an angle of measure $45°$ and in clock wise direction makes an angle of measure $-45°$.





7.3.2 (a) Quadrants

The Cartesian plane is divided into four quadrants and the angle θ is said to be in the quadrant where OP lies. In the diagram, θ is in the first quadrant.

As shown in figure (b).

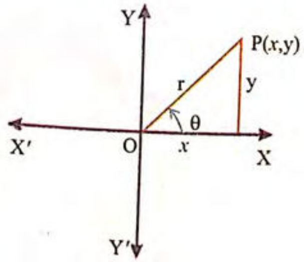


Fig.(a)

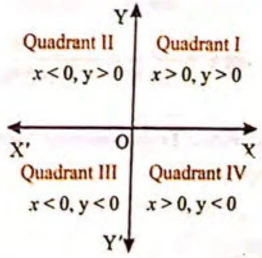
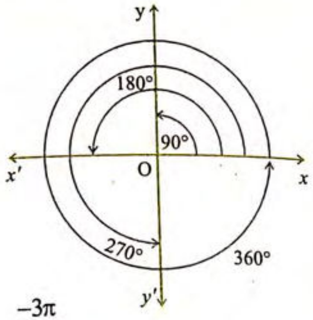


Fig.(b)

(b) Quadrantal angles

Quadrantal angles are those angles whose terminal sides coincide with co-ordinate axes i.e. x-axis or y-axis. The Quadrantal measures of angles are 0°, 90°, 180°, 270° and 360° or 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π radians as shown in figure.



Exercise 7.3

1. Find coterminal angles of the following angles.
   (i). 55° (ii). −45° (iii). $\frac{\pi}{6}$ (iv). $\frac{-3\pi}{4}$
2. State the quadrant in which the following angles lie?
   (i). $\frac{8\pi}{5}$ (ii). 75° (iii). −818° (iv). $\frac{-5\pi}{4}$ (v). 103°

7.3.3 Trigonometric ratios

A trigonometric ratio is a ratio of lengths of two sides in a right triangle. Trigonometric ratios are used to find the measure of a side or an acute angle in a right triangle. The trigonometric ratios for any acute angle θ of a right triangle ABC are

$$\sin\theta = \frac{\text{opposite side of }\theta}{\text{hypotenuse}}$$

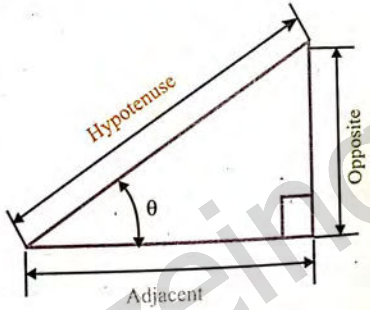
$$\cos\theta = \frac{\text{adjacent side of }\theta}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side of }\theta}{\text{adjacent side of }\theta}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite site of }\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adjacent}}{\text{opposite}}$$



A trick to remember Trigonometric formulas

some people have
curly brown hair
through proper brushing

7.3.3.1 Trigonometric ratios with the help a unit circle

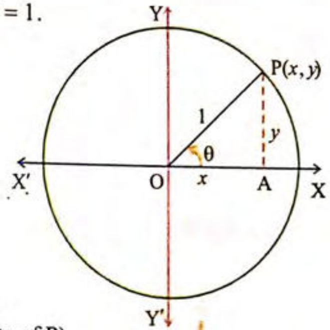
Consider a circle with centre 0 and radius 1 (unit circle). P (x, y) is any point on the circle. Radius $\overline{OP}$ makes an angle θ with the positive x-axis. θ can be measured either in degrees or in radians. Draw a perpendicular $\overline{PA}$ on x-axis, so that OAP is a right triangle in which $m\angle A = 90°$ as shown in the figure.

In ΔOAP opposite side of θ is $\overline{AP} = y$, adjacent side of θ is $\overline{OA} = x$ and hypotenuse is $\overline{OP} = 1$.
Therefore by definition

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{\overline{AP}}{\overline{OP}} = \frac{y}{1} = y$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{\overline{OA}}{\overline{OP}} = \frac{x}{1} = x$$

and $$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\overline{AP}}{\overline{OP}} = \frac{y}{x}$$



Hence trigonometric ratios of a unit circle are

$\sin\theta = y$ (the y co-ordinate of P)
$\cos\theta = x$ (the x co-ordinate of P)
$\tan\theta = \frac{y}{x}$ $\left(\frac{y \text{ co-ordinate of P}}{x \text{ co-ordinate of P}}\right)$

Reciprocal trigonometric ratios

$$\text{cosec}\,\theta = \frac{\overline{OP}}{\overline{AP}} = \frac{1}{y} = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{\overline{OP}}{\overline{OA}} = \frac{1}{x} = \frac{1}{\cos\theta}$$

and $$\cot\theta = \frac{\overline{OA}}{\overline{AP}} = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}$$

Note (Important points to be noted)

1. If θ lies in 1st Quadrant, then we can write as $0 < \theta < \frac{\pi}{2}$ or $0 < \theta < 90°$.
2. If θ lies in 2nd Quadrant, then we can write as $\frac{\pi}{2} < \theta < \pi$ or $90° < \theta < 180°$.
3. If θ lies in 3rd Quadrant, then we can write as $\pi < \theta < \frac{3\pi}{2}$ or $180° < \theta < 270°$.
4. If θ lies in 4th Quadrant, then we can write as $\frac{\pi}{2} < \theta < 2\pi$ or $270° < \theta < 360°$.

7.3.4 Values of trigonometric ratios for 45°, 30° and 60°

The values of trigonometric ratios for 45°, 30° and 60° are given in the following table.

T-ratio\ θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\text{cosec}\,\theta$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$
$\sec\theta$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2
$\cot\theta$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

7.3.5 Recognizing signs of trigonometric ratios in different Quadrants

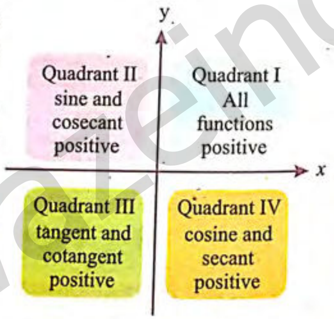
Signs of trigonometric ratios are different in different quadrants depending upon the terminal side of the angle θ.

(i). If θ lies in first quadrant, then $x > 0$, $y > 0$ therefore sinθ, cosθ, tanθ, cosecθ, secθ, tanθ are all positive.

(ii). If θ lies in 2^{nd} quadrant, then $x < 0$, $y > 0$ therefore sinθ and cosecθ are positive and the remaining four trigonometric ratios are negative.

(iii). If θ lies in 3^{rd} quadrant, then $x < 0$, $y < 0$ therefore tanθ and cotθ are positive and the remaining four trigonometric ratios are negative.

(iv). If θ lies in 4^{th} quadrant, then $x > 0$, $y < 0$ therefore cosθ and secθ are positive and the remaining trigonometric ratios are negative.



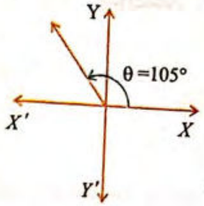
These four results are summarized in the figure at the right.

Example 11 Find the signs of the following trigonometric ratios and tell in which quadrant they lie?

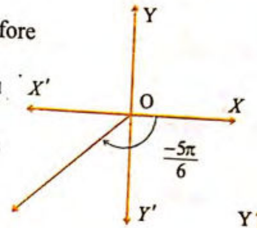
(i) $\sin 105°$ (ii) $\tan\frac{-5\pi}{6}$ (iii) $\sec 1030°$ (iv) $\cot 710°$

Solution

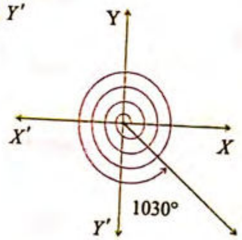
(i) $\theta = 105°$ lies in 2^{nd} quadrant, therefore sin 105° also lies in 2^{nd} Quadrant and hence sign of sin 105° is positive.



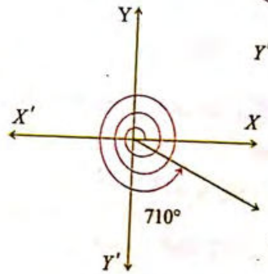
(ii) Since $\frac{-5\pi}{6}$ lies in the 3^{rd} quadrant, therefore $\tan\left(\frac{-5\pi}{6}\right) = -\tan\frac{5\pi}{6}$ also lies in 3^{rd} quadrant and it has positive sign. As shown in figure.



(iii) Since $\theta = 1030° = 720° + 310°$, therefore it lies in 4^{th} quadrant and sec 1030° in 4^{th} quadrant has positive sign.



(iv) Since $\theta = 710° = 360° + 350°$, therefore it lies in 4^{th} quadrant and cot 710° in 4^{th} quadrant is negative.



Example 12 If $\tan\theta < 0$ and $\cos\theta > 0$, name the quadrant in which angle θ lies.

Solution

When $\tan\theta < 0$, θ lies in quadrant II or IV. When $\cos\theta > 0$, θ lies in quadrant I or IV. When both conditions are met ($\tan\theta < 0$ and $\cos\theta > 0$), θ must lie in quadrant IV.

Try This

If $\sin\theta < 0$ and $\cos\theta < 0$, name the quadrant in which angle θ lies.

7.3.6 Finding the values of remaining trigonometric ratios if one trigonometric ratio is given

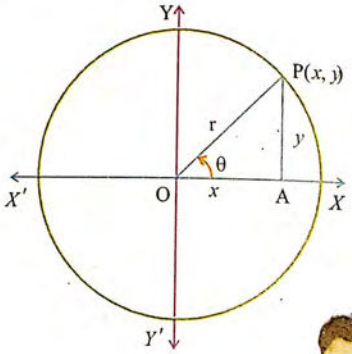
Example 13 If $\tan\theta = 1$, find the other trigonometric ratios, when θ lies in first quadrant.

Solution $\tan\theta = 1 = \frac{y}{x}$

$\Rightarrow \quad y = 1 \text{ and } x = 1$

Hence by Pythagoras theorem;

$$r^2 = x^2 + y^2$$
$$r^2 = (1)^2 + (1)^2 = 1 + 1$$
$$r^2 = 2$$
$$r = \sqrt{2}$$



Therefore $\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$$
$$\cot\theta = \frac{x}{y} = \frac{1}{1} = 1$$
$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{\sqrt{2}}{1} = \sqrt{2}$$
$$\sec\theta = \frac{r}{x} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

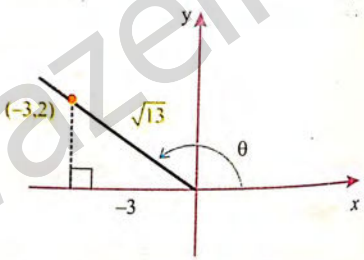
Example 14 Given that $\tan\theta = -\frac{2}{3}$ and θ is in the second quadrant, find the other function values.

Solution $\tan\theta = \frac{y}{x} = -\frac{2}{3} = \frac{2}{-3}$,

$\therefore y = 2$ and $x = -3$

Hence by Pythagoras theorem;

$$r^2 = x^2 + y^2$$
$$r^2 = (-3)^2 + (2)^2 = 9 + 4 = 13$$
$$r^2 = 13$$
$$r = \sqrt{13}$$



Expressing $-\frac{2}{3}$ as $\frac{2}{-3}$ sine θ is in quadrant II

Therefore,

$$\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{13}} = -\frac{3}{\sqrt{13}}$$
$$\cot\theta = \frac{x}{y} = \frac{-3}{2} = -\frac{3}{2}$$
$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{\sqrt{13}}{2}$$
$$\sec\theta = \frac{r}{x} = \frac{\sqrt{13}}{-3} = -\frac{\sqrt{13}}{3}$$

Example 15 If $\cos\theta = \frac{4}{5}$ where θ lies in 4^{th} quadrant, find the values of other trigonometric ratios.

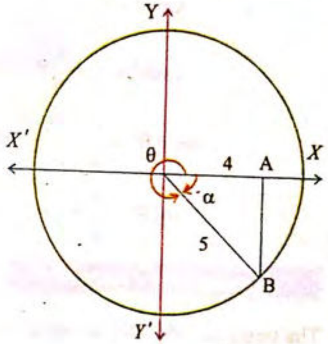
Solution Given

$$\cos\theta = \frac{4}{5} = \frac{x}{r}$$

$\Rightarrow \quad x = 4 \text{ and } r = 5$

Hence by Pythagoras theorem;

$$r^2 = x^2 + y^2$$
$$y^2 = r^2 - x^2 = (5)^2 + (4)^2 = 25 - 16 = 9$$
$$y^2 = 9 \Rightarrow y = \pm 3$$
$$y = -3 \; (\because \theta \text{ lies in } 4^{th} \text{ quadrant.})$$



Therefore,

$$\sin\theta = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5}$$
$$\tan\theta = \frac{y}{x} = \frac{-3}{4} = -\frac{3}{4}$$
$$\cot\theta = \frac{x}{y} = \frac{4}{-3} = -\frac{4}{3}$$
$$\sec\theta = \frac{r}{x} = \frac{5}{4}$$
$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{5}{-3} = -\frac{5}{3}$$

Try This

If $\sin\theta = \frac{1}{2}$ and θ lies in 2^{nd} Quadrant, find the other trigonometric ratios.

7.3.7 Calculating the values of trigonometric ratios of 0°, 90°, 180°, 270° and 360°

Trigonometric ratio of 0°

In XY-plane, the terminal side $\overrightarrow{OP}$ of the angle θ° coincides with $\overrightarrow{OX}$.

Therefore radius $\overline{OP} = r = 1$

or $x = 1$ and $y = 0$

so $\sin 0° = \frac{y}{r} = \frac{0}{1} = 0$

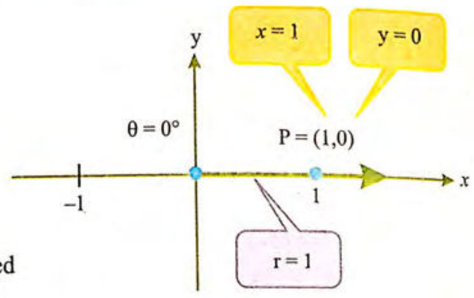
$\cos 0° = \frac{x}{r} = \frac{1}{1} = 1$

$\tan 0° = \frac{y}{x} = \frac{0}{1} = 0$

$\cot 0° = \frac{x}{y} = \frac{1}{0} =$ undefined

$\text{cosec}\, 0° = \frac{r}{y} = \frac{1}{0} =$ undefined

$\sec 0° = \frac{r}{x} = \frac{1}{1} = 1$



Trigonometric ratio of 90°

The terminal side $\overrightarrow{OP}$ of angle 90° coincides with $\overrightarrow{OY}$.

Therefore $\overrightarrow{OP} = r = 1$

or $x = 0$ and $y = 1$

Hence $\sin 90° = \frac{y}{r} = \frac{1}{1} = 1$

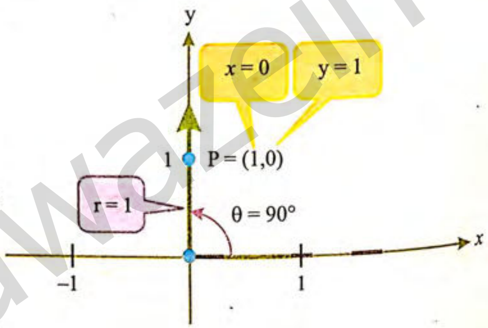
$\cos 90° = \frac{x}{r} = \frac{0}{1} = 0$

$\tan 90° = \frac{y}{x} = \frac{1}{0} =$ undefined

$\text{cosec} 90° = \frac{r}{y} = \frac{1}{1} = 1$

$\sec 90° = \frac{r}{x} = \frac{1}{0} =$ undefined

$\cot 90° = \frac{x}{y} = \frac{0}{1} = 0$



Trigonometric ratio of 180°

In this case, the terminal side $\overrightarrow{OP}$ of angle 180° coincides with $\overrightarrow{OX}$

Therefore $\overrightarrow{OP} = r = 1$

or $x = -1$ *and* $y = 0$

$\sin 180° = \frac{y}{r} = \frac{0}{r} = 0$

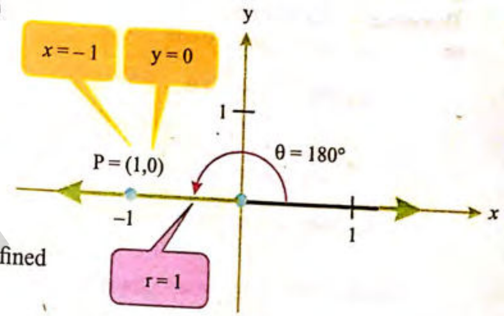
$\cos 180° = \frac{x}{r} = \frac{-1}{1} = -1$

$\tan 180° = \frac{y}{x} = \frac{0}{-1} = 0$

$\text{cosec} 180° = \frac{r}{y} = \frac{1}{0} =$ undefined

$\sec 180° = \frac{r}{x} = \frac{1}{-1} = -1$

$\cot 180° = \frac{x}{r} = \frac{-1}{0} =$ undefined



Trigonometric ratio of 270°

In this case, the terminal side $\overrightarrow{OP}$ of angle 270° coincides with $\overrightarrow{OY'}$

Therefore $\overrightarrow{OP} = r = 1$

or $x = 0$ *and* $y = -1$

$\sin 270° = \frac{y}{r} = \frac{-1}{1} = -1$

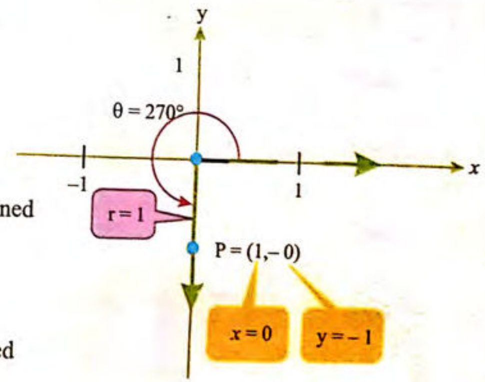
$\cos 270° = \frac{x}{r} = \frac{0}{1} = 0$

$\tan 270° = \frac{y}{x} = \frac{-1}{0} =$ undefined

$\text{cosec} 270° = \frac{r}{y} = \frac{1}{-1} = -1$

$\sec 270° = \frac{r}{x} = \frac{1}{0} =$ undefined

$\cot 270° = \frac{x}{r} = \frac{0}{-r} = 0$



Trigonometric ratio of 360°

In this case, the terminal side $\overrightarrow{OP}$ of angle 360° coincides with $\overrightarrow{OX}$.

Therefore $\overrightarrow{OP} = r = 1$

or $x = 1$ *and* $y = 0$

$$\sin 360° = \frac{y}{r} = \frac{0}{1} = 0$$

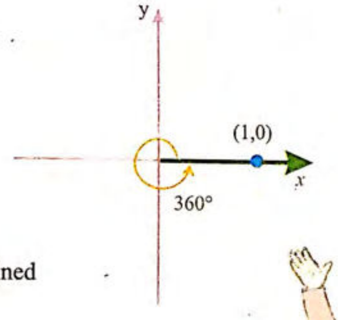
$$\cos 360° = \frac{x}{r} = \frac{1}{1} = 1$$

$$\tan 360° = \frac{y}{x} = \frac{0}{1} = 0$$

$$\text{cosec}360° = \frac{r}{y} = \frac{1}{0} = \text{undefined}$$

$$\sec 360° = \frac{r}{x} = \frac{1}{1} = 1$$

$$\cot 360° = \frac{x}{r} = \frac{1}{0} = \text{undefined}$$



Summarizing all the results, we have a table

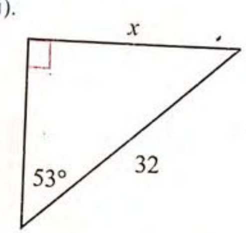
The trigonometric table

θ	0°	30°	30°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
cot θ	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
cosec θ	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

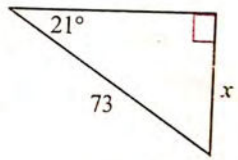
Exercise 7.4

1. Find the signs of the following trigonometric ratios and tell in which quadrant they lie?
   (i). sin 98° (ii). sin 160° (iii). tan 200°
   (iv). sec 120° (v). cosec 198° (vi). sin 460°
2. Find the trigonometric ratios of the following angles.
   (i). −180° (ii). −270° (iii). 720°
   (iv). 1470°
3. If $\sec\theta = 2$ where θ lies in 4th quadrant, find the other values of trigonometric ratios.
4. If $\sin\ \theta = \frac{4}{5}$ and $\frac{\pi}{2} < \theta < \pi$ then find other trigonometric ratios.
5. Find the values of
   (i). $2\sin 45° \cos 45°$ (ii). $\frac{\tan60° - \tan30°}{1 + \tan60° \tan30°}$ (iii). $\frac{\cos45°}{\sin45° + \tan45°}$
   (iv). $\tan30° \tan60° + \tan45°$ (v). $\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6}$
6. In which quadrant θ lies?
   (i). $\sin\theta > 0,\ \tan\theta > 0$ (ii). $\sin\theta < 0,\ \cot\theta > 0$ (iii). $\sin\theta > 0,\ \cos\theta < 0$
   (iv). $\cos\theta > 0,\ \text{cosec}\theta < 0$ (v). $\tan\theta < 0,\ \sec\theta > 0$ (vi). $\cos\theta < 0,\ \tan\theta < 0$
7. For each triangle, find each missing measure to two decimal places.

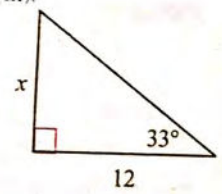
(i).



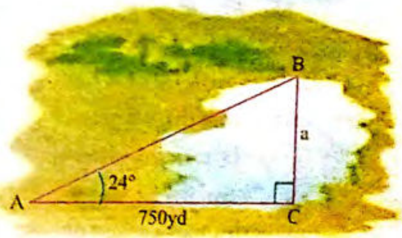
(ii).



(iii).



8. The irregular blue shape in the diagram represents a lake. The distance across the lake, *a*, is unknown. To find this distance, a surveyor took the measurements shown in the figure. What is the distance across the lake?



7.4 Trigonometric identities

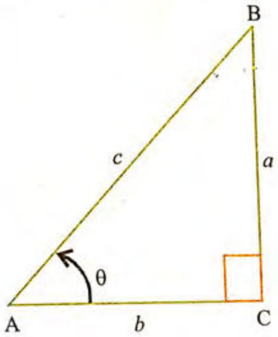
For any real number θ, we have following fundamental trigonometric identities.

(i) $\cos^2\theta + \sin^2\theta = 1$ (ii) $1 + \tan^2\theta = \sec^2\theta$ (iii) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Proof Consider a right triangle ABC in which

$m\angle ACB = 90^\circ$, $m\angle BAC = \theta$

$m\overline{AB} = c$, $m\overline{AC} = b$ and $m\overline{CB} = a$



Then we have $\sin\theta = \dfrac{a}{c}$, $\cos\theta = \dfrac{b}{c}$ (i)

$\tan\theta = \dfrac{a}{b}$, $\sec\theta = \dfrac{c}{b}$ (ii)

$\cot\theta = \dfrac{b}{a}$, $\operatorname{cosec}\theta = \dfrac{c}{a}$ (iii)

By Pythagoras theorem

$c^2 = b^2 + a^2$ (iv)

Dividing both sides by c^2, we have

$$\frac{c^2}{c^2} = \frac{b^2}{c^2} + \frac{a^2}{c^2}$$

$$\Rightarrow \quad 1 = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2$$

Putting the values of $\dfrac{b}{c}$ and $\dfrac{a}{c}$ from (i), we get

$$1 = (\cos\theta)^2 + (\sin\theta)^2$$

$$\Rightarrow \quad 1 = \cos^2\theta + \sin^2\theta$$

or $\boxed{\cos^2\theta + \sin^2\theta = 1}$ (v)

Now dividing equation (iv) by b^2, we have

$$\frac{c^2}{b^2} = \frac{b^2}{b^2} + \frac{a^2}{b^2}$$

$$\left(\frac{c}{b}\right)^2 = 1 + \left(\frac{a}{b}\right)^2$$

Putting values from (ii), we get

$$(\sec\theta)^2 = 1 + (\tan\theta)^2$$

$$\Rightarrow \quad \sec^2\theta = 1 + \tan^2\theta$$

or $\boxed{1 + \tan^2\theta = \sec^2\theta}$ (vi)

Again dividing Eq. (iv) by a^2, we have

$$\frac{c^2}{a^2} = \frac{b^2}{a^2} + \frac{a^2}{a^2}$$

$$\left(\frac{c}{a}\right)^2 = \left(\frac{b}{a}\right)^2 + 1$$

Putting values from (3), we get

$$(\operatorname{cosec}\theta)^2 = (\cot\theta)^2 + 1$$

$$\Rightarrow \quad \operatorname{cosec}^2\theta = \cot^2\theta + 1$$

or $\boxed{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$ (vii)

Example 16 Show that $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Solution L.H.S $= (\sin\theta + \cos\theta)^2$

$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$

$= 1 + 2\sin\theta\cos\theta$ $[\because \sin^2\theta + \cos^2\theta = 1]$

$= R.H.S$

Hence $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Example 17 Prove that $\sin\theta = \sqrt{1 - \cos^2\theta}$

Solution As we know that

$\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \sin^2\theta = 1 - \cos^2\theta$

$\Rightarrow \sin\theta = \sqrt{1 - \cos^2\theta}$

Which is the required result.

Example 18 Prove that $\dfrac{\sqrt{1 - \sin^2\theta}}{\sin\theta} = \cot\theta$

Solution L.H.S $= \dfrac{\sqrt{1 - \sin^2\theta}}{\sin\theta}$

As we know that

$\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \cos^2\theta = 1 - \sin^2\theta$

Hence L.H.S $= \dfrac{\sqrt{\cos^2\theta}}{\sin\theta} = \dfrac{\cos\theta}{\sin\theta} = \cot\theta = R.H.S$

Hence, $\dfrac{\sqrt{1 - \sin^2\theta}}{\sin\theta} = \cot\theta$

Example 19 Prove that $\sec^2\theta+\tan^2\theta=\dfrac{1+\sin^2\theta}{1-\sin^2\theta}$

Solution

$$\text{L.H.S} =\sec^2\theta+\tan^2\theta$$
$$=\frac{1}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}$$
$$=\frac{1+\sin^2\theta}{\cos^2\theta} \quad \text{.............................. (i)}$$

As we know that,

$$\cos^2\theta+\sin^2\theta=1$$
$$\Rightarrow \quad \cos^2\theta=1-\sin^2\theta$$

Putting value of $\cos^2\theta$ in (i), we get

$$\text{L.H.S}=\frac{1+\sin^2\theta}{1-\sin^2\theta}=\text{R.H.S}$$

Hence,

$$\sec^2\theta+\tan^2\theta=\frac{1+\sin^2\theta}{1-\sin^2\theta}$$

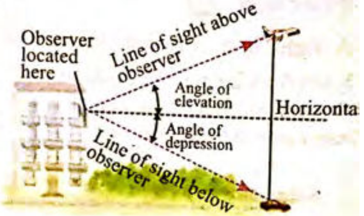
Exercise 7.5

Prove the following trigonometric identities.

1. $(\sec^2\theta-1)\cos^2\theta=\sin^2\theta$
2. $\tan\theta+\sec\theta=\dfrac{1+\sin\theta}{\cos\theta}$
3. $(\cos\theta-\sin\theta)^2=1-2\sin\theta\cos\theta$
4. $\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$
5. $\tan\theta+\cot\theta=\sec\theta\,\text{cosec}\theta$
6. $\dfrac{1-\sin\theta}{\cos\theta}=\dfrac{\cos\theta}{1+\sin\theta}$
7. $\sin\theta\sqrt{1+\tan^2\theta}=\tan\theta$
8. $\cos\theta=\sqrt{1-\sin^2\theta}$
9. $(1+\cos\theta)(1-\cos\theta)=\dfrac{1}{\text{cosec}^2\theta}$
10. $\cos x-\cos x\sin^2 x=\cos^3 x$
11. $\dfrac{\sin x}{1+\cos x}+\dfrac{1+\cos x}{\sin x}=2\,\text{cosec}\,x$
12. $\dfrac{\sin x}{1+\cos x}=\dfrac{1-\cos x}{\sin x}$
13. $\dfrac{1}{1+\cos a}+\dfrac{1}{1-\cos a}=2+2\cot^2 a$
14. $\cos^4 b-\sin^4 b=1-2\sin^2 b$
15. $\dfrac{\sin y+\cos y}{\sin y}+\dfrac{\cos y-\sin y}{\cos y}=\sec y\,\text{cosec}\,y$
16. $(\sec x-\tan x)^2=\dfrac{1-\sin x}{1+\sin x}$
17. $\sin x\tan x+\cos x=\sec x.$

7.5 Angle of Elevation and Depression

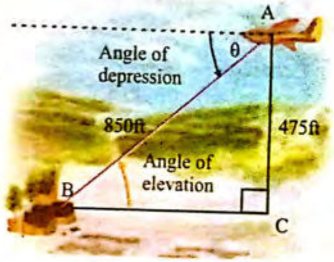
Many applications of right triangle trigonometry involve the angle made with an imaginary horizontal line. As shown in the figure, an angle formed by a horizontal line and the line of sight to an object that is above the horizontal line is called the angle of elevation. The angle formed by a horizontal line and the line of sight to an object that is below the horizontal line is called the angle of depression.



Example 20 An aerial photographer who photographs a farm house for a company has determined from experience that the best photo is taken at a height of approximately 475 ft and a distance of 850 ft from the farmhouse. What is the angle of depression from the plane to the house?

Solution

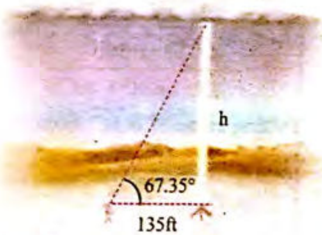
When parallel lines are cut by a transversal, alternate interior angles are equal. Thus the angle of depression from the plane to the house, θ, is equal to the angle of elevation from the house to the plane, so we can use the right triangle shown in the figure. Since we know the side opposite ∠B and the hypotenuse, we can find ∠B by using the sine function.



$$\sin\theta=\sin B=\frac{475\,ft}{850\,ft}\approx 0.5588$$
$$\theta\approx 34°$$

Thus the angle of depression is approximately 34°.

Example 21 To measure cloud height at night, a vertical beam of light is directed on a spot on the cloud. From a point 135 ft away from the light source, the angle of elevation to the spot is found to be 67.35°. Find the height of the cloud.



Solution From the figure, we have tan 67.35°

$$\tan 67.35°=\frac{h}{135}$$
$$h=135\tan 67.35°\approx 324\,ft$$

The height of the cloud is about 324 ft.

Tidbit

Trigonometry helps to find the height of buildings and mountains. Clinometer is used to obtain the angle of elevation of the top of an object.

Example 22

A light house is 300 meter above the sea level. Angles of depression of two boats from the top of light house are 30° and 45° respectively. If line joining the boats passes through the foot of the light house, find the distance between the boats when they are on the same side of the light house.

Solution

Let boats are at point C and D respectively.

Height of light house $m\overline{AB} = 300m$

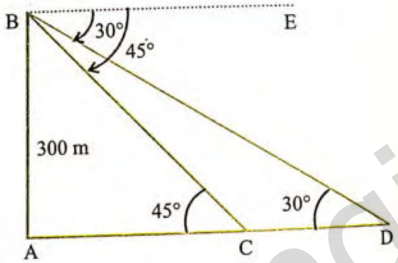
Distance between two boats $m\overline{CD} = ?$

Angles of depression are $m\angle EBD = 30°$ and $m\angle EBC = 45°$

But $m\angle EBD = m\angle BDA$ and $m\angle EBC = m\angle BCA$

Therefore $m\angle BDA = 30°$ and $m\angle BCA = 45°$

Consider ΔABC



$$\tan 45° = \frac{m\overline{AB}}{m\overline{AC}}$$

$$1 = \frac{300\,m}{m\overline{AC}}$$

or $m\overline{AC} = 300\,m$ (i)

Now

Consider ΔABD

$$\tan 30° = \frac{m\overline{AB}}{m\overline{AD}}$$

$$\frac{1}{\sqrt{3}} = \frac{300}{m\overline{AC} + m\overline{CD}} \qquad \because m\overline{AD} = m\overline{AC} + m\overline{CD}$$

Putting value of $m\overline{AC}$ from (i).

$$\frac{1}{\sqrt{3}} = \frac{300}{300 + \overline{CD}}$$

$\Rightarrow \quad 300 + m\overline{CD} = 300\sqrt{3}$

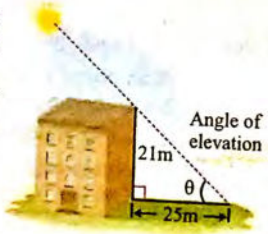
$\Rightarrow \quad m\overline{CD} = 300\sqrt{3} - 300$

$m\overline{CD} = 300\left(\sqrt{3}-1\right)m = 219.6\,m$

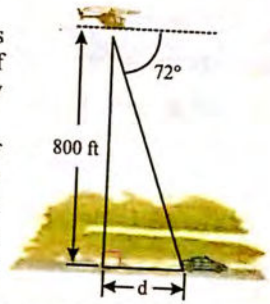
Hence distance between two boats is 219.6 m.

Exercise 7.6

1. A building that is 21 metres tall casts a shadow 25 metres long. Find the angle of elevation of the sun to the nearest degree.



2. A light house is 150 m above the sea level. Angle of depression of a boat from its top is 60°. Find the distance between the boat and the lighthouse.
3. A tree is 50 m high. Find the angle of elevation of its top to a point, on the ground 100 m away from the foot of tree.
4. From the top of hill 240 m high, measure of the angles of depression of the top and bottom of minaret are 30° and 60° respectively. Find the height of minaret.
5. A police helicopter is flying at 800 feet. A stolen car is sighted at an angle of depression of 72°. Find the distance of the stolen car, to the nearest foot, from a point directly below the helicopter.



6. A light house is 300 m above the sea level. The angle of depression of two boats from the top of light house are 30° and 45° respectively. If the line of joining of the boats passes through the foot of light house, find the distance between two boats when they are on the opposite side of the light house.
7. The angle of elevation of the top of a cliff is 30°. Walking 210 metre from the point towards the cliff, the angle of elevation becomes 45°. Find the height of cliff.

Activity

Find the error.

Zia and Rabia are finding the height of the hill.

Zia

$\sin 15° = \frac{x}{580}$

$580(\sin 15°) = x$

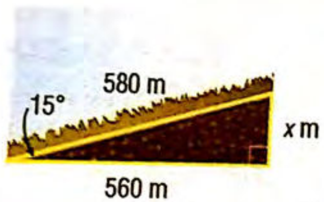
$150 \approx x$

Rabia

$\sin 15° = \frac{x}{560}$

$560(\sin 15°) = x$

$145 \approx x$



Who is correct?

Review Exercise 7

1. At the end of each question, four circles are given. Fill in the correct circle only.

(i). If an object is above the level of observer then the angle formed between the horizontal line and observer's line of sight is called

- ○ angle of depression
- ○ angle of elevation
- ○ obtuse angle
- ○ none of the above

(ii). cot = ________.

○ $\frac{\sin\theta}{\cos\theta}$ ○ $\frac{1}{\cos\theta}$ ○ $\frac{\cos\theta}{\sin\theta}$ ○ $\frac{1}{\sin\theta}$

(iii). $1+\tan^2\theta =$ ______

○ $\sin^2\theta$ ○ $\cos^2\theta$ ○ $\text{cosec}^2\theta$ ○ $\sec^2\theta$

(iv). If $\tan\theta = 1$ then $\sin\theta =$ ________ when θ lies in 3rd quadrant.

○ $\frac{1}{2}$ ○ $\frac{-1}{2}$ ○ $-\frac{1}{\sqrt{2}}$ ○ $\frac{1}{\sqrt{2}}$

(v). Sin (–350°) lies in ________.

○ 1st quadrant ○ 2nd quadrant ○ 3rd quadrant ○ 4th quadrant

(vi). 45°= ________ radian.

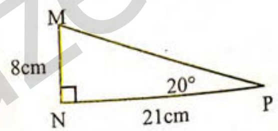
○ $\frac{\pi}{3}$ ○ $\frac{\pi}{4}$ ○ $\frac{\pi}{6}$ ○ $\frac{\pi}{2}$

(vii). If the measure of the hypotenuse of a right triangle is 5 feet and $m\angle B = 58°$, what is the measure of the leg adjacent to ∠B?

○ 4.2402 ○ 8.0017 ○ 0.10060 ○ 2.6496

(viii). Find the value of tan P to the nearest tenth.

○ 2.6 ○ 0.5
○ 0.4 ○ 0.1

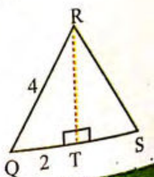


(ix). RT is equal to TS. What is ∠S?

○ $2\sqrt{6}$ ○ $2\sqrt{3}$ ○ $4\sqrt{3}$ ○ $2\sqrt{2}$

(x). RT is equal to TS. What is ∠S?

○ 25° ○ 30° ○ 45° ○ 60°



2. Convert 45°35′30″ into decimal form.

3. Convert 216.67° into $D°M'S''$ form.

4. Through how many radians does a minute of a clock turn through
(i). in 45 minutes (ii). in one hour.

5. Find coterminal angle of 190° and –250°.

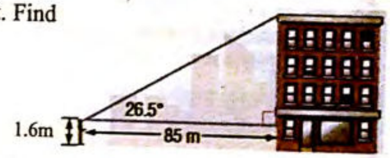
6. Find trigonometric ratios of
(i). 390° (ii). –240°

7. Prove that

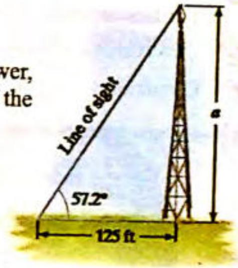
(i). $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$ (ii). $2\cos\theta\sec\theta - \tan\theta\cot\theta = 1$

8. If $\sec\theta = 2$ and θ does not lie in first quadrant. Find remaining trigonometric ratios.

9. Refer to the diagram shown. What is the height of the building?



10. From a point on level ground 125 feet from the base of a tower, the angle of elevation is 57.2°. Approximate the height of the tower to the nearest foot.



The Amazing number 1,089

1089 • 1 = 1089	1089 • 9 = 9801
1089 • 2 = 2178	1089 • 8 = 8712
1089 • 3 = 3267	1089 • 7 = 7623
1089 • 4 = 4356	1089 • 6 = 6534

A unique property among two digit numbers.
$33^2 = 1{,}089 = 65^2 - 56^2$

Summary

- An angle is a union of two rays which have a common point (vertex). One of the ray is called 'initial side' and other ray is called 'terminal side'.
- Sexagesimal system (degrees, minutes, seconds) is the system of measurement of an angle in which one complete rotation is divided into 360 parts called degrees, written as 360°. One degree is divided into 60 parts called minutes, written as 60′ and one minute is again divided into 60 parts called seconds, written as 60″.
- In Circular system (radians) unit of measure of angle is radian. One radian is an angle subtended at the centre of a circle an arc whose length is equal to radius of the circle.
- Coterminal angles are angles having the same initial and terminal sides and differ by a multiple of 2 radians or 360°. They are also called general angles.
- Angles are in standard position if the vertex of an angle lies at the origin, and initial side lies on positive x-axis.
- Quadrants are obtained when XY-plane is divided into four equal parts.
- Quadrant angles are 0°, 90°, 180°, 270, 360°.
- Relationship between radian and degree measure

  $1° = \frac{\pi}{180}$ radian = 0.0175 radian and 1 radian = $\left(\frac{180}{\pi}\right)^0$ = 57.295 degrees.

  Relation between central angle and arc length of a circle: $l = r\theta$

  Area of a circular sector, $A = \frac{1}{2}r^2\theta$

  two or more than two angles with the same initial and terminal sides are called coterminal angles.
- There are six trigonometric ratios: $\sin\theta, \cos\theta, \tan\theta, \sec\theta, \cot\theta, \text{cosec}\,\theta$.
- Trigonometric identities are
  - (a) $\cos^2\theta + \sin^2\theta = 1$
  - (b) $1 + \tan^2\theta = \sec^2\theta$
  - (c) $1 + \cot^2\theta = \text{cosec}^2\theta$
- Angle of elevation is the angle between the horizontal line and observer's line of sight when looking at the top of a wall.
- Angle of depression is the angle between the horizontal line and observer's line of sight when looking at the bottom of a wall.

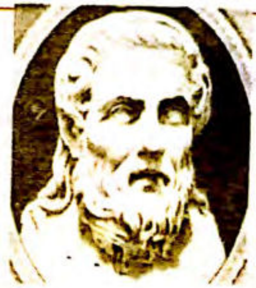
Unit 8 PROJECTION OF A SIDE OF A TRIANGLE

In this unit the students will be able to

Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' Theorem).

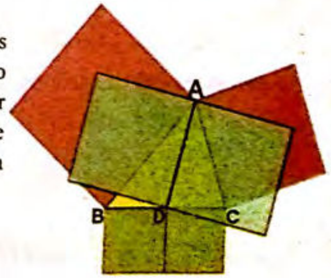
Why it's important



In geometry, **Apollonius's theorem** is a theorem relating the length of a median of a triangle to the lengths of its side. It states that "the sum of the squares of any two sides of any triangle equals twice the square on half the third side, together with twice the square on the median bisecting the third side".

Specifically, in any triangle ABC, if AD is a median, then

$$|AB|^2 + |AC|^2 = 2(|AD|^2 + |BD|^2)$$



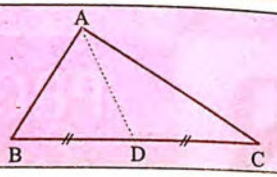
It is a special case of Stewart's theorem. For an isosceles triangle with $|AB| = |AC|$, the median AD is perpendicular to BC and the theorem reduces to the Pythagorean Theorem for triangle ADB (or triangle ADC). From the fact that the diagonals of a parallelogram bisect each other, the theorem is equivalent to the parallelogram law.

The theorem is named after Apollonius of Perga.

Math formula (Apollonius theorem)

If AD is the median, then:

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$



Theorem 8.1

In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.

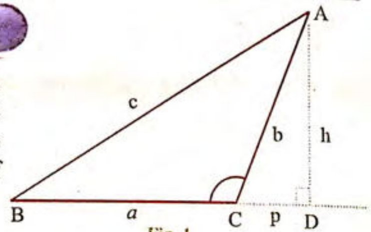


Fig. 1

Given

In ΔABC, $\angle C$ is the obtuse angle. Let $m\overline{BC} = a$, $m\overline{AC} = b$ and $m\overline{AB} = c$. $\overline{AD}$ is the perpendicular from A to $\overline{BC}$ (produced) so that $\overline{CD}$ is the projection of $\overline{AC}$ on $\overrightarrow{BC}$. Let $m\overline{CD} = p$ and $m\overline{AD} = h$.

To prove

$$(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{CA})^2 + 2(m\overline{BC})\cdot(m\overline{CD})$$

or $c^2 = a^2 + b^2 + 2ap$

Proof

Statements	Reasons
In a right angled triangle ΔADB	$\overline{AD} \perp \overrightarrow{BC}$
$\therefore \quad (m\overline{AB})^2 = (m\overline{BD})^2 + (m\overline{AD})^2$	Pythagoras theorem
or $\quad (m\overline{AB})^2 = (m\overline{BC} + m\overline{CD})^2 + (m\overline{AD})^2$	$m\overline{BD} = m\overline{BC} + m\overline{CD}$
or $\quad (m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{CD})^2 + 2(m\overline{BC})\cdot(m\overline{CD}) + (m\overline{AD})^2 \longrightarrow$(i)	$(x+y)^2 = x^2 + y^2 + 2xy$
Again in the right angled triangle ΔADC $\quad (m\overline{AC})^2 = (m\overline{CD})^2 + (m\overline{AD})^2$	Pythagoras theorem.
or $\quad (m\overline{AC})^2 - (m\overline{CD})^2 = (m\overline{AD})^2 \longrightarrow$(ii)	Equal can be subtracted from the equal without changing the value.
so that (i) becomes $(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{CD})^2$	Putting the value of $(m\overline{AD})^2$ from (ii) in (i).

Statements	Reasons
$+2(m\overline{BC})\cdot(m\overline{CD}) + (m\overline{AC})^2 - (m\overline{CD})^2$	
$(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{AC})^2 +2(m\overline{BC})\cdot(m\overline{CD}) + (m\overline{CD})^2 - (m\overline{CD})^2$	Commutative property of addition of real numbers.
$\therefore (m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{AC})^2 + 2(m\overline{BC})(m\overline{CD})$	The difference in two equal numbers is zero.
$c^2 = a^2 + b^2 + 2ap$	

Corollary – I:

In a ΔABC with obtuse angle at C. If $\overline{BD}$ is perpendicular on $\overline{AC}$ produced and m $\overline{BC}$ = m $\overline{AC}$ then prove that

$(\overline{AB})^2 = 2\ (\overline{AC})\ (\overline{AD})$

or

$c^2 = 2(b)\ (b + p)$

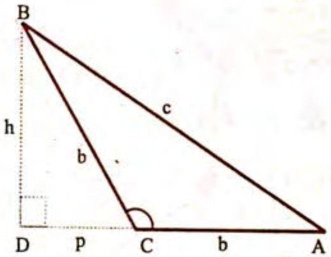


Fig. 2

Given

A ΔABC having on obtuse angle $m\angle ACB$, m $\overline{BC}$ = m $\overline{AC}$ and $\overline{BD}$ is perpendicular on $\overleftrightarrow{AC}$ produced.

To prove

$(m\overline{AB})^2 = 2(m\overline{AC})\ (m\overline{AD})$

or

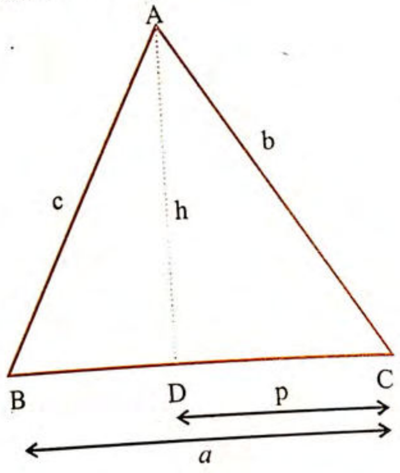
$c^2 = 2b\ (b + p)$

Proof

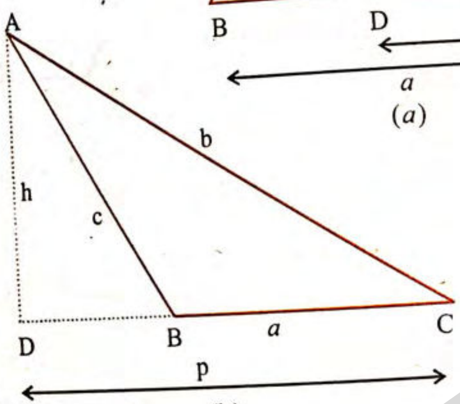
Statement	Reasons
$(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{AC})^2 + 2(m\overline{AC})\ (m\overline{CD})$	By Theorem 8.1
$= (m\overline{AC})^2 + (m\overline{AC})^2 + 2(m\overline{AC})\ (m\overline{CD})$	
$= 2(m\overline{AC})^2 + 2(m\overline{AC})\ (\overline{CD})$	
$= 2(m\overline{AC})\ (m\overline{AC} + m\overline{CD})$	
$= 2(m\overline{AC})\ (m\overline{AD})$	Point C is inbetween $\overline{AD}$
$(m\overline{AB})^2 = 2(m\overline{AC})\ (m\overline{AD})$ or $c^2 = 2b\ (b + p)$	

Theorem 8.2

In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished twice the rectangle contained by one of those sides, and the projection on it of the other.



(a)



(b)

Given

The given triangle is either acute as in figure (a) or obtuse as in figure (b). In the ΔABC, $m\angle C$ is acute. $\overline{AD}$ is the perpendicular from A to $\overline{CB}$ (produced if necessary as in figure b) so that $\overline{CD}$ is the projection of $\overline{CA}$ on $\overline{CB}$. Let us denote $m\overline{BC} = a$, $m\overline{AC} = b$, $m\overline{AB} = c$, $m\overline{CD} = p$, the projection and $m\overline{AD} = h$.

To prove

$$(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{AC})^2 - 2(m\overline{BC})\cdot(m\overline{CD})$$

or $\quad c^2 = a^2 + b^2 - 2ap$

Proof

Statements	Reasons
ΔADB is a right-triangle	$\because \overline{AD} \perp \overline{BC}$ (produced where necessary)
$\therefore (m\overline{AB})^2 = (m\overline{BD})^2 + (m\overline{AD})^2$	Pythagoras theorem
Also $m\overline{BD} + m\overline{CD} = m\overline{BC}$	Segment addition postulate (fig. a)
$\therefore (m\overline{AB})^2 = (m\overline{BC} - m\overline{CD})^2 + (m\overline{AD})^2 \longrightarrow$ (I)	$\because m\overline{BD} = m\overline{BC} - m\overline{CD}$
Again $m\overline{BD} + m\overline{BC} = m\overline{CD}$	Segment addition postulate (fig. b)
$\therefore (m\overline{AB})^2 = (m\overline{CD} - m\overline{BC})^2 + (m\overline{AD})^2 \longrightarrow$ (II)	Since $m\overline{BD} = m\overline{CD} - m\overline{BC}$
But the R.H.S of (i) and (ii) are the same so we can select any one of the above equations. Thus from (i), we have	$\because (m\overline{BC} - m\overline{CD})^2 = (m\overline{CD} - m\overline{BC})^2$
$(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{CD})^2 - 2(m\overline{BC})\cdot(m\overline{CD}) + (m\overline{AD})^2$	
or $(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{CD})^2 + (m\overline{AD})^2 - 2(m\overline{BC})\cdot(m\overline{CD}) \longrightarrow$ (III)	Commutative property of addition of real numbers.
But $(m\overline{CD})^2 + (m\overline{AD})^2 = (m\overline{AC})^2 \longrightarrow$ (IV)	By Pythagoras theorem.
$\therefore (m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{AC})^2 - 2(m\overline{BC})\cdot(m\overline{CD})$	Using IV in III.
or $\quad c^2 = a^2 + b^2 - 2ap$	

Did You Know?

Some odd multiplications

$37037037037 \times 9 = 333333333333$

$13717421 \times 9 = 123456789$

Exercise 8.1

1. In ΔABC, $m\overline{AB} = 6cm$, $m\overline{BC} = 10cm$, $m\angle B = 120°$. The projection of $\overline{BC}$ on $\overline{AB}$ is 5 cm. Find $m\overline{AC}$.
2. In ΔABC, $m\overline{AB} = 3cm$, $m\overline{BC} = 5cm$, $m\overline{AC} = 7cm$. Find the projection of $\overline{BC}$ on $\overline{AB}$.
3. In ΔABC, $a=7$, $b=11$, $c=8$. Calculate the projection of $\overline{AC}$ on $\overline{AB}$.
4. In a parallelogram ABCD, m $\overline{AB}$ = 4cm, m $\overline{AC}$ = 7cm, m $\overline{AD}$ = 5cm. Find which of the angles of the parallelogram are obtuse.

Theorem 8.3

(Apollonius Theorem)

In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' theorem)

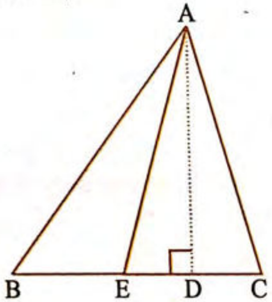


Fig. (a)

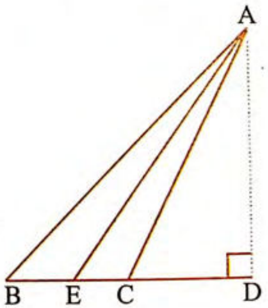


Fig. (b)

Given

In the ΔABC, $\overline{AE}$ is the median drawn from the vertex A to $\overline{BC}$.

To prove

$$(m\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{EB})^2 + 2(m\overline{EA})^2$$

Construction

From A, draw

$\overline{AD} \perp \overline{BC}$ (produced if necessary)

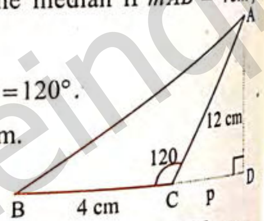
Proof

Statements	Reasons
Since ΔABD and ΔADC are right triangles,	Construction
$\therefore (m\overline{AB})^2 = (m\overline{AD})^2 + (m\overline{BD})^2 \longrightarrow (1)$	Pythagoras theorem
$(m\overline{AC})^2 = (m\overline{AD})^2 + (m\overline{DC})^2 \longrightarrow (2)$	Pythagoras theorem
Therefore, $(m\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{AD})^2 + (m\overline{BD})^2 + (m\overline{DC})^2$	Adding (1) and (2)
But $m\overline{BD} = m\overline{BE} + m\overline{ED}$	Segments addition postulate
So that the last result becomes $m(\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{AD})^2 + (m\overline{BE} + m\overline{ED})^2 + (m\overline{DC})^2 \longrightarrow (3)$	
Since $m\overline{DC} = m\overline{EC} - m\overline{ED}$	From construction in figure (a)
and $m\overline{DC} = m\overline{ED} - m\overline{EC}$	From construction in figure (b)
$\therefore (m\overline{DC})^2 = (m\overline{EC} - m\overline{ED})^2 = (m\overline{ED} - m\overline{EC})^2 \longrightarrow (4)$	From algebra $(a-b)^2 = (b-a)^2$
so any one of these values can be put in place of $(m\overline{DC})^2$ and thus (3) becomes $(m\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{AD})^2 + (m\overline{BE} + m\overline{ED})^2 + (m\overline{EC} - m\overline{ED})^2$	Using (4) in (3)
or $(m\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{AD})^2 + (m\overline{BE})^2 + (m\overline{ED})^2 + 2(m\overline{BE})\cdot(m\overline{ED}) + (m\overline{EC})^2 + (m\overline{ED})^2 - 2(m\overline{EC})\cdot(m\overline{ED})$	$\because$ from algebra $(a+b)^2 = a^2 + b^2 + 2ab$ and $(a-b)^2 = a^2 + b^2 - 2ab$
or $(m\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{AD})^2 + (m\overline{ED})^2 + (m\overline{ED})^2 + (m\overline{BE})^2 + (m\overline{EC})^2$	Using commutative property of addition for real numbers.

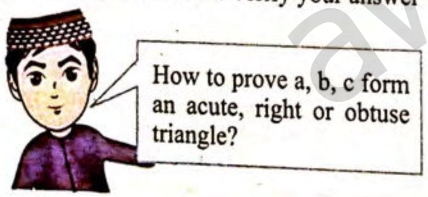
$+2(m\overline{BE})\cdot(m\overline{ED})-2(m\overline{EC})\cdot(m\overline{ED})$	
or $(m\overline{AB})^2+(m\overline{AC})^2=2(m\overline{AD})^2$	$\because m\overline{BE}=m\overline{EC}$ as $\overline{AE}$ is the median.
$+2(m\overline{ED})^2+(m\overline{BE})^2+(m\overline{BE})^2$	
$+2(m\overline{BE})\cdot(m\overline{ED})-2(m\overline{BE})\cdot(m\overline{ED})$	
or $(m\overline{AB})^2+(m\overline{AC})^2=2\left[(m\overline{AD})^2+(m\overline{ED})^2\right]$	$\because x-x=0 \quad \forall x\in R$
$+2(m\overline{BE})^2 \longrightarrow (5)$	
But $(m\overline{AD})^2+(m\overline{ED})^2=(m\overline{AE})^2 \longrightarrow (6)$	ΔADE is a right-triangle. Using (6) in (5).
$\therefore (m\overline{AB})^2+(m\overline{AC})^2=2(m\overline{AE})^2+2(m\overline{BE})^2$	$\because m\overline{EB}=m\overline{BE}$ and
or $(m\overline{AB})^2+(m\overline{AC})^2=2(m\overline{EB})^2+2(m\overline{EA})^2$	$m\overline{AE}=m\overline{EA}$

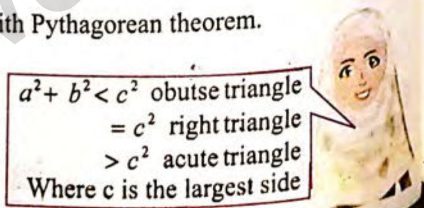
Exercise 8.2

1. Using Apollonius theorem find the length of the medians of a triangle having sides 10 cm, 12 cm and 16 cm respectively.
2. In ΔABC, D is the mid-point of $\overline{BC}$. Find the length of the median if $m\overline{AB}=4cm$, $m\overline{BC}=5cm$ and $m\overline{AC}=6cm$.
3. ΔABC is given with $m\overline{BC}=4cm$, $m\overline{AC}=12cm$ and $m\angle C=120°$. Find $m\overline{CD}$, $m\overline{AD}$, $m\overline{AB}$ and then verify Apollonius theorem.



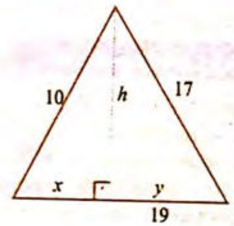
4. ΔABC is a right triangle with $m\angle B=90°$ and $\overline{BD}\perp\overline{AC}$. If $m\overline{AB}=6cm$ and $m\overline{BC}=5cm$, find $m\overline{AD}$ and $m\overline{CD}$. Verify your answer with Pythagorean theorem.





Review Exercise 8

1. At the end of each question, four circles are given. Fill in the correct circle only.
   (i). Phythagoras was a /an ________ mathematician.
   ○ Indian ○ Greek ○ Pakistani ○ Chinese
   (ii). Apollonius was born in ________.
   ○ Perga ○ Istanbul ○ Islamabad ○ Kabul
   (iii). Apollonius is the name of a ________.
   ○ City ○ Town ○ Country ○ Mathematician
2. Find the projection of the side of measure 10 units upon the side of measure 17 units in a triangle whose sides respectively have measures 10, 17 and 21 units.
3. ΔABC is a right -triangle with $m\angle A=90°$. From the vertex A perpendicular $\overline{AD}$ is drawn on $\overline{BC}$. If $m\overline{AB}=5\ cm$, $m\overline{AC}=8\ cm$. Find $m\overline{BC}$, $m\overline{AD}$ and $m\overline{BD}$.
4. In the above question, $\overline{BE}$ is a median. Find $m\overline{BE}$ using
   a. Pythagoras theorem
   b. Apollonius theorem
5. Find h, x and y in the figure.



Summary

- A median of a triangle is a segment from one vertex of the triangle to the midpoint of the opposite side.
- In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' Theorem).

Unit

9 CHORDS OF A CIRCLE

In this unit the students will be able to

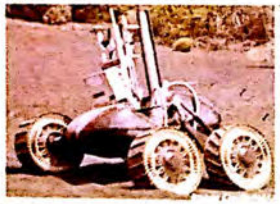
To prove the following theorems along with corollaries and apply them to solve appropriate problems.

- One and only one circle can pass through three non- collinear points.
- A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.
- Perpendicular from the centre of a circle on a chord bisects it.
- If two chords of a circle are congruent then they will be equidistant from the centre.
- Two chords of a circle which are equidistant from the centre are congruent.

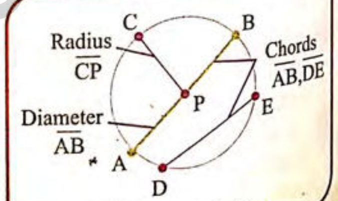
Why it's important

One of the most useful applications of the circles is the wheel. From ancient times until today, the wheel has made an enormous contribution to progress in travel, transportation, industry, and other elements of civilization.

A **circle** is the set of all points in a plane that are equidistant from a given point in the plane known as the center of the circle. A **radius** (plural, radii) is a line segment from the centre of the circle to a point on the circle.



A **chord** is a line segment whose endpoints lie on the circumference of a circle. It divides a circle into two segments. The larger part is called major segment and the smaller part, the minor segment. If the chord happens to be a diameter, each segment is a semicircle. A **diameter** is a chord that contains the centre of a circle. A circle can be named by using the symbol ⊙ and the center of the circle. The circle in the illustration is ⊙ P, or circle P.



⊙ P, or circle P

Theorem 9.1

One and only one circle can pass through three non-collinear points.

Given

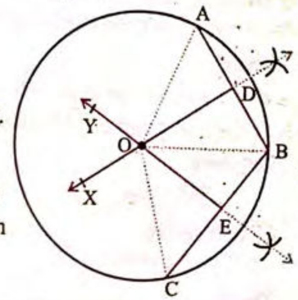
A, B and C are three non-collinear points.

To prove

One and only one circle can pass through the points A, B and C.

Construction

Join B to A and C. Draw perpendicular bisectors $\overleftrightarrow{XD}$ and $\overleftrightarrow{YE}$ of $\overline{AB}$ and $\overline{BC}$ respectively, which intersect at O. Join O to A, B and C.

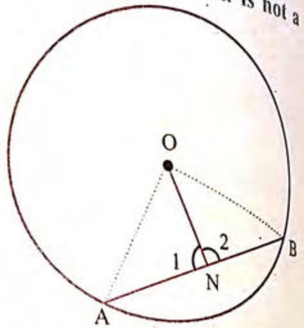


Proof

Statements	Reasons
In $\Delta OAD \longleftrightarrow \Delta OBD$	
$\angle ODA \cong \angle ODB$	Both are right angles.
$\overline{AD} \cong \overline{BD}$	D is the mid-point of $\overline{AB}$.
$\overline{OD} \cong \overline{OD}$	Common.
$\therefore \Delta OAD \cong \Delta OBD$	S.A.S postulate.
$\therefore \overline{OA} \cong \overline{OB} \longrightarrow (i)$	Corresponding sides of congruent triangles.
Again in $\Delta OBE \longleftrightarrow \Delta OCE$	
$\angle OEB \cong \angle OEC$	Both are right angles.
$\overline{BE} \cong \overline{EC}$	E is the midpoint of $\overline{BC}$
$\overline{OE} \cong \overline{OE}$	Common.
$\therefore \Delta OBE \cong \Delta OCE$	S.A.S postulate.
$\therefore \overline{OB} \cong \overline{OC} \longrightarrow (ii)$	Corresponding sides of congruent triangles.
From (i) and (ii), $\overline{OA} \cong \overline{OB} \cong \overline{OC}$ $\Rightarrow m\overline{OA} = m\overline{OB} = m\overline{OC}$ It means that the point O is equidistant from the three points A, B, C. Therefore a circle with centre O and radius $m\overline{OA}$ or $m\overline{OB}$ or $m\overline{OC}$ will pass through the points A, B and C. Since O is the only point which is equidistant from the points A,B and C. Therefore one and only one circle can pass through these three non-collinear points.	Transitive property of congruence.

Theorem 9.2

A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.



Given

A circle with centre at O, $\overline{AB}$ is a chord of the circle.
N is the mid point of $\overline{AB}$ which is joined to O.

To prove

$\overline{ON} \perp \overline{AB}$

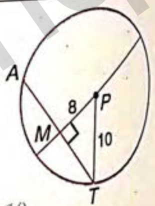
Construction

Join O to A and B.

Proof

Statements	Reasons
In $\Delta OAN \longleftrightarrow \Delta OBN$	
$\overline{OA} \cong \overline{OB}$	Radii of a given circle
$\overline{AN} \cong \overline{BN}$	Given
$\overline{ON} \cong \overline{ON}$	Common
$\therefore \Delta OAN \cong \Delta OBN$	$S.S.S \cong S.S.S$
$\therefore \angle 1 \cong \angle 2$	Corresponding angles of the congruent triangles
But $m\angle 1 + m\angle 2 = 180° \longrightarrow (i)$	Supplementary angles postulate
$\therefore m\angle 1 = m\angle 2 = 90°$	
Hence $\overline{ON} \perp \overline{AB}$	$\because m\angle 1 = m\angle 2$

Example 1 In $\odot P$, if $\overline{PM} \perp \overline{AT}$, $PT = 10$, and $PM = 8$, find MT.

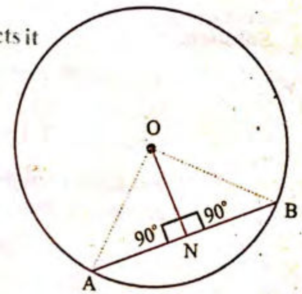


Solution

$\angle PMT$ is a right angle.	*Def. of perpendicular*
ΔPMT is a right triangle.	*Def. of right triangle*
$(MT)^2 + (PM)^2 = (PT)^2$	*Pythagorean Theorem*
$(MT)^2 + 8^2 = 10^2$	*Replace PM with 8 and PT with 10.*
$(MT)^2 + 64 = 100$	*$8^2 = 64; 10^2 = 100$*
$(MT)^2 + 64 - 64 = 100 - 64$	*Subtract 64 from each side.*
$(MT)^2 = 36$	*Simplify.*
$\sqrt{(MT)^2} = \sqrt{36}$	*Take the square root of each side.*
$MT = 6$	*Simplify.*

Theorem 9.3

Perpendicular from the centre of a circle on a chord bisects it



Given

A circle with centre at O, $\overline{AB}$ is a chord. $\overline{ON} \perp \overline{AB}$.

To prove

$\overline{AN} \cong \overline{BN}$

Construction

Join O to A and B.

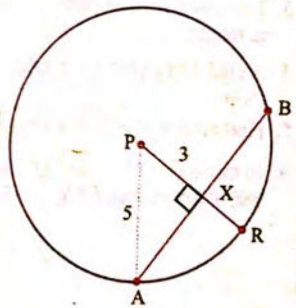
Proof

Statements	Reasons
In $\Delta AON \longleftrightarrow \Delta BON$	
$\angle ONA \cong \angle ONB$	Right angles
$\overline{ON} \cong \overline{ON}$	common
$\overline{OA} \cong \overline{OB}$	Radii of the same circle
$\therefore \Delta AON \cong \Delta BON$	H.S $\cong$ H.S
Hence $\overline{AN} \cong \overline{BN}$	Corresponding sides of congruent triangles

Example 2

$\odot$P has a radius of 5 cm. and PX is 3 cm
$\overline{PR}$ is perpendicular to $\overline{AB}$ at point X. Find AB.

Solution



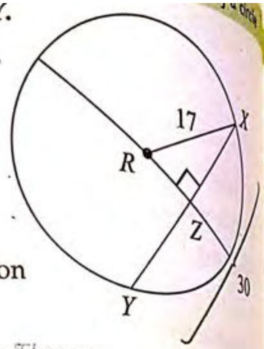
By the Pythagorean theorem:

$$(AX)^2 + 3^2 = 5^2$$
$$(AX)^2 = 5^2 - 3^2$$
$$(AX)^2 = 16$$
$$AX = 4$$

By the radius and chord theorem, $\overline{PR}$ bisects $\overline{AB}$, so BX = AX = 4.

Therefore, AB = AX + BX = 4 + 4 = 8 cm.

Example 3 In $\odot R$, $XY = 30$, $RX = 17$, and $\overline{RZ} \perp \overline{XY}$. Find the distance from R to $\overline{XY}$.



Solution

The measure of the distance from R to $\overline{XY}$ is RZ. Since $\overline{RZ} \perp \overline{XY}$, $\overline{RZ}$ bisects $\overline{XY}$, by Theorem 9.3. Thus, $XZ = \frac{1}{2}(30)$ or 15.

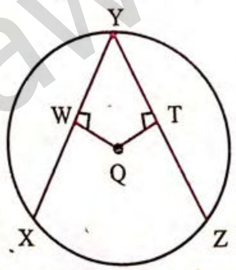
For right triangle RZX, the following equation can be written.

$(RZ)^2 + (XZ)^2 = (RX)^2$	*Pythagorean Theorem*
$(RZ)^2 + 15^2 = 17^2$	*Replace XZ with 15 and RX with 17.*
$(RZ)^2 + 225 = 289$	*$15^2 = 225$; $17^2 = 289$*
$(RZ)^2 + 225 - 225 = 289 - 225$	*Subtract 225 from each side.*
$(RZ)^2 = 64$	*Simplify.*
$\sqrt{(RZ)^2} = \sqrt{64}$	*Take the square root of each side.*
$RZ = 8$	*Simplify.*

The distance from R to $\overline{XY}$, or $\overline{RZ}$, is 8 units.

Exercise 9.1

1. If the radius of a circle is 30 cm, find the length of a chord which is 10 cm from the centre.
2. If a chord of a circle is 48 cm long and its distance from the centre is 18 cm. Find the diameter of the circle.
3. The diameter of a circle is 5 units long. How far from the centre is a chord which is 4 units in length.
4. A chord of a circle is 8 cm in length is drawn 5 cm from the centre. Find the length of the radius.
5. Find the length of a chord that is at a distance of 5cm from the centre of a circle with radius 8cm.
6. In circle Q, $\overline{QW} \perp \overline{XY}$, $\overline{QT} \perp \overline{YZ}$, QW = 5, QT = 5 and YT = 12. Find XY.

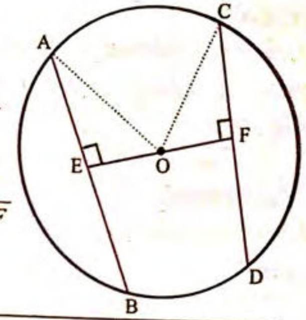


Theorem 9.4

If two chords of a circle are congruent then they will be equidistant from the centre.

Given

A circle with centre O, $\overline{AB}$ and $\overline{CD}$ are two congruent chords of the circle.



To prove

$\overline{AB}$ and $\overline{CD}$ are equidistant from the centre O.

Construction

Join O to A and C. Also draw perpendiculars $\overline{OE}$ and $\overline{OF}$ on the given chords $\overline{AB}$ and $\overline{CD}$ respectively.

Proof

Statements	Reasons
Since $\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$	Construction
$\therefore \overline{AE} \cong \overline{EB}$ and $\overline{CF} \cong \overline{DF}$	By the use of Theorem 9.3.
Or $m\overline{AE} = m\overline{EB}$ and $m\overline{CF} = m\overline{DF}$	
But $m\overline{AB} = m\overline{CD}$	Given
or $m\overline{AE} + m\overline{EB} = m\overline{CF} + m\overline{DF}$	Segment addition postulate
$m\overline{AE} + m\overline{AE} = m\overline{CF} + m\overline{CF}$	$\because m\overline{EB} = m\overline{AE}$ and $m\overline{DF} = m\overline{CF}$
$2m\overline{AE} = 2m\overline{CF}$	Adding equal quantities
$m\overline{AE} = m\overline{CF}$	Dividing both sides by 2.
or $\overline{AE} \cong \overline{CF} \longrightarrow (i)$	
Now in $\Delta AOE \longleftrightarrow \Delta COF$	
$\overline{OA} \cong \overline{OC}$	Radii of the same circle
$\overline{AE} \cong \overline{CF} \longrightarrow (ii)$	From (i) proved above
$\angle AEO \cong \angle CFO$	Right angles
$\therefore \Delta AOE \cong \Delta COF$	H.S $\cong$ H.S
$\therefore \overline{OE} \cong \overline{OF}$ or $m\overline{OE} = m\overline{OF}$	Corresponding sides of the congruent triangles.
$\therefore \overline{AB}$ and $\overline{CD}$ are equidistant from the centre of the circle .	

Corollary

In congruent circles, congruent chords are equidistant from the centres.

Theorem 9.5

...chords of a circle which are equidistant from the centre are congruent.

...ven

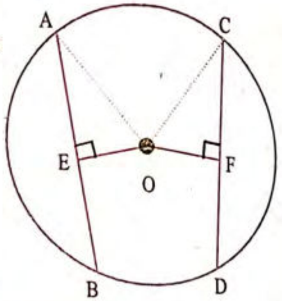
...rcle with centre at O, $\overline{AB}$ and $\overline{CD}$ are two chords of the ...e. $\overline{OE} \perp \overline{AB}$, $\overline{OF} \perp \overline{CD}$ and $\overline{OE} \cong \overline{OF}$.

...prove

$\overline{AB} \cong \overline{CD}$

...nstruction

...O with A and C.



...oof

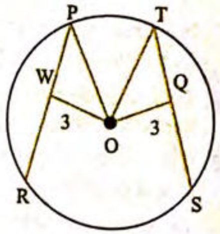
Statements	Reasons
$\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$	Given
$\therefore m\overline{AE} = m\overline{EB}$ and $m\overline{CF} = m\overline{FD} \longrightarrow (i)$	Theorem 9.3
...Now in $\Delta AEO \longleftrightarrow \Delta CFO$	
$\angle AEO \cong \angle CFO$	Both are right angles
$\overline{OA} \cong \overline{OC}$	Radii of the same circle
$\overline{OE} \cong \overline{OF}$	Given
$\therefore$ $\Delta AEO \cong \Delta CFO$	H.S ≅ H.S
$\therefore$ $\overline{AE} \cong \overline{CF}$	Corresponding sides of congruent triangles
...or $m\overline{AE} = m\overline{CF}$	Double of equal lengths
$\therefore$ $2m\overline{AE} = 2m\overline{CF} \longrightarrow (ii)$	
Also $m\overline{AB} = m\overline{AE} + m\overline{EB}$	Segment addition postulate
$m\overline{AB} = m\overline{AE} + m\overline{AE} = 2m\overline{AE} \longrightarrow (iii)$	$\because m\overline{EB} = m\overline{AE}$
Similarly $m\overline{CD} = 2m\overline{CF} \longrightarrow (iv)$	From (ii)
Since $2m\overline{AE} = 2m\overline{CF}$	From (iii) and (iv)
Therefore $m\overline{AB} = m\overline{CD}$	
or $\overline{AB} \cong \overline{CD}$	

Hence the two chords are congruent.

...ollary

...ongruent circles, chords equidistant from the centre are also congruent.

Example 4 Circle O has a radius of 5 centimeters. Chords ST and RP are each 3 centimeters from 0. Compare the lengths of $\overline{ST}$ and $\overline{RP}$.



Solution

Draw $\overline{OT}$ and $\overline{OP}$.
OT = OP = 5

$\overline{OT}$ and $\overline{OP}$ are radii of circle O. Radii of the same circle have the same measure.

$(OW)^2 + (WP)^2 = (OP)^2$
$3^2 + (WP)^2 = 5^2$
$(WP)^2 = 25 - 9$
$(WP)^2 = 16$
$WP = 4$

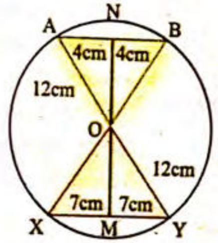
$(OQ)^2 + (QT)^2 = (OT)^2$
$3^2 + (OT)^2 = 5^2$
$(OT)^2 = 25 - 9$
$(OT)^2 = 16$
$QT = 4$

PR = 2(WP) = B ← Theorem 9.3 → ST = 2(QT) = 8

Chord ST and RP are congruent.

Example 5 The length of two parallel chords of a circle of radius 12cm are 14cm and 8cm respectively. Calculate the distance between the chords.

Solution

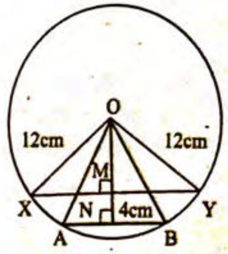


In ΔAON, $(ON)^2 = 12^2 - 4^2$
$= 144 - 16$
$ON = \sqrt{128}$
$ON = 11.31cm$

In ΔYOM, $(OM)^2 = 12^2 - 7^2$
$= 144 - 49$
$OM = \sqrt{95}$
$OM = 9.747$

In first case NM = ON + OM
= 11.31 + 9.747
= 21.1 cm

In second case, NM = ON – OM
= 11.31 – 9.747
= 1.56cm



Therefore the distance between the chords can either be about 21.1cm or 1.56 cm.

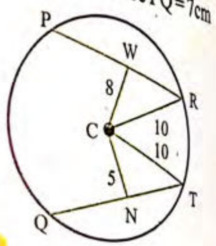
... a circle of radius 5cm, there are two parallel chords of length 8cm and 6cm respectively. Calculate the distance between the chords.

Two Parallel chords $\overline{PQ}$ and $\overline{MN}$ are 3cm apart on the same side of a circle where PQ = 7cm and MN = 14 cm. Calculate the radius of the circle.

Circle C has a radius of 10. Chord QT is 5 units from C and chord PR is 8 units from C.

a. Compare the lengths of $\overline{PR}$ and $\overline{QT}$.

b. Compare the distances of $\overline{PR}$ and $\overline{QT}$ from C.



Review Exercise 9

At the end of each question, four circles are given. Fill in the correct circle only.

(i). In a circle, two chords are equally distant from the centre of the circle. The chords are

○ congruent ○ not congruent ○ parallel ○ non parallel

(ii). $\overline{AB}$ and $\overline{CD}$ are two chords of the same circle ⊙ and $AB < CD$. Then

○ AB is closer to O ○ $\overline{AB}$ must be parallel to $\overline{CD}$
○ CD is closer to O ○ Can't decide

(iii). A chord is 5 cm from the centre of a circle of radius 13cm. The length of the chord is

○ 6 centimeters ○ 12 centimeters
○ 24 centimeters ○ 30 centimeters

(iv). A chord 40 units long is contained in a circle of radius 25. The distance of the chord from the centre of the circle is

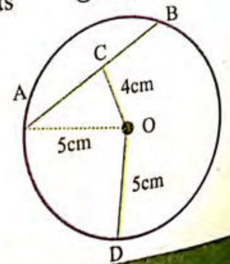
○ 15 units ○ 31.2 units ○ 47.1 units ○ 50.1 units

(v). A chord $8\sqrt{3}$ units long is 4 units from the centre of a circle. The length of the radius of the circle is

○ 14.4 units ○ 8 units ○ $8\sqrt{2}$ units ○ $2\sqrt{8}$ units

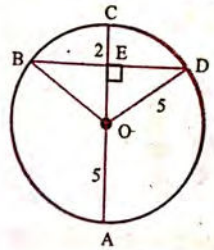
(vi). In the adjacent circular figure with centre O ad radius 5cm, the length of the chord intercepted at 4cm away from the centre of this circle is:

○ 4 cm ○ 6 cm
○ 7 cm ○ 9 cm



(vii). In the diagram, circle O has a radius of 5, and CE = 2. Diameter $\overline{AC}$ is perpendicular to chord $\overline{BD}$ at E. What is the length of $\overline{BD}$?

○ 12 ○ 8
○ 10 ○ 4



2. In circle O, $\overline{AB}$ is a chord, $\overline{OA}$ is a radius and $\overline{AB} \perp \overline{OC}$. (Leave radicals in your answers in simplest form).

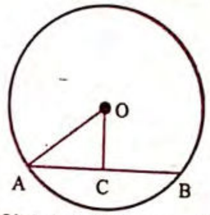
(i). If AO= 13m and OC = 5m, find AC and AB.

(ii). If AB is 16 cm long and is 6 cm from O. Find the radius and diameter of the circle.

(iii). If the diameter of circle O is 34 cm, how far from the centre is a chord 30 cm long?

(iv). The radius of circle O is 25 units long. How long is $\overline{AB}$ if it is 7 units from point O?

(v). Chord AB is 10 m long and is 5 m from O. Find OA.



3. The perpendicular bisector of a chord $\overline{XY}$ cuts $\overline{XY}$ at N and the circle at P. If XY = 16cm and NP = 2cm, calculate the radius of the circle.

Summary

- A **circle** is the set of all points in a plane that are equidistant from a given point in the plane known as the center of the circle.
- A **radius** (plural, radii) is a line segment from the centre of the circle to a point on the circle.
- A **chord** is a line segment whose endpoints lie on the circumference of a circle. It divides a circle into two segments. The larger part is called major segment and the smaller part, the minor segment. If the chord happens to be a diameter, each segment is a semicircle.
- A **diameter** is a chord that contains the centre of a circle.
- One and only one circle can pass through three non-collinear points.
- A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.
- Perpendicular from the centre of a circle on a chord bisects it.
- If two chords of a circle are congruent then they will be equidistant from the centre.
- Two chords of a circle which are equidistant from the centre are congruent.

Unit 10

TANGENT TO A CIRCLE

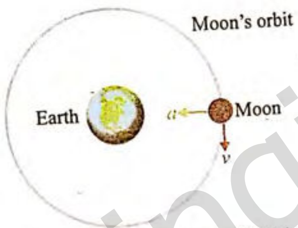
In this unit the students will be able to

To prove the following theorems along with corollaries and apply them to solve appropriate problems.

- If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.
- The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.
- The two tangents drawn to a circle from a point outside it, are equal in length.
- If two circles touch externally or internally, the distance between their centres is respectively equal to the sum or difference of their radii.

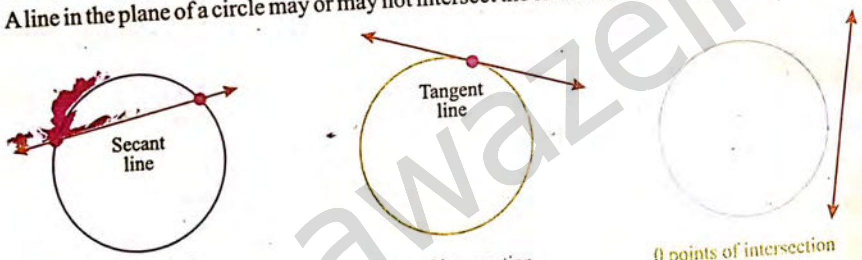
Why it's important

The moment of an object revolving around Earth in a circular orbit is in the direction of a line tangent to the orbit of the object. You will encounter such lines frequently.



Secants and Tangents

A line in the plane of a circle may or may not intersect the circle. There are three possibilities.

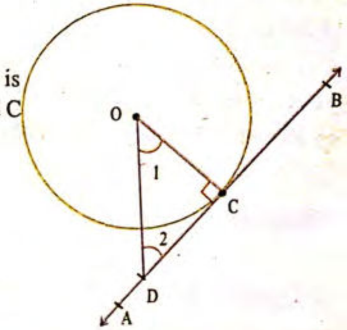


2 points of intersection 1 point of intersection 0 points of intersection

A **secant** to a circle is the line that intersects the circle at two distinct points. If a straight line and a circle have only one point of contact then that line is called a **tangent** and the point of intersection is known as point of tangency/contact.

Theorem 10.1

If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.



Given

A circle having centre O with a line $\overleftrightarrow{AB}$ which is perpendicular to the radial segment $\overline{OC}$ at its outer end C

i.e. $\overleftrightarrow{AB} \perp \overline{OC}$.

To prove $\overleftrightarrow{AB}$ is a tangent to the circle.

Construction

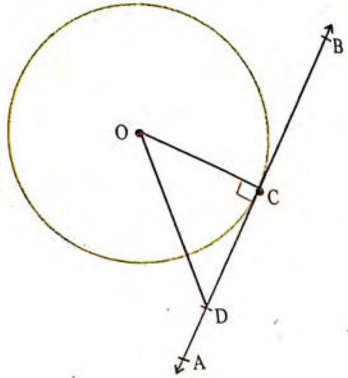
Take any point D on $\overleftrightarrow{AB}$ other than C. Join O to D.

Proof

Statements	Reasons
$\because$ $\overline{OC} \perp \overleftrightarrow{AB}$	Given
So the ΔOCD is a right triangle	$\because m\angle OCD = 90°$
$\therefore \angle 1$ and $\angle 2$ are acute	i.e. $\angle 1 + \angle 2 = 90°$
Thus $m\angle OCD > m\angle 2$	$\because \angle OCD$ is a right angle and $\angle 2$ is an acute angle.
$\therefore m\overline{OD} > m\overline{OC}$ (i)	In a triangle, greater angle has greater side opposite to it.
But $\overline{OC}$ is the radial segment with C as its outer end.	Given
$\therefore$ D lies in the exterior of the circle.	
Hence $\overleftrightarrow{AB}$ meets the circle at one and only one point which is C.	Definition of tangent to a circle.
$\therefore \overleftrightarrow{AB}$ is tangent to the circle.	

Theorem 10.2

The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.



Given

A circle with centre O. $\overleftrightarrow{AB}$ is tangent to the circle at point C. $\overline{OC}$ is the radial segment which is obtained by joining O with the point of contact C of the tangent $\overleftrightarrow{AB}$.

To prove

$\overline{OC} \perp \overleftrightarrow{AB}$

Construction

Take any point D on $\overleftrightarrow{AB}$ except C. Join O and D.

Proof

Statements	Reasons
$\because \overleftrightarrow{AB}$ is tangent to the circle at the point C.	Given
So C is the only point common to the circle and the line $\overleftrightarrow{AB}$.	Definition of tangent to a circle.
D is a point on $\overleftrightarrow{AB}$ other than C	Construction
$\therefore$ D is an exterior point of the circle.	Except C every point of $\overleftrightarrow{AB}$ is outside the circle.
It means, $m\overline{OD} > m\overline{OC}$	
Hence $m\overline{OC}$ is the shortest distance between the point O and the line $\overleftrightarrow{AB}$.	By definition of shortest distance.
$\therefore$ $\overrightarrow{OC} \perp \overleftrightarrow{AB}$	

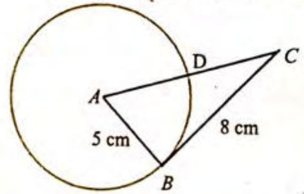
Corollaries

1. At any point on the circumference of a circle one and only one tangent to the circle can be drawn.
2. Perpendicular to a tangent at the point of contact passes through the centre of the circle.

Example 1 $\overline{CB}$ is tangent in the figure to the circle. Find the length of segment CD.

Solution Since CB is tangent to the circle, then $\overline{AB} \perp \overline{CB}$.

This means that ΔABC is a right triangle and we can apply the Pythagorean theorem to find the length of line segment AC.

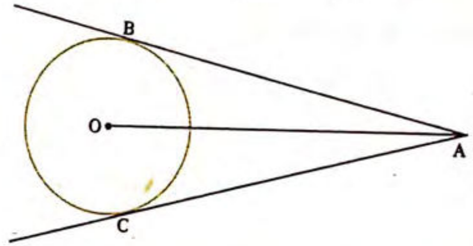


$$AC^2 = AB^2 + BC^2$$
$$AC^2 = 25 + 64 = 89$$
$$AC = \sqrt{89} \approx 9.43 \text{ cm.}$$

$CD = AC - AD = 9.43 \text{ cm} - 5 \text{ cm} = 4.43 \text{ cm}.$

Theorem 10.3

The two tangents drawn to a circle from a point outside it, are equal in length.



Given

A circle with centre O. A is any point outside the circle. $\overline{AB}$ and $\overline{AC}$ are drawn two tangents from point A.

To prove $m\overline{AB} = m\overline{AC}$

Construction

Join O to A, B and C.

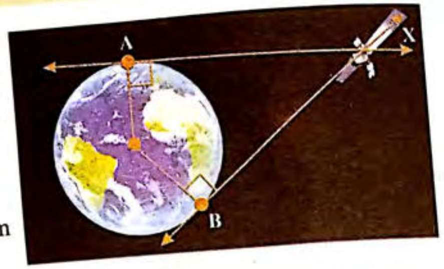
Proof

Statements	Reasons
In $\Delta AOB \leftrightarrow \Delta AOC$	
$\overline{AO} \cong \overline{AO}$	Common
$\overline{OB} \cong \overline{OC}$	Radial segments.
$\angle ABO \cong \angle ACO = 90^\circ$	Right angles because tangent to a circle is perpendicular to the radial segment at the point of contact.
$\therefore$ $\Delta AOB \cong \Delta AOC$	H.S $\cong$ H.S
$\therefore$ $\overline{AB} \cong \overline{AC}$	Corresponding sides of two congruent triangles.
Or $m\overline{AB} = m\overline{AC}$	

Corollary

The two tangents drawn to a circle from an external point subtend equal angles at the centre.

Example 2 Global positioning satellites are used in navigation. If the range of the satellite. AX is 16,000 miles. What is the range of BX?



Solution

$\overline{AX}$ and $\overline{BX}$ and are tangents to a circle from the same external point, so they are equal.

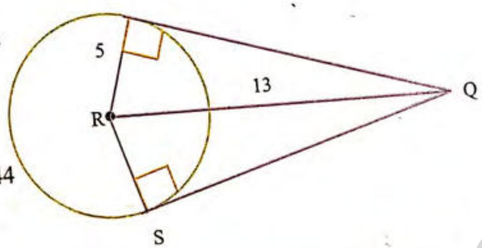
$$AX = BX = 16{,}000 \text{ miles}$$

Example 3 QT and QS are tangent segments to circle R. $TR = 5$. And $RQ = 13$. Find QT and QS.

Solution

Since QRT is a right triangle.

$$\therefore (QR)^2 = (QT)^2 + (TR)^2$$
$$(13)^2 = (QT)^2 + 5^2$$
$$169 = (QT)^2 + 25$$
$$(QT)^2 = 169 - 25 = 144$$
$$QT = 12$$
$$QT = QS = 12$$

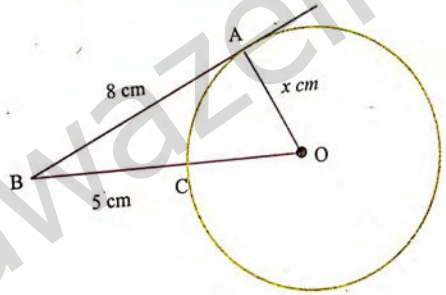


Example 4 In the figure $\overline{AB}$ is a tangent to the circle, with centre O. Given that AB = 8cm, BC = 5cm and OA = x cm, find the value of x, and ∠AOB.

Solution

$$\angle AOB = 90^\circ$$
$$OB = (x+5)\text{cm}$$

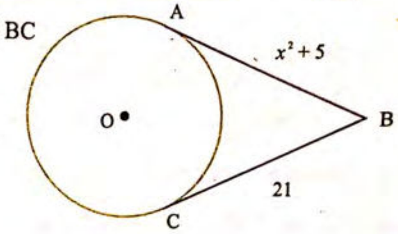
Now
$$(x+5)^2 = x^2 + 8^2$$
$$x^2 + 10x + 25 = x^2 + 64$$
$$10x = 64 - 25$$
$$10x = 39$$
$$x = 3.9$$



In ΔAOB, $\tan \angle AOB = \dfrac{8}{3.9}$

Therefore $\angle AOB = 64.01^\circ$

Example 5 In the adjacent figure, AB and BC are two tangents. Evaluate x.

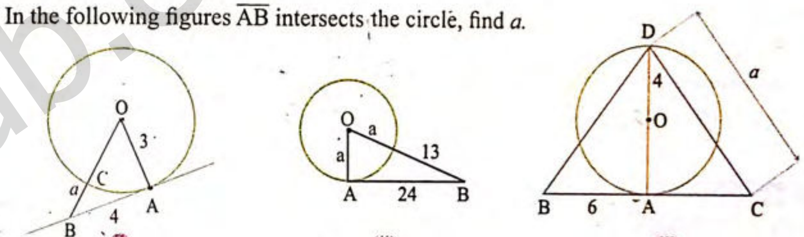


Solution We know that tangent segments from an external point are equal.

So, AB = BC
$$x^2 + 5 = 21$$
$$x^2 = 21 - 5$$
$$x^2 = 16$$
$$x = 4$$

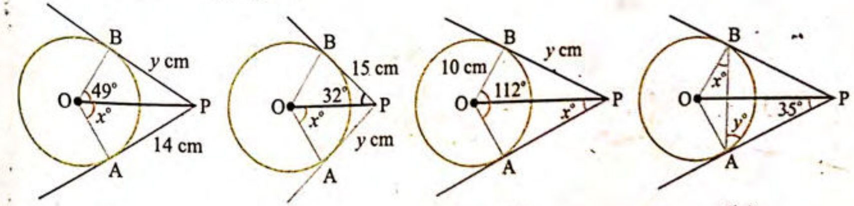
Exercise 10.1

1. Find the length of a tangent drawn to a circle of radius 6 cm from a point at a distance of 10 cm from the centre of the circle.
2. Find the radius of a circle if the length of the tangent drawn is 8 cm and distance from the centre of the circle to the point outside the circle is 9 cm.
3. A chord $\overline{AC}$ of a circle is produced to P. From P a tangent $\overline{PB}$ to the circle is drawn. Prove that $m\angle PBC = m\angle BAP$.
4. In the following figures $\overline{AB}$ intersects the circle, find a.



(i) (ii) (iii)

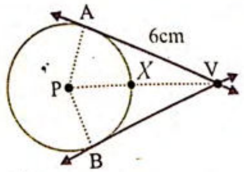
5. If the radius of a circle is 8 cm. Tangents drawn from an external point P make an angle of 60°. Find the distance between the centre of the circle and the point P.
6. Given that $\overline{PA}$ and $\overline{PB}$ are tangents to the circle, with centre O, find the values of x and y in each of the following figure.



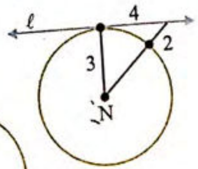
(i) (iii) (iv)

7. In the figure below, $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are tangent to P, the radius of P is 3cm, and VA = 6 cm. Find the following.

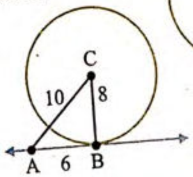
(i). VB (ii). AP (iii). PV (iv). XV



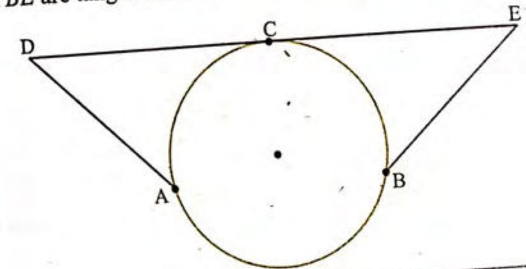
8. In circle N. verify that line ℓ is a tangent.



9. Verify that $\overleftrightarrow{AB}$ is tangent to C at B.



10. If AD, DE and BE are tangents to a circle as shown prove that $AD + BE = DE$.



Note

Two circles that intersect in one point are called tangent circles.

Two circles that have a common centre are called concentric circles.

Theorem 10.4(a)

If two circles touch externally, the distance between their centres is equal to the sum of their radii.

Given

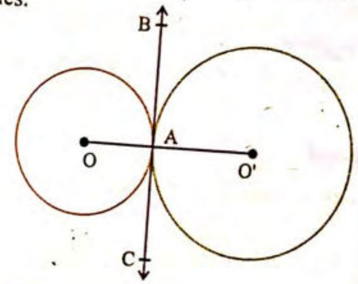
Two circles with centres O and O' which are touching each other externally at the point A, $\overline{OA}$ and $\overline{O'A}$ are the radial segments of the circles.

To prove

$$m\overline{OO'} = m\overline{OA} + m\overline{AO'}$$

Construction

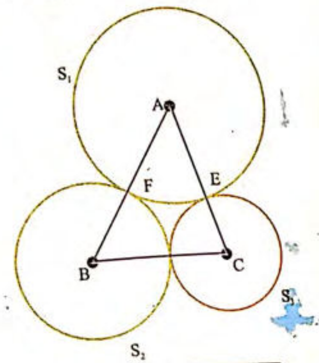
Draw a common tangent $\overleftrightarrow{BC}$ at the point A which is the common point of contact of the given two circles.



Proof

Statements	Reasons
$m\angle BAO = 90°$ (i)	∵ A tangent to a circle is perpendicular to the radial segment at the point of contact.
Similarly $m\angle BAO' = 90°$ (ii)	∵ A tangent to a circle is perpendicular to the radial segment at the point of contact.
$\because m\angle BAO + m\angle BAO' = 90° + 90° = 180°$	Adding (i) and (ii).
$m\angle BAO$ and $m\angle BAO'$ are supplementary angles with the common vertex A.	∵ Their sum is 180°.
$\therefore \overrightarrow{AO}$ and $\overrightarrow{AO'}$ are opposite rays.	Postulate of supplementary angles.
This makes O, A and O', the three different collinear points.	Consequence of being $\overrightarrow{AO}$ and $\overrightarrow{AO'}$, the opposite rays
$\therefore m\overline{OO'} = m\overline{OA} + m\overline{AO'}$	segments addition postulate.

Example 6 Three circles touch in pairs externally. Prove that the perimeter of a triangle formed by joining centres is equal to the sum of their diameters.



Given

Three circles have centres A, B and C their radii are r_1, r_2, and r_3 respectively. They touch in pairs externally at D, E and F. So that ΔABC is formed by joining the centres of these circles.

To prove

Perimeter of ΔABC = Sum of the diameters of these circles

Proof

Statements	Reasons
Three circles with centres A, B and C touch in pairs externally at the points, D, E and F.	Given
$\therefore$ $m\overline{AB} = m\overline{AF} + m\overline{FB}$ (i) $m\overline{BC} = m\overline{BD} + m\overline{DC}$ (ii) and $m\overline{CA} = m\overline{CE} + m\overline{EA}$ (iii)	
$m\overline{AB} + m\overline{BC} + m\overline{CA}$ $= m\overline{AF} + m\overline{FB} + m\overline{BD} + m\overline{DC} + m\overline{CE} + m\overline{EA}$ $= (m\overline{AF} + m\overline{EA}) + (m\overline{FB} + m\overline{BD}) + (m\overline{CD} + m\overline{CE})$	Adding (i), (ii) and (iii).
Perimeter of $\Delta ABC = 2r_1 + 2r_2 + 2r_3$ $= d_1 + d_2 + d_3$ = Sum of diameters of the circles.	$d_1 = 2r_1$, $d_2 = 2r_2$ and $d_3 = 2r_3$ are diameters of the circles.

Math Fun

Using only addition, how do you add eight 8's and get the number 1000?

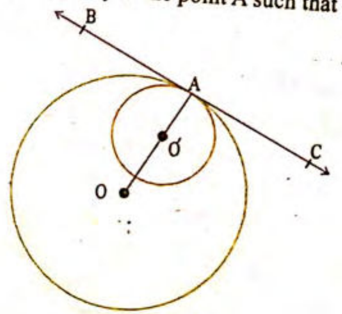
Answer: 888+88+8+8+8 = 1000.

Theorem 10.4(b)

If two circles touch internally, the distance between their centres is the difference of their radii.

Given

Two circles with centre O and O' touch each other internally at the point A such that $\overline{OA}$ and $\overline{O'A}$ are radial segments of these circles.



To prove

$m\overline{OO'} = m\overline{OA} - m\overline{O'A}$

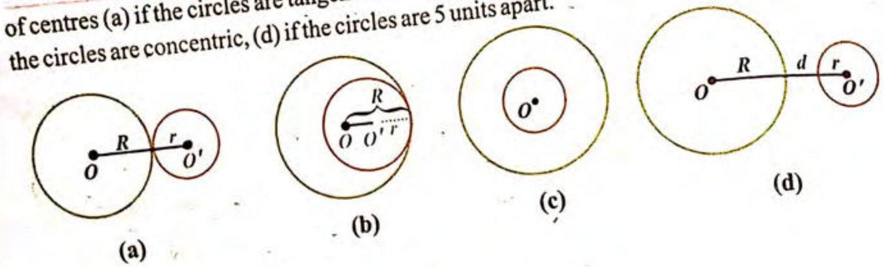
Construction

Draw a common tangent $\overleftrightarrow{BC}$ at the point A which is the common point of contact of the two circles.

Proof

Statements	Reasons
Since the circles touch each other at A therefore both $\overline{OA}$ and $\overline{O'A}$ are perpendicular to $\overleftrightarrow{BC}$.	$\because$ A tangent to a circle is perpendicular to the radius drawn through the point of contact.
$\therefore$ $m\angle BAO = m\angle BAO' = 90°$	
i.e. $\overline{OA} \perp \overleftrightarrow{BC}$ at A.	
and $\overline{O'A} \perp \overleftrightarrow{BC}$ at A.	
$\therefore$ O, O' and A lie on the same straight line.	At a point on a line a unique perpendicular can be drawn on it.
$\therefore$ $m\overline{OA} = m\overline{OO'} + m\overline{O'A}$	Segment addition postulate.
or $m\overline{OO'} = m\overline{OA} - m\overline{O'A}$	From a law of equation.

Example 7 Two circles have radii of 9 and 4, respectively. Find the length of their line of centres (a) if the circles are tangent externally, (b) if the circles are tangent internally, (c) if the circles are concentric, (d) if the circles are 5 units apart.



Solution

Let R = radius of larger circle, r = radius of smaller circle.

(a) Since $R = 9$ and $r = 4$, $OO' = R + r = 9 + 4 = 13$.

(b) Since $R = 9$ and $r = 4$, $OO' = R - r = 9 - 4 = 5$.

(c) Since the circles have the same centre, their line of centres has zero length.

(d) Since $R = 9$ and $r = 4$, and $d = 5$, $OO' = R + d + r = 9 + 5 + 4 = 18$.

Did You Know?

$$32 = 2^{(3+2)}$$

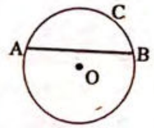
Exercise 10.2

1. If two circles with radii 8 cm and 3 cm respectively touch each other externally, then what is the distance between their centres?
2. The distance between the centres of two circles touching internally is 5 cm. The radius of the bigger circle is 17 cm. What is the radius of the smaller circle?
3. In a circle a 10 cm long chord is at a distance of 12 cm from the centre. What is the length of a chord at a distance of 5 cm from the centre?
4. A chord is 18 cm long. The radius of the circle is 15 cm. What is the distance of the midpoint of the chord from the centre of the circle?
5. What is the length of a chord at distance of 6cm from the centre of a circle of radius 10 cm?
6. The distance of a chord from the centre of a circle is 3 cm, and the length of the chord is 8 cm. What is the diameter?
7. The radius of a circle is 8 cm and one of the chords is of 12 cm. What is the distance of the chord from the centre?
8. The length of a chord of a circle of radius 7.5 cm is 9 cm. What is its distance from the centre of the circle?

Review Exercise 10

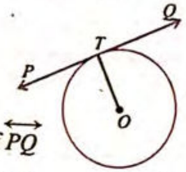
1. At the end of each question, four circles are given. Fill in the correct circle only.

(i) In the figure, $\overset{\frown}{ACB}$ is called



○ an arc ○ a secant
○ a chord ○ a diameter

(ii) In a circle with centre O, if $\overline{OT}$ is the radial segment and $\overleftrightarrow{PTQ}$ is the tangent line, then

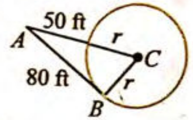


○ $\overline{OT} \perp \overleftrightarrow{PQ}$ ○ $\overline{OT} \not\perp \overleftrightarrow{PQ}$
○ $\overline{OT} \parallel \overleftrightarrow{PQ}$ ○ $\overline{OT}$ is right bisector of $\overleftrightarrow{PQ}$

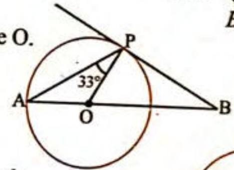
(iii) Two tangents drawn to a circle from a point outside it are of.......... in length.

○ half ○ equal ○ double ○ triple

2. In the diagram, B is a point of tangency. Find the radius r of ⊙ C.

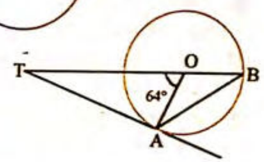


3. In the figure, $\overline{BP}$ is a tangent to the circle with centre O. Given that ∠APO = 33°, find ∠PBA.

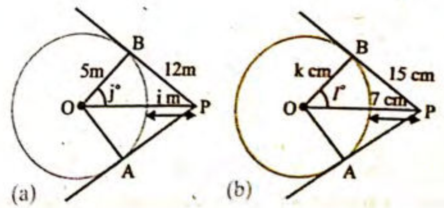


4. In the figure, O is the centre of the circle passing through the points A and B. $\overline{TA}$ is a tangent to the circle at A and TOB is a straight line.

Given that ∠AOT = 64°, Find (i) ∠ATB, (ii) ∠TAB



5. Given that $\overline{PA}$ and $\overline{PB}$ are tangents to each of the following circles with centre O, find the values of the unknowns.

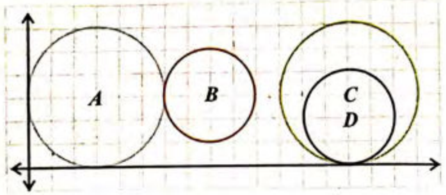


Activity

Lengths in circles in a coordinate plane

Use the diagram for lengths.

a. Radius of ⊙A
b. Diameter of ⊙A
c. Radius of ⊙B
d. Diameter of ⊙B
e. Radius of ⊙C
f. Diameter of ⊙C
g. Radius of ⊙D
h. Diameter of ⊙D



Summary

- A secant to a circle is a line that intersects the circle at two distinct points. If a straight line and a circle have only one point of contact then that line is called a tangent and the point of intersection is known as point of tangency/contact.
- A line is tangent to a circle if and only if the line is perpendicular to a radius of the circle at its end point on the circle.
- If a line is perpendicular to a radius at its end points the line is a tangent
- Tangent segments from a common external point are congruent.
- The tangents subtend equal angles at the centre.
- The line joining the external point to the centre of the circle bisects the angle between the tangents
- If two circles touch externally, the distance between their centres is equal to the sum of their radii.
- If two circles touch internally, the distance between their centres is the difference of their radii.

Did You Know?

$3025 = (30 + 25)^2$

and

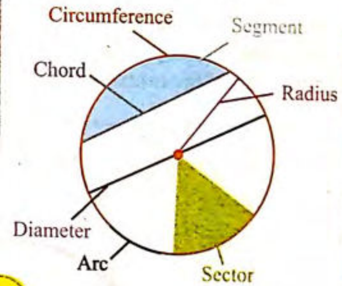
$2025 = (20 + 25)^2$

Unit 11 CHORDS AND ARCS

In this unit the students will be able to

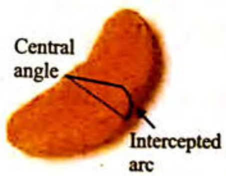
To prove the following theorems along with corollaries and apply them to solve appropriate problems.

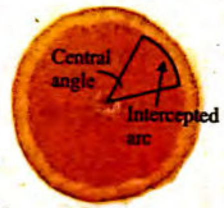
- If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.
- If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
- If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.



Why it's important

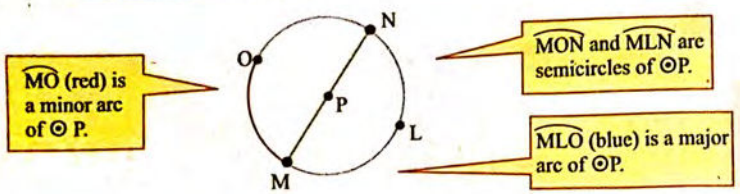
An orange may consist of nine wedges, seen in cross section here. Thus, an average wedge would form a central angle of about one-ninth of the full circle, or 40°. (When you look at a typical orange wedge, does it seem to be about 40°?)





Major and Minor Arcs

An arc is an unbroken part of a circle. Any two distinct points on a circle divide the circle into two arcs. The points are called the endpoints of the arcs.

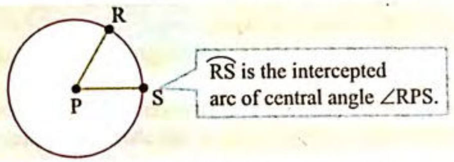


A **semicircle** is an arc whose endpoints are endpoints of the diameter of the circle. A semicircle is informally called a half-circle. A semicircle is named by its endpoints and another point that lies on the arc.

A minor arc of a circle is an arc that is shorter than a semicircle of that circle
A minor arc is named by its endpoints.

A major arc of a circle is an arc that is longer than a semicircle of that circle.
A major arc is named by its endpoints and another point that lies on the arc.

Central angles of circles are used to find the measures of arcs.



A central angle of a circle is an angle in the plane of a circle whose vertex is the centre of the circle. An arc whose endpoints lie on the sides of the angle and whose other points lie in the interior of the angle is the **intercepted arc** of the central angle.

Math Fun

$1 \times 9 + 2 = 11$

$12 \times 9 + 3 = 111$

$123 \times 9 + 4 = 1111$

$1234 \times 9 + 5 = 11111$

$12345 \times 9 + 6 = 111111$

$123456 \times 9 + 7 = 1111111$

$1234567 \times 9 + 8 = 11111111$

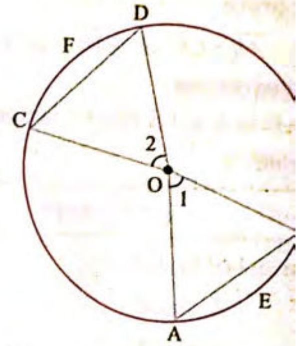
$12345678 \times 9 + 9 = 111111111$

$123456789 \times 9 + 10 = 1111111111$

Theorem 11.1

If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.

Case (a) For one circle



Given

A circle with centre O $\overset{\frown}{AEB}$ and $\overset{\frown}{CFD}$ are congruent arcs i.e. $\overset{\frown}{AEB} \cong \overset{\frown}{CFD}$. $\overline{AB}$ and $\overline{CD}$ are the corresponding chords of the given congruent arcs.

To prove

$\overline{AB} \cong \overline{CD}$.

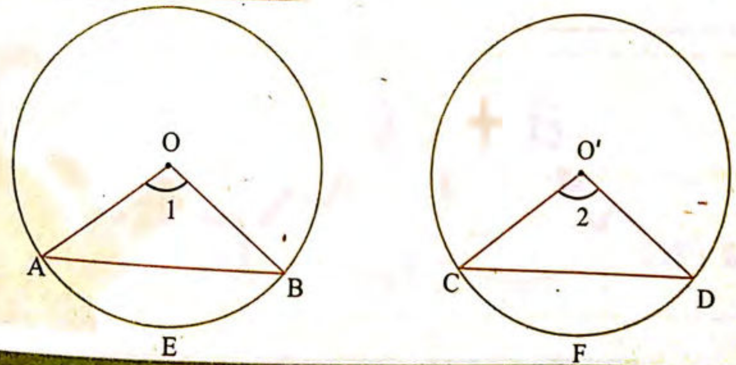
Construction

Join O to A, B, C and D respectively and name the central ∠1 and ∠2.

Proof

Statements	Reasons
In $\Delta OAB \leftrightarrow \Delta OCD$	
$\overline{OA} \cong \overline{OC}$	Radii of the same circle.
$\overline{OB} \cong \overline{OD}$	Radii of the same circle.
$\angle 1 \cong \angle 2$	Central angles of two congruent arcs.
$\therefore$ $\Delta OAB \cong \Delta OCD$	S.A.S Postulate
$\therefore$ $\overline{AB} \cong \overline{CD}$	Corresponding sides of two congruent triangles.

Case (b) For two congruent circles



Given

Two congruent circles with centres O and O' respectively. AEB and CFD are two congruent arcs of these circles where $\overline{AB}$ and $\overline{CD}$ are the corresponding chords.

To prove

$\overline{AB} \cong \overline{CD}$ or $m\overline{AB} = m\overline{CD}$

Construction

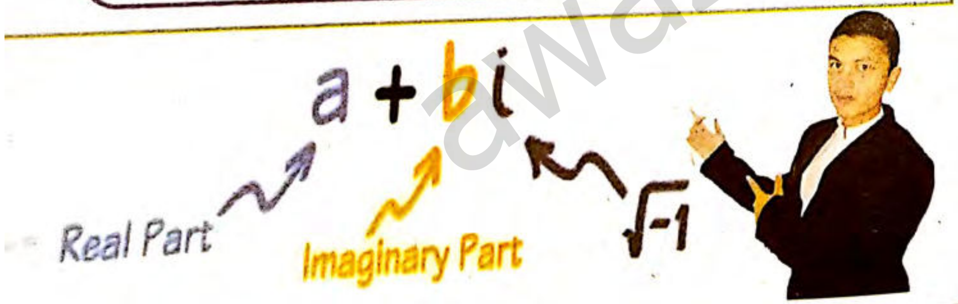
Join O to A and B and O' to C and D respectively.

Proof

Statements	Reasons
In $\Delta OAB \leftrightarrow \Delta O'CD$	
$\overline{OA} \cong \overline{O'C}$	Radii of two congruent circles.
$\overline{OB} \cong \overline{O'D}$	Radii of two congruent circles.
$\angle 1 \cong \angle 2$	Central angles of two congruent arcs
$\therefore$ $\Delta OAB \cong \Delta O'CD$	S.A.S Postulate
$\therefore$ $\overline{AB} \cong \overline{CD}$	Corresponding sides of two congruent triangles
Or $m\overline{AB} = m\overline{CD}$	

Tidbit

The sum of any two odd numbers is even; the product of any two odd numbers is odd.



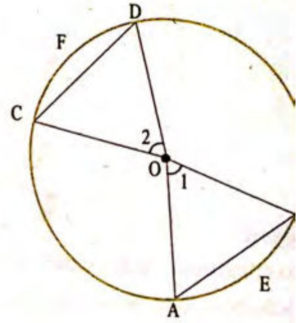
Theorem 11.2

If two chords of a circle (or of congruent circles) are equal, then their correspondi arcs (minor, major or semi-circular) are congruent.

Case (a) For one circle

Given

A circle with centre O having two chords $\overline{AB}$ and $\overline{CD}$ such that $\overline{AB} \cong \overline{CD}$.



To prove

$AEB \cong CFD$ (These are minor arcs of the chords $\overline{AB}$ and $\overline{CD}$ respectively).

and $AFB \cong CED$ (These are major arcs of the chords $\overline{AB}$ and $\overline{CD}$ respectively).

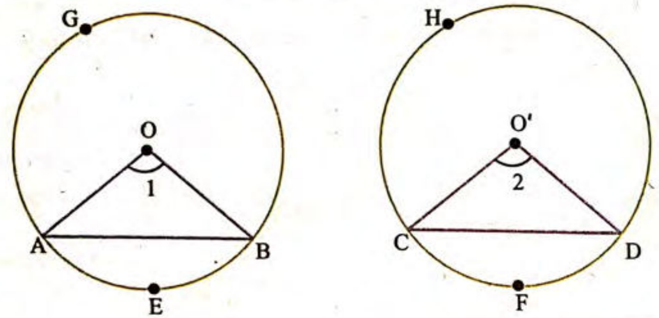
Construction

Join O with A, B, C and D respectively.

Proof

Statements	Reasons
In the $\Delta AOB \leftrightarrow \Delta COD$	
$\overline{OA} \cong \overline{OC}$	Radii of the same circle.
$\overline{OB} \cong \overline{OD}$	Radii of the same circle.
$\overline{AB} \cong \overline{CD}$	Given
$\therefore$ $\Delta AOB \cong \Delta COD$	S.S.S $\cong$ S.S.S
$\therefore$ $\angle 1 \cong \angle 2$	Corresponding angles of two congruent triangles.
but $\angle 1$ and $\angle 2$ are central angles	
$\therefore$ $\overset{\frown}{AEB} \cong \overset{\frown}{CFD}$	Definition of equal or congruent arcs.
Thus the corresponding minor arcs of two equal chords $\overline{AB}$ and $\overline{CD}$ of a circle are congruent.	

ase (b) Let's prove the same result for two congruent circles



Given

Two congruent circles with centres O and O' with two chords $\overline{AB}$ and $\overline{CD}$ respectively which are not the diameters such that $m\overline{AB} = m\overline{CD}$ or $\overline{AB} \cong \overline{CD}$.

To prove

$AEB \cong CFD$ (these are minor arcs corresponding to the chords $\overline{AB}$ and $\overline{CD}$) and

$AGB \cong CHD$ (these are the major arcs corresponding to the chords $\overline{AB}$ and $\overline{CD}$)

Construction

Join O with A and B and O' with C and D. This gives central angles labeled 1 and 2.

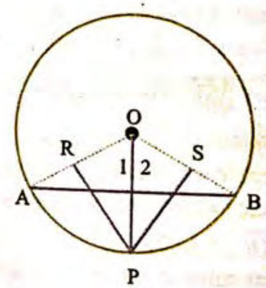
Proof

Statements	Reasons
In the $\Delta OAB \leftrightarrow \Delta O'CD$	
$\overline{OA} \cong \overline{O'C}$	Radii of two congruent circles.
$\overline{OB} \cong \overline{O'D}$	Radii of two congruent circles.
$\overline{AB} \cong \overline{CD}$	Given
∴ $\Delta OAB \cong \Delta O'CD$	S.S.S ≅ S.S.S
$\angle 1 \cong \angle 2$	Corresponding angles of two congruent triangles.
$m\angle 1 = m\angle 2$	
$m\widehat{AEB} = m\widehat{CFD}$	Condition of equality of two arcs.
$\widehat{AEB} \cong \widehat{CFD}$	
s the corresponding minor arcs of two al chords $\overline{AB}$ and $\overline{CD}$ of two gruent circles are congruent.	

Example 1 A point P on the circumfernece is equidistant from the radii $\overline{OA}$ and O
Prove that $m\widehat{AP} = m\widehat{BP}$

Given

AB is the chord of a circle with centre O. Point P on the circumference of the circle is equidistant from the radii $\overline{OA} = \overline{OB}$ so that $m\overline{PR} = m\overline{PS}$



To prove

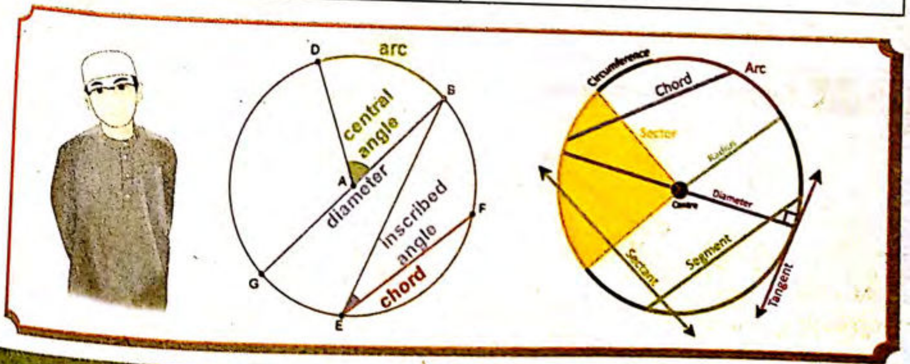
$m\widehat{AP} = m\widehat{BP}$

Construction

Join O with P. write ∠1 and ∠2 as shown in the figure.

Proof

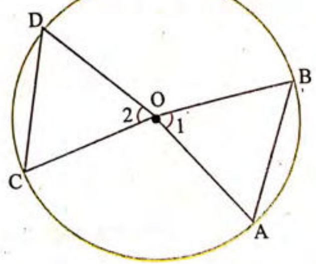
Statements	Reasons
In ∠rt ΔOPR and ∠rt ΔOPS	
$m\overline{OP} = m\overline{OP}$	Common
$m\overline{PR} = m\overline{PS}$	Point P is equidistant from radii. (Given)
∴ $\Delta OPR \cong \Delta OPS$	(In ∠rtΔ^s H.S ≅ H.S)
So $m\angle 1 \cong m\angle 2$	Central angles of a circle.
Chord AP ≅ Chord BP	
Hence $m\widehat{AP} = m\widehat{BP}$	Arcs corresponding to equal chords in a circle.



Theorem 11.3

Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).

Case (a) For one circle



Given

A circle with centre O. $\overline{AB}$ and $\overline{CD}$ are two chords of the circle (which are not diameters) such that $\overline{AB} \cong \overline{CD}$ or $m\overline{AB} = m\overline{CD}$.

Arcs subtend $\angle 1$ and $\angle 2$ at the centre.

To prove

$\angle 1 \cong \angle 2$

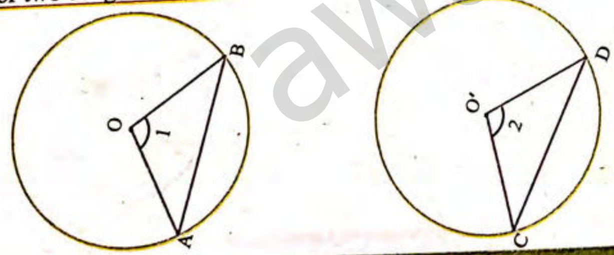
Construction

We join O to A, B, C and D respectively so that $m\overline{OA} = m\overline{OB} = m\overline{OC} = m\overline{OD}$ = radii of a circle.

Proof

Statements	Reasons
In the $\Delta OAB \leftrightarrow \Delta OCD$	
$\overline{OA} \cong \overline{OC}$	Radii of the same circle.
$\overline{OB} \cong \overline{OD}$	Radii of the same circle.
$\overline{AB} \cong \overline{CD}$	Given
$\therefore$ $\Delta OAB \cong \Delta OCD$	S.S.S $\cong$ S.S.S
$\therefore$ $\angle 1 \cong \angle 2$	Corresponding angles of congruent triangles.

Case (b) For two congruent circles



Given

Two congruent circles with centres O and O' having two equal chords $\overline{AB}$ and $\overline{CD}$ i.e $\overline{AB} \cong \overline{CD}$.

To prove

These chords subtend equal angles at the centre i.e. $\angle 1 \cong \angle 2$.

Construction

Join O with A and B, and O' with C and D.

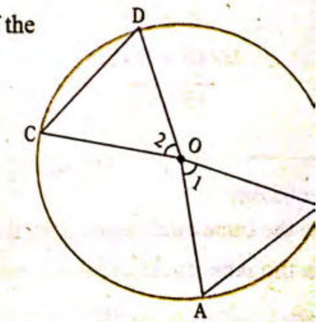
Proof

Statements	Reasons
In the $\Delta OAB \leftrightarrow \Delta O'CD$	
$\overline{OA} \cong \overline{O'C}$	Radii of two congruent circles.
$\overline{OB} \cong \overline{O'D}$	Radii of two congruent circles.
$\overline{AB} \cong \overline{CD}$	Given
$\therefore$ $\Delta OAB \cong \Delta O'CD$	S.S.S $\cong$ S.S.S
$\therefore$ $\angle 1 \cong \angle 2$	Corresponding angles of congruent triangles.

Theorem 11.4

If the angles subtended by two chords of a circle (or congruent circles) at the cent (corresponding centres) are equal in measures, then the chords are equal in measures.

Case (a) For one circle



Given

A circle with centre O. $\overline{AB}$ and $\overline{CD}$ are two chords of the circle and $\angle 1 \cong \angle 2$.

To prove

$\overline{AB} \cong \overline{CD}$

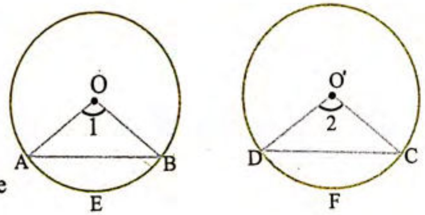
Proof

Statements	Reasons
In the $\Delta OAB \leftrightarrow \Delta OCD$	
$\overline{OA} \cong \overline{OC}$	Radii of the same circle.
$\overline{OB} \cong \overline{OD}$	Radii of the same circle.
$\angle 1 \cong \angle 2$	Given
$\therefore$ $\Delta AOB \cong \Delta COD$	S.A.S Postulate
$\therefore$ $\overline{AB} \cong \overline{CD}$	Corresponding sides of congruent triangles.

Case (b) For two congruent circles

Given

Two congruent circles with centres O and O', $\overline{AB}$ and $\overline{CD}$ are two chords of these circles such that they subtend equal angles at the centre i.e. $\angle 1 \cong \angle 2$.



To prove

$\overline{AB} \cong \overline{CD}$

Proof

Statements	Reasons
In the $\Delta OAB \leftrightarrow \Delta O'CD$	
$\overline{OA} \cong \overline{O'C}$	Radii of two congruent circles.
$\overline{OB} \cong \overline{O'C}$	Radii of two congruent circles.
$\angle 1 \cong \angle 2$	Given
$\therefore$ $\Delta OAB \cong \Delta O'CD$	S.A.S Postulate
$\therefore$ $\overline{AB} \cong \overline{CD}$	Corresponding sides of two congruent triangles.

orollaries

In the same circle equal central angles have equal arcs.

In the same circle equal arcs have equal central angles.

Example 2 The internal bisector of a central angle in a circle bisects an arc on w it stands.

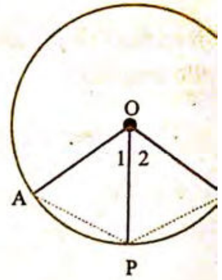
Given

In a circle with centre O. $\overline{OP}$ is an internal bisector of central angle AOB.

To prove $\widehat{AP} \cong \widehat{BP}$

Construction

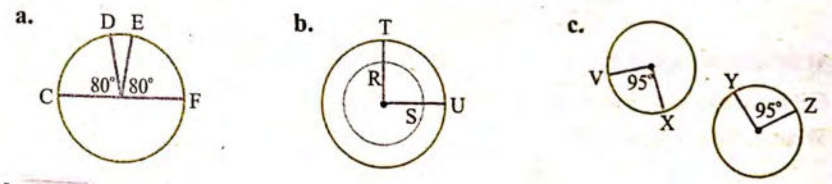
Draw AP and BP, then write $\angle 1$ and $\angle 2$ as shown in the figure.



Proof

Statements	Reasons
In $\Delta OAP \leftrightarrow \Delta OBP$	
$m\overline{OA} = m\overline{OB}$	Radii of the same circle.
$m\angle 1 = m\angle 2$	Given $\overline{OP}$ as an angle bisector of $\angle AOB$
and $m\overline{OP} = m\overline{OP}$	Common
$\Delta OAP \cong \Delta OBP$	(S.A.S $\cong$ S.A.S)
Hence $\overline{AP} \cong \overline{BP}$	
$\Rightarrow$ $\widehat{AP} \cong \widehat{BP}$	Areas corresponding to equal chords in a circle.

Example 3 Tell whether the red arcs are congruent. Explain why or why not.



Solution

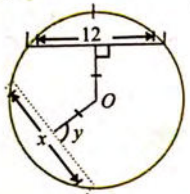
a. $\widehat{CD} \cong \widehat{EF}$ because they are in the same circle and $m\widehat{CD} = m\widehat{EF}$.

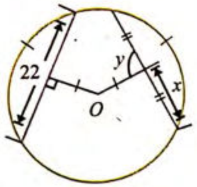
b. $\widehat{RS}$ and $\widehat{TU}$ have the same measure, but are not congruent because they are arcs of circles that are not congruent.

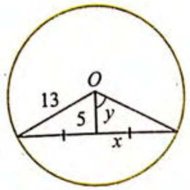
c. $\widehat{VX} = \widehat{YZ}$ because they are in congruent circles and $m\widehat{VX} = m\widehat{YZ}$

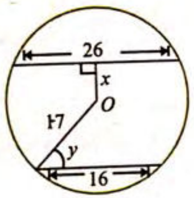
Exercise 11

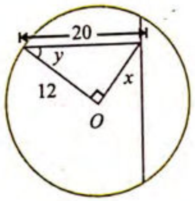
1. Given that O is the centre of each of the following circles, find the value of x and y in the following cases.

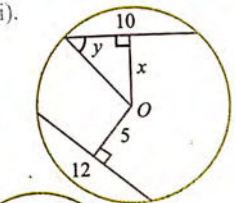
(i).


(ii).


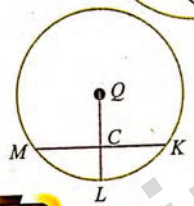
(iii).


(iv).


(v).


(vi).


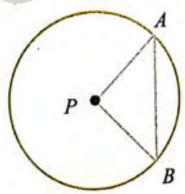
2. In $\odot Q$, $\widehat{KL} = \widehat{LM}$. If $CK = 2x + 3$ and $CM = 4x$, find x.



Review Exercise 11

1. At the end of each question, four circles are given. Fill in the correct circle only.

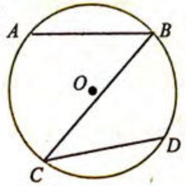
(i) $\odot P$ has a radius of 3 and $\widehat{AB}$ has a measure of 90°.
What is the length of $\overline{AB}$?



○ $3\sqrt{2}$ ○ $3\sqrt{3}$

○ 6 ○ 9

(ii) In the accompanying diagram of circles O, $\widehat{AB} \cong \widehat{CD}$.
Which statement is true?

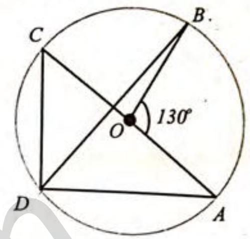


○ $\overline{AB} \cong \overline{CD}$ ○ $\widehat{AB} \parallel \widehat{CD}$

○ $\overline{AC} \cong \overline{BD}$ ○ $\angle ABC \cong \angle BCD$

2. In a circle if any pair of diameters are ⊥ each other then the lines joining its ends in order, form a square.

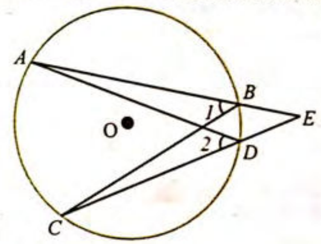
3. In a circle with centre O, AOC is diameter. $\angle AOB = 130°$, find $\angle ADB$, $\angle BDC$ and $m\widehat{BC}$



4. Chords AB and CD of a circle meet at the point E outside the circle. Prove that:

(a) $\angle A = \angle C$ (b) $\angle 1 = \angle 2$

(c) ΔADE and ΔCBE are equiangular.



Summary

- A central angle of a circle is an angle whose vertex is the centre of the circle.
- The measure of a minor arc is the measure of its central angle.
- The measure of the entire circle is 360.
- The measure of major arc is the difference between 360 and the measure of the relate minor arc.
- The measure of the semi circle is 180°.
- Two arcs of the same circle are adjacent if they have a common end points.
- The measure of an arc formed by two adjacent arcs is the sum of the measure of the two arcs
- Two circles are congruent circles if they have same radius.
- Two arcs are congruent arcs if they have the same measure and they are arcs of the sam circle or of congruent circles.
- In the same circle, or in congruent circles, two minor arcs are congruent if and only their corresponding chords are congruent.
- In the same circle, or in congruent circles, two chords are congruent if and only if the are equidistant from the center.

Unit 12

ANGLE IN A SEGMENT OF A CIRCLE

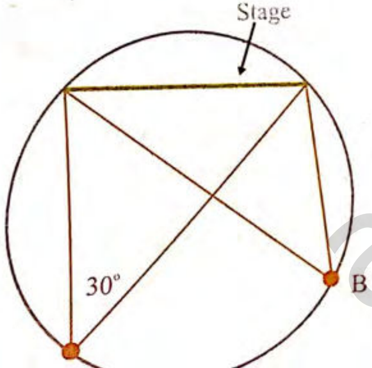
In this unit the students will be able to

To prove the following theorems along with corollaries and apply them to solve appropriate problems.

- The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.
- Any two angles in the same segment of a circle are equal.
- The angle in a semi-circle is a right angle,
- The angle in a segment greater than a semi-circle is less than a right angle,
- The angle in a segment less than a semi-circle is greater than a right angle.
- The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Why it's important

A person's effective field of vision is about 30°. In the diagram of the amphitheater, a person sitting at point A can see the entire stage. What is the measure of $\angle B$? Can the person sitting at point B view the entire stage?
Angles A and B intercept the same arc. By Theorem 2 of this unit, the angles must have the same measure, so $m\angle A = m\angle B = 30°$.
The person sitting at point B can view the entire stage.



APPLICATION
OPTICS

Theorem 12.1

The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.

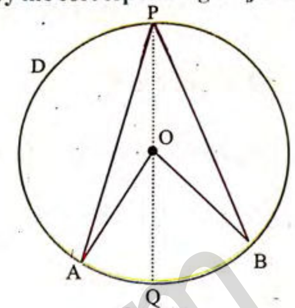


fig: (i)

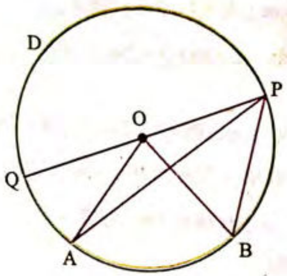


fig: (ii)

Given

A circle with centre O, $\overset{\frown}{AB}$ is a minor arc whose central angle is $\angle AOB$.
P is any point on the major arc $\overset{\frown}{ADB}$. $\angle APB$ is the angle subtended by the arc at P.

To prove

$m\angle AOB = 2m\angle APB$

Case 1: When P is an arbitrary point of the major arc (General Case) figure (i).
Case 2: When PQ is an extremity of the semi-circle containing the minor arc $\overset{\frown}{AB}$ figure (ii).

Construction

Draw $\overrightarrow{PO}$ to meet the circumference at Q (for both cases).

Proof

Statements	
Case (i)	
In the ΔOAP	
$m\overline{OA} = m\overline{OP}$	Radii of the same circle.
$\therefore$ ΔOAP is an isosceles triangle	Definition of an isosceles triangle.
and $m\angle OAP = m\angle OPA$	$\because$ If two sides of a triangle are equal, the angles which are opposite to them are also equal.

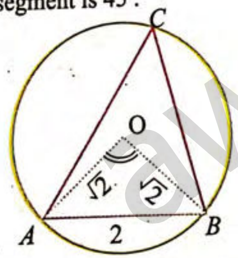
Statements	Reasons
Also $m\angle AOQ = m\angle OAP + m\angle OPA$	$\because$ The measure of exterior angle of a triangle is equal to the sum of the measures of opposite interior angles.
$\therefore$ $m\angle AOQ = 2m\angle OPA$ (I)	$\because$ $m\angle OAP = m\angle OPA$ (Proved above)
Similarly $m\angle BOQ = 2m\angle OPB$ (II)	Adding I and II.
$\therefore$ $m\angle AOQ + m\angle BOQ = 2m\angle OPA + 2m\angle OPB$	Postulate of addition of angles.
or $m\angle AOB = 2[m\angle OPA + m\angle OPB]$	
or $m\angle AOB = 2m\angle APB$	
Case (ii) All the statements and reasons are same up to obtaining the equations I and II.	
Subtracting I from II we get $m\angle BOQ - m\angle AOQ = 2m\angle OPB - 2m\angle OPA$	Postulate of subtraction of angles.
$m\angle BOQ - m\angle AOQ = 2[m\angle OPB - m\angle OPA]$	
$\therefore$ $m\angle AOB = 2m\angle APB$	

Corollary

The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.

Example 1

The radius of a circle is $\sqrt{2}$ cm. A chord 2cm in length divides the circle into two segments. Prove that the angle of larger segment is 45°.



Solution

Given

a circle with centre O and radius $m\overline{OA} = m\overline{OB} = \sqrt{2}$ cm,
length of chord $\overline{AB} = 2$ cm divides the circle into two segments with ACB as
larger one.

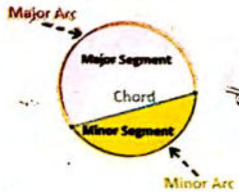
To prove

$m\angle ACB = 45°$

Construction

Join O with A and B.

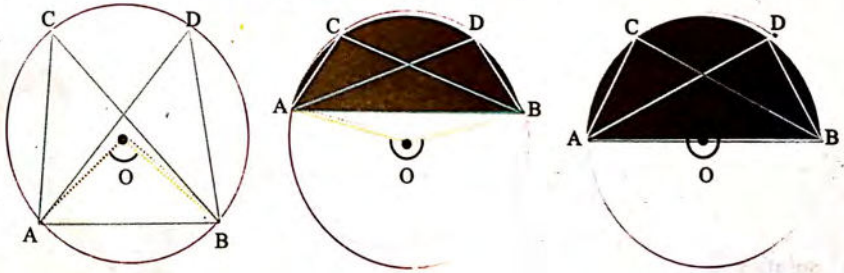
Proof



Statements	Reasons
In ΔOAB $(OA)^2 + (OB)^2 = (\sqrt{2})^2 + (\sqrt{2})^2$ $= 2 + 2$ $= 4 = (2)^2 = (AB)^2$	$m\overline{OA} = m\overline{OB} = \sqrt{2}$ cm $\because$ $m\overline{AB} = 2$ cm
$\therefore$ ΔOAB is right angled triangle. With $m\angle AOB = 90°$	Which being a central angle standing on an arc AB.
$m\angle ACB = \frac{1}{2} m\angle AOB$ $= \frac{1}{2}(90°) = 45°$	By Theorem 12.1 Circum-angle is half of the central angle.

Theorem 12.2

Any two angles in the same segment of a circle are equal.



Given

A circle with centre O. $\angle ACB$ and $\angle ADB$ are any two angles in the same segment ACDB (Shaded) of the circle.

To prove

$m\angle ACB = m\angle ADB$

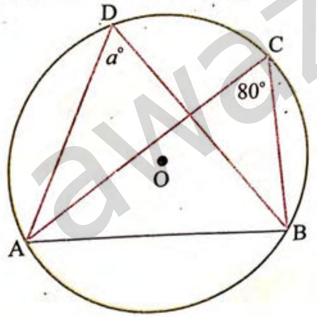
Construction

Join O to A and B respectively.

Proof

Statements	Reasons
In all the three figures, we have $m\angle AOB = 2m\angle ACB$ (I)	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
Similarly $m\angle AOB = 2m\angle ADB$ (II)	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
$\therefore 2m\angle ACB = 2m\angle ADB$	The RHS of (I) and (II) are equal to the same quantity.
$\therefore m\angle ACB = m\angle ADB$	Halves of equal quantities arc again equal.

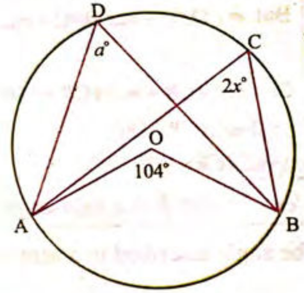
Example 2 Find the value of a in the following circle centred at O.



Solution

a = 80 (Angles in the same segment).

Example 3 Find the value of each of the variables in the following circle centred at O.



Solution

$2a = 104$ (Angle at centre theorem)

$\frac{2a}{2} = \frac{104}{2}$

$a = 52$

Also, $2x = a$ (Angles in the same segment)

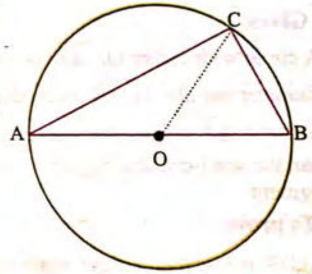
$\therefore 2x = 52$

$\frac{2x}{2} = \frac{52}{2}$

$x = 26$

Theorem 12.3(a)

The angle in a semi-circle is a right angle.



Given

A circle with centre O, $\overline{AB}$ is a diameter of the circle and $\angle ACB$ is any angle in the semi-circle.

To prove

$\angle ACB$ is a right angle i.e. $m\angle ACB = 90°$.

Construction

Join O to C.

Proof

Statements	Reasons
In ΔOAC,	
$m\overline{OA} = m\overline{OC}$	Radii of the same circle
$\therefore \Delta OAC$ is an isosceles triangle	Definition of an isosceles triangle.
and $m\angle OAC = m\angle OCA$ (I)	If two sides of a triangle are equal, the angles which are opposite to them are also equal.
Similarly in the ΔOCB	
$m\overline{OB} = m\overline{OC}$	Radii of a circle.
$\therefore m\angle OBC = m\angle OCB$ (II)	
$\therefore m\angle OAC + m\angle OBC = m\angle OCA + m\angle OCB$	Adding (I) and (II).

$m\angle OAC + m\angle OBC = m\angle ACB$ (III)	$\because m\angle OCA + m\angle OCB = m\angle ACB$
But $m\angle OAC + m\angle OBC + m\angle ACB = 180^\circ$	$\because$ The sum of three angles of a triangle is equal to 180°.
Or $m\angle ACB + m\angle ACB = 180^\circ$	$\because m\angle OAC + m\angle OBC = m\angle ACB$ Adding two equal numbers.
$\Rightarrow 2m\angle ACB = 180^\circ$	
$\therefore m\angle ACB = 90^\circ$	
or $\angle ACB$ is a right-angle.	

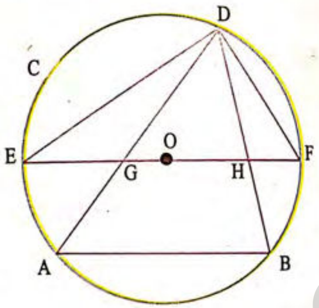
The angle inscribed in a semicircle is always a right angle (90°).

Theorem 12.3(b)

The angle in a segment greater than a semi-circle is less than a right angle.

Given

A circle with centre O, $\overline{AB}$ is a chord and $\overline{EF}$ is the diameter parallel to $\overline{AB}$ such that $\overline{AB}$ lies below the diameter $\overline{EF}$. Thus the segment AECDFB is greater than the semi-circular region. $\angle ADB$ is any angle in the segment.



To prove

$\angle ADB$ is less than a right angle i.e. $m\angle ADB < 90^\circ$.

Construction

Join D with E and F such that $\angle EDF$ is in the semi-circle.

Proof

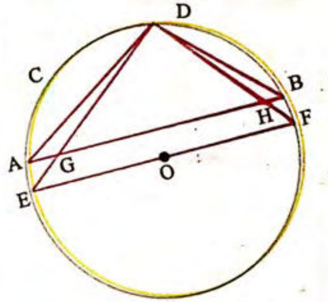
	Reasons
$m\angle EDF = 90^\circ$	The measure of an angle in a semi-circle is a right angle.
$\therefore m\angle EDG + m\angle GDH + m\angle HDF = 90^\circ$	$\because m\angle EDG + m\angle GDH + m\angle HDF = m\angle EDF$
$\therefore m\angle GDH = 90^\circ - [m\angle EDG + m\angle HDF]$	
$\Rightarrow m\angle GDH < 90^\circ$	
or $m\angle ADB < 90^\circ$	$\because \angle GDH$ and $\angle ADB$ are same.

Theorem 12.3(c)

The angle in a segment less than a semi-circle is greater than a right angle.

Given

A circle with centre O, $\overline{AB}$ is a chord and $\overline{EF}$ is the diameter parallel to $\overline{AB}$ such that $\overline{AB}$ lies above the diameter $\overline{EF}$. The segment ACDB is less than the semi-circular region. $\angle ADB$ is any angle in the segment.



To prove

$\angle ADB$ is greater than a right angle i.e. $m\angle ADB > 90^\circ$.

Construction

Join D to E and F. $\angle EDF$ is in the semi-circle.

Proof

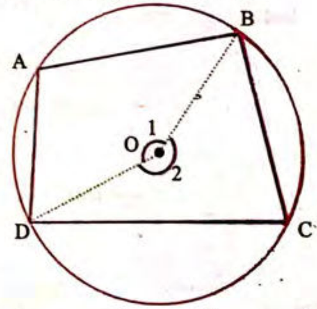
Statements	Reasons
$m\angle EDF = 90^\circ$	The measure of an angle in a semi -circle is a right angle.
or $m\angle GDH = 90^\circ$	$\because \angle EDF$ and $\angle GDH$ are same.
or $m\angle ADG + m\angle GDH + m\angle HDB = m\angle ADG + 90^\circ + m\angle HDB$	Adding equal quantities on both sides of an equation.
$\Rightarrow m\angle ADB = 90^\circ + m\angle ADG + m\angle HDB$	Postulate of addition of angles.
$\Rightarrow m\angle ADB > 90^\circ$	

Theorem 12.4

The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Given

A circle with centre at O and ABCD is a quadrilateral inscribed in the given circle.



To prove

$m\angle BCD + m\angle BAD = 180^\circ$ and $m\angle ABC + m\angle ADC = 180^\circ$

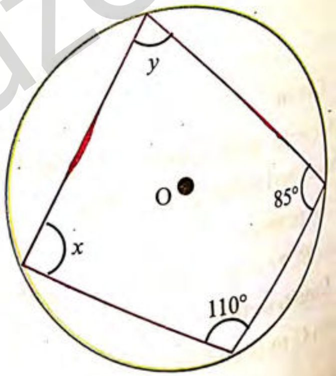
Construction

Join O to B and D.

Proof

Statements	Reasons
$m\angle 1 = 2\, m\angle BCD$	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
$m\angle 2 = 2m\angle BAD$	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
$m\angle 1 + m\angle 2 = 2m\angle BCD + 2m\angle BAD$ (I) But $m\angle 1 + m\angle 2 = 360°$ $\therefore$ (I) becomes $2[m\angle BCD + 2m\angle BAD] = 360°$	Adding the above two results. $\because$ circumference of a circle subtends an angle equal t o four right angles at the centre.
$\therefore\ m\angle BCD + m\angle BAD = \frac{360°}{2}$	Dividing both sides of an equation by 2.
or $m\angle BCD + m\angle BAD = 180°$ $\therefore \angle BCD$ and $\angle BAD$ are supplementary. Similarly we can prove that $m\angle ABC$ and $m\angle ADC$ are supplementary.	Definition of supplementary angles.

Example 4 O is the centre of the circle. Find the unknowns in the figure.

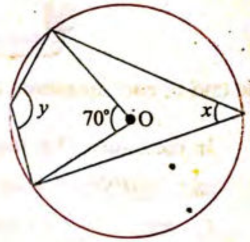


Solution Since

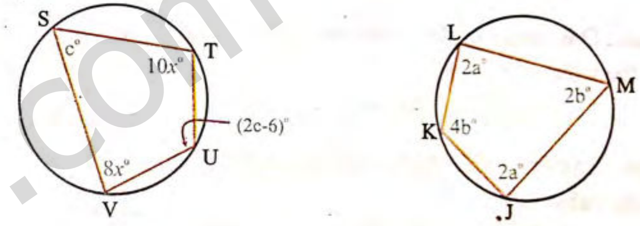
$$x + 85° = 180°$$
$$\therefore x = 180° - 85° = 95°$$
$$y + 110° = 180°$$
$$\therefore y = 180° - 110° = 70°$$

Exercise 12

1. O is the centre of the circle.
   Find the unknowns in the figure.

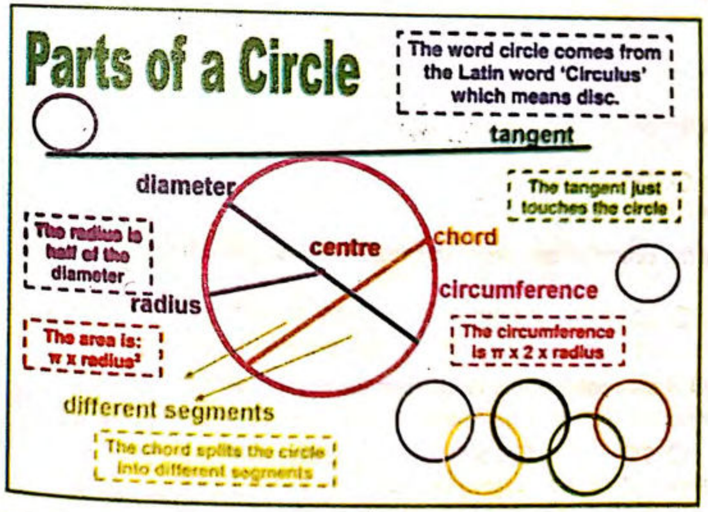


2. A regular hexagon is inscribed in a circle. Each side of the hexagon is $5\sqrt{3}$ units fro[m] centre of the circle. Find the radius of the circle.
3. Find the value of each variable.



(a) (b)

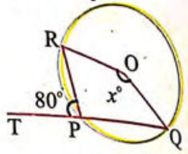
4. Show that a parallelogram inscribed in a circle will be a rectangle.



Review Exercise 12

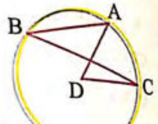
1. At the end of each question, four circles are given. Fill in the correct circle only.

(i). In the figure, O is the centre of the circle, $\angle TPR = 80°$ and $\angle QPS = x°$. Find x.



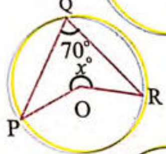
○ 80° ○ 160° ○ 100° ○ 120°

(ii). In the diagram, $\angle ADC$ is a central angle and $m\angle ADC = 60°$. What is $m\angle ABC$?



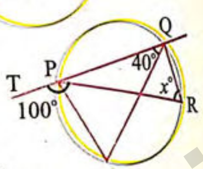
○ 15° ○ 60° ○ 120° ○ 30°

(iii). In the figure, O is centre of the circle and $\angle PQR = 70°$. Calculate the value of x.



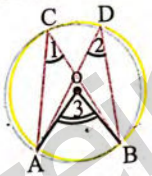
○ 140° ○ 220° ○ 290° ○ 110°

(iv). In the figure, $\angle SPT = 100°$, $\angle PQS = 40°$ and $\angle PRQ = x°$. Calculate the value of x.



○ 30° ○ 40° ○ 50° ○ 60°

(v). In the adjacent figure if $m\angle 3 = 75°$, then find $m\angle 1$ and $m\angle 2$.



○ $37\frac{1}{2}°$, $37\frac{1}{2}°$ ○ $37\frac{1}{2}°$, 75°

○ 75°, $37\frac{1}{2}°$ ○ 75°, 75°

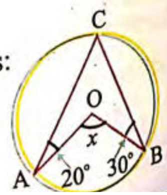
(vi). Given that O is the centre of the circle. The angle marked x will be:

○ $12\frac{1}{2}°$ ○ 25° ○ 50° ○ 75°

(vii). Given that O is the centre of the circle, the angle marked y will be:

○ $12\frac{1}{2}°$ ○ 25° ○ 50° ○ 75°

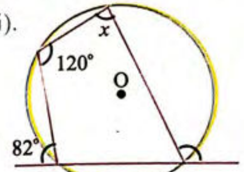
(viii). In the figure, O is the centre of the circle then the angle x is:



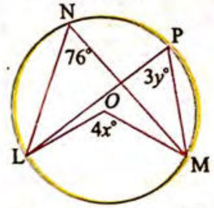
○ 50° ○ 75° ○ 100° ○ 125°

2. O is the centre of the circle. Find the unknowns in the figure.

(i).



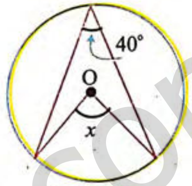
(ii).



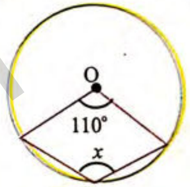
3. ABCD is a quadrilateral circumscribed about a circle. Show that $m\overline{AB} + m\overline{CD} = m\overline{BC} + m$

4. Find the value of x in each of the following figures where O is the centre of the circle.

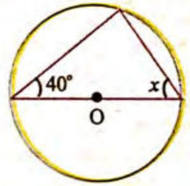
(i).



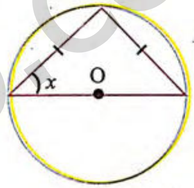
(ii).



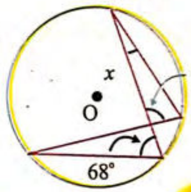
(iii).



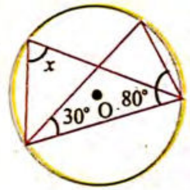
(iv).



(v).



(vi).



Summary

- The inscribed angle is an angle whose vertex is on a circle and whose sides cont chords of the circle.
- The arc that lies in the interior of an inscribed angle and has end points on the angl called the intercepted arc of the angle.
- The measure of an inscribed angle is one half the measure of its intercepted arc.
- If two inscribed angles of a circle intercept the same arc, then the angles are congruent
- A polygon is an inscribed polygon if all its vertices lie on a circle.
- The circle that contain the vertices is a circumscribed circle.
- A quadrilateral with its four vertices lying on the circumference of the circle is calle cyclic quadrilateral.
- A quadrilateral can be inscribed in a circle if and only if the opposite angles are supple
- In a circle:
  - the angle in a semicircle is right
  - the angle in a segment greater than a semicircle is acute
  - the angle in a segment less than a semicircle is obtuse.

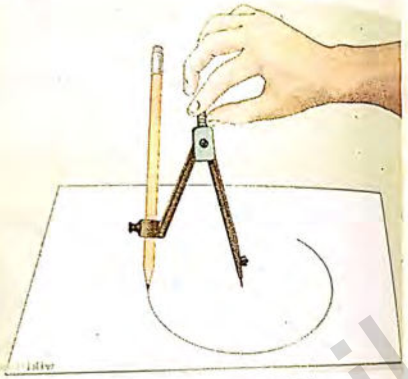
Unit 13

PRACTICAL GEOMETRY CIRCLE

In this unit the students will be able to

- Locate the centre of a given circle.
- Draw a circle passing through three given non-collinear points.
- Complete the circle:
  - by finding the centre,
  - without finding the centre,

  When a part of its circumference is given.
- Circumscribe a circle about a given triangle.
- Inscribe a circle in a given triangle.
- Escribe a circle to a given triangle.
- Circumscribe an equilateral triangle about a given circle.
- Inscribe an equilateral triangle in a given circle.
- Circumscribe a square about a given circle.
- Inscribe a square in a given circle.
- Circumscribe a regular hexagon about a given circle.
- Inscribe a regular hexagon in a given circle..
- Draw a tangent to a given arc, without using the centre, through a given point P when P is
  - the middle point of the arc,
  - at the end of the arc,
  - outside the arc.
- Draw a tangent to a given circle from a point *P* when
  - on the circumference,
  - outside the circle.
- Draw two tangents to a circle meeting each other at a given angle.
- Draw direct common tangent or external tangent,
- Transverse common tangent or internal tangent to two equal circles.
- Draw
  - direct common tangent or external tangent,
  - transverse common tangent or internal tangent to two unequal circles.
- Draw a tangent to
  - two unequal touching circles,
  - two unequal intersecting circles
- Draw a circle which touches
  - both the arms of a given angle,
  - two converging lines and passes through a given point between them,
  - three converging lines



Why it's important

Greek geometers at the time of Euclid believed that circles have a special perfection. With the rediscovery of Euclid's Elements by English philosopher Adelard (twelfth century), this way of thinking made its way into the European world. The designs in many early buildings and churches were based on geometric principles learned from Euclid. Today the practical geometry of circles is at its best in masajids.

Practical geometry is a very important branch of mathematics. In this branch we mechanical methods of constructing various geometrical figures. It is the branch whi highly essential in all draftsmanship necessary in the work of engineers, architect, surv and others. The huge buildings, bridges and dams around us are all indebted to pra geometry.

Historical Bhong Masjid Rahim Yar Khan,Pakistan

13.1 Practical Geometry

13.1.1 Locate the centre of a given circle

Given: A circle without any mention of its centre.

Required: To locate the centre of the circle.

Steps of Construction:

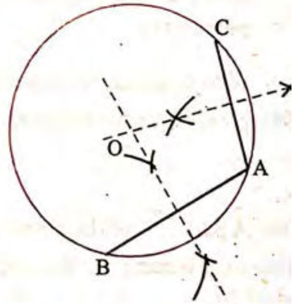
(i). Choose any three points A, B and C on the circumference of the given circle.

(ii). Join A with B and C.

(iii). Draw the perpendicular bisector of $\overline{AB}$ and $\overline{AC}$.

(iv). These bisectors intersect at O.

(v). O is the required center of the given circle.



13.1.2 Draw a circle passing through three given non–collinear points

Given: Any three points A, B and C in a plane which are non–collinear.

Required: To draw a circle passing through these points.

Steps of Construction:

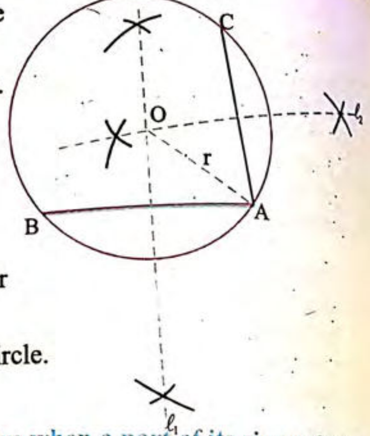
(i). Join A to B and C respectively.

(ii). Draw the perpendicular bisectors ℓ_1 and ℓ_2 of $\overline{AB}$ and $\overline{AC}$.

(iii). Point of intersection of these perpendicular bisectors is O.

(iv). Taking radius $= m\overline{OA} = m\overline{OB} = m\overline{OC}$, draw a circle.

This is the required circle.



13.1.3 (a) Complete the circle, by finding the centre when a part of its circumference is given

Given: A part $\widehat{AB}$ of the circumference of a circle.

Required: To complete the circle by finding its centre.

Steps of Construction:

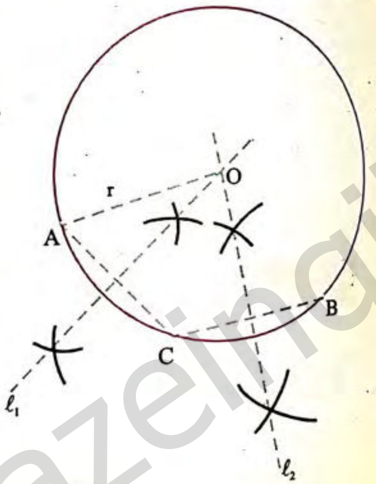
(i). Choose any point C in the given arc $\widehat{AB}$ at reasonable distance both from A and B with C as such we get $\overline{AC}$ and $\overline{BC}$.

(ii). Find the mid – points of $\overline{AC}$ and $\overline{BC}$ and draw their perpendicular bisectors ℓ_1 and ℓ_2.

(iii). The point of intersection of ℓ_1 and ℓ_2 is the centre of the required circle. Denote this point by O.

(iv). With O as centre draw a circle with radius equal to $m\overline{OA}$ or $m\overline{OB}$ or $m\overline{OC}$.

This gives the required circle.



(b) Complete the circle without finding the center when a part of its circumference is given

Given: A part $\widehat{AC}$ of the circumference of a circle.

Required: To complete the circle without finding the centre.

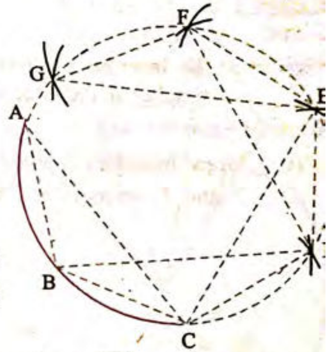
Steps of Construction:

(i). Take any point B on the arc $\widehat{AC}$.

(ii). Join A to B, B to C and A to C.

(iii). With C as a centre and radius equal to $m\overline{AB}$, draw an arc. Again taking B as centre and radius radius equal to $m\overline{AC}$, draw another ar c, which cuts the first arc at point D.

(iv). Join B and C to D. With D as centre and radius equal to $m\overline{BC}$, draw an arc and with C as centre and radius equal to $m\overline{BD}$, draw another arc which cuts the first arc at point E.



(v). Join C and D to E. With E as centre and radius equal to $m\overline{CD}$, draw an arc and w D as centre and radius equal to $m\overline{CE}$, draw another arc which cuts the first arc a

(vi). Continue the same process which gives a sequence of points getting nearer a nearer to the point A. All the points D, E, F, G ......so obtained are the points lying the circumference of the required circle.

(vii). By free –hand drawing of arcs joining the above sequence of points.

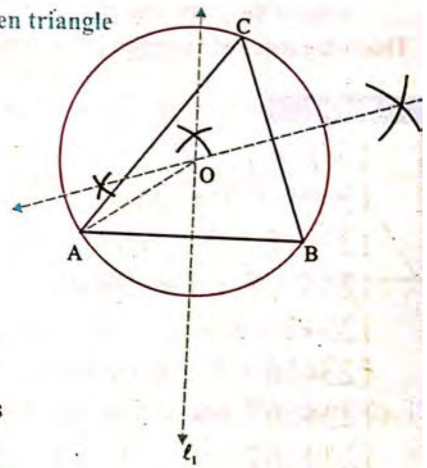
This gives the required circle.

13.2 Circles Attached to Polygons

13.2.1 Circumscribe a circle about a given triangle

Given: ΔABC

Required: To circumscribe a circle about the given ΔABC.



Steps of Construction:

(i). Draw ΔABC.

(ii). Draw perpendicular bisectors ℓ_1 and ℓ_2 of two sides $\overline{AB}$ and $\overline{BC}$, which intersect each other at O.

(iii). Draw a circle with centre O and radius equal to $m\overline{OA}$ or $m\overline{OB}$ or $m\overline{OC}$..

(iv). This is the required circumscribed circle about the given triangle.

13.2.2 Inscribe a circle in a given triangle

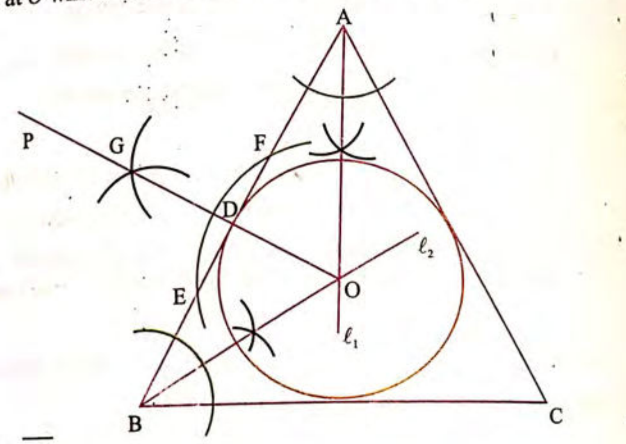
Given: ΔABC

Required: To inscribe a circle in the given triangle i.e. to construct a circle which touches all the three sides of the given triangle.

Steps of Construction:

(i). Draw bisectors ℓ_1 and ℓ_2 of any two angles say $\angle A$ and $\angle B$ of the given ΔABC.

(ii). ℓ_1 and ℓ_2 intersect at O which is the centre of the required circle.



(iii). From O draw $\overline{OD} \perp \overline{AB}$.

(iv). With O as centre and $m\overline{OD}$ as radius draw a circle, which touches all the three sides of the given triangle.

This is the required inscribed circle in the given triangle.

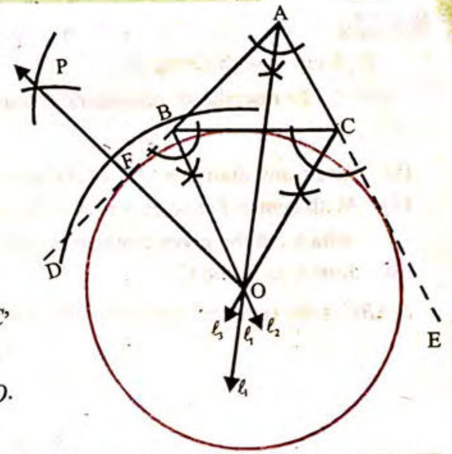
Math Fun

$1 \times 8 + 1 = 9$
$12 \times 8 + 2 = 98$
$123 \times 8 + 3 = 987$
$1234 \times 8 + 4 = 9876$
$12345 \times 8 + 5 = 98765$
$123456 \times 8 + 6 = 987654$
$1234567 \times 8 + 7 = 9876543$
$12345678 \times 8 + 8 = 98765432$
$123456789 \times 8 + 9 = 987654321$

13.2.3 Escribe a circle to a given triangle

Given: ΔABC

Required: To draw an escribed circle opposite to vertex A of the ΔABC.



Steps of Construction:

(i). Produce $\overline{AB}$ and $\overline{AC}$ to form two exterior angles $\angle CBD$ and $\angle BCE$.

(ii). Draw bisectors ℓ_1, ℓ_2 and ℓ_3 of $\angle BAC$, $\angle CBD$ and $\angle BCE$ respectively.

(iii). All these angle bisectors intersect at O.

(iv). Draw $\overline{OF} \perp \overline{AB}$.

(v). With O as centre and $m\overline{OF}$ as the radius, draw a circle, which touches $\overline{BC}$, $\overline{BD}$ and $\overline{CE}$.

This circle is the required escribed circle. Similarly escribed circles opposite to the vertices B and C can be drawn.

13.2.4 Circumscribe an equilateral triangle about a given circle

Given: A circle with centre O.

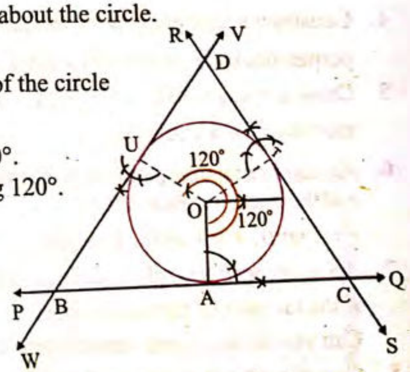
Required: To circumscribe an equilateral triangle about the circle.

Steps of Construction:

(i). Take any point A on the circumference of the circle and join O to A. $\overline{OA}$ is radial segment.

(ii). Construct an angle $\angle AOT$ of measure 120°.

(iii). Construct another angle $\angle TOU$ measuring 120°.



(iv). The three points A, T, U which are lying on the circumference of the given circle are such that they divide the boundary of the circle into three equal arcs which are $\overset{\frown}{AT}$, $\overset{\frown}{TU}$ and $\overset{\frown}{UA}$.

(v). Draw perpendiculars to $\overline{OA}$, $\overline{OT}$ and $\overline{OU}$ at A, T and U respectively. These perpendicular lines are $\overleftrightarrow{PQ}$, $\overleftrightarrow{RS}$ and $\overleftrightarrow{VW}$.

(vi). These lines intersect each other at B, C and D forming a triangle BCD.

The ΔBCD is the required equilateral triangle circumscribed about the given circle.

13.2.5 Inscribe an equilateral triangle in a given circle

Given: A circle with centre O.

Required: To inscribe an equilateral triangle in the circle.

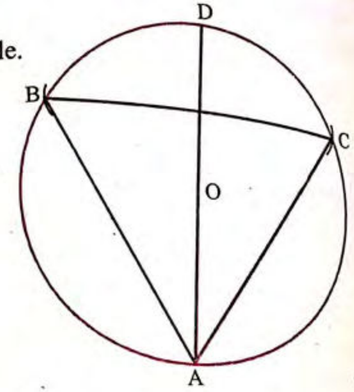
Steps of Construction:

(i). Draw any diameter $\overline{AD}$ of the given circle.

(ii). With centre D and radius $m\ \overline{OD}$, draw two arcs which cut the given circle at B and C.

(iii). Join A to B and C.

ΔABC is the required equilateral triangle.



Exercise 13.1

1. Construct a triangle with sides 2 cm, 2.5 cm and 3 cm. Also draw its circumcircle.
2. Construct a triangle ABC such that $m\overline{AB} = 3"$ $m\overline{AC} = 4"$ and $m\angle A = 60°$. Draw circumcircle to this triangle.
3. Suppose we have a triangle whose sides are 3 cm, 4 cm and 6 cm respectively. Draw its inscribed circle.
4. Construct a triangle ABC with sides $m\overline{AB} = 5cm,\ m\overline{BC} = 6cm\ \ and\ \ m\overline{CA} = 8cm$. Draw perpendicular bisectors of its sides and then circumscribe a circle.
5. Draw a triangle ABC with $m\angle A = 60°$ and $m\angle B = 45°$. Draw three angle bisectors and then inscribe a circle in it.
6. An equilateral triangle is inscribed in a circle. Find the altitude of the triangle if the radius r of the circle varies as under.
   $r = 3\ units,\ r = 4\ units,\ r = 6 units,\ r = 12\ units$. Can you deduce some result from this?
7. An equilateral triangle is circumscribed about a circle. Find the altitude of the triangle if the radius r of the circle varies as r = 2 units, r = 5 units r =10 units. Can you deduce some result from this?
8. Circumscribe an equilateral triangle about a circle of radius 2", 3" and 1".
9. Draw a triangle with sides 2.5 cm, 3.5 cm and 4.5 cm long. Draw an escribed circle to the triangle touching the longest side of the triangle.
10. For the problem in Q,9 draw an escribed circle to the triangle touching the smallest side.

Unit 13 Practical geometry circle

13.2.6 Circumscribe a square about a given circle

Given: A circle centre at O.

Required: To circumscribe a square about the given circle.

Steps of construction:

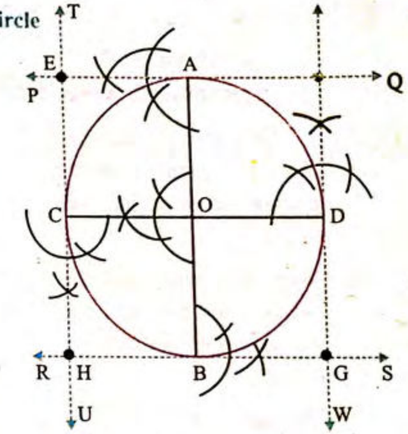
(i). Draw any diameter $\overline{AB}$ of the circle.

(ii). Draw another diameter $\overline{CD}$ which is perpendicular to $\overline{AB}$.

(iii). Draw perpendicular $\overleftrightarrow{PQ}$, $\overleftrightarrow{RS}$, $\overleftrightarrow{TU}$ and $\overleftrightarrow{VW}$ at the extremities A, B, C, D of the diameters $\overline{AB}$ and $\overline{CD}$.

(iv). These lines cut each other at the points E, F, G and H.

(v). EFGH is the required circumscribed square about the given circle.



13.2.7 Inscribe a square in a given circle

Given: A circle with centre O.

Required: To inscribe a square in the given circle.

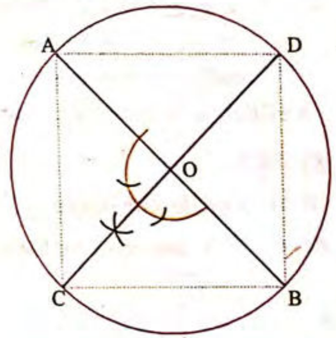
Steps of construction:

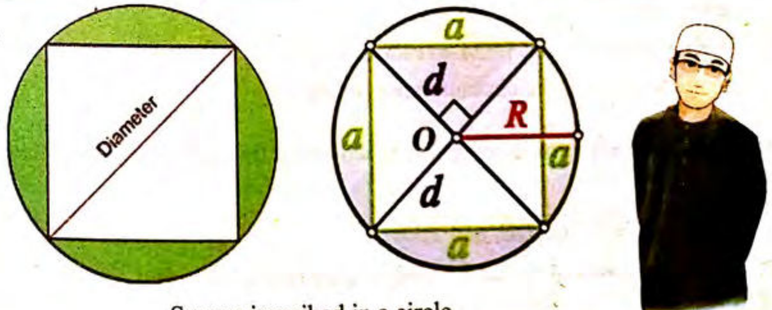
(i). Draw any diameter, say $\overline{AB}$, of the circle.

(ii). Draw another diameter $\overline{CD}$ of the circle which is perpendicular to $\overline{AB}$.

(iii). Draw $\overline{AC}$, $\overline{CB}$, $\overline{BD}$ and $\overline{DA}$.

$ACBD$ is the required inscribed square.





Square inscribed in a circle.

12.8 Circumscribe a regular hexagon about a given circle

Given: A circle with centre O.

Required: To circumscribe a regular hexagon about the given circle.

Steps of construction:

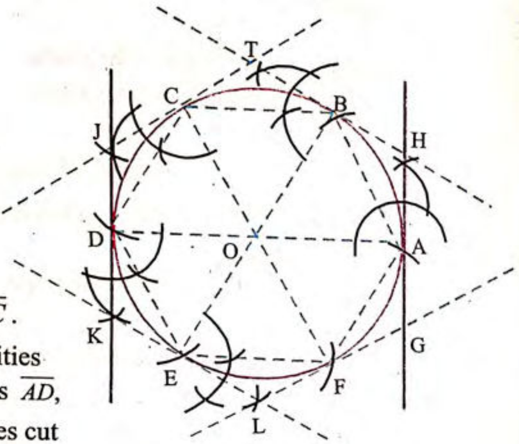
(i). Take any point A on the circumference of the given circle.

(ii). With A as centre and $m\overline{OA}$ as radius, draw two arcs which cut the circumference of the circle at B and F.

(iii). Through O, draw $\overline{AD}$, $\overline{BE}$ and $\overline{FC}$.

(iv). Draw perpendiculars at the extremities A, B, C, D, E and F of the diameters $\overline{AD}$, $\overline{EB}$ and $\overline{FC}$ of the circle. These lines cut each other at points G, H, I, J, K and L.

GHIJKL is the required circumscribed hexagon.



12.9 Inscribe a regular hexagon in given circle

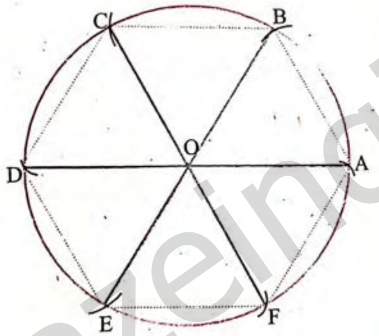
Given: A circle with centre O.

Required: To inscribe a regular hexagon in the given circle.

Steps of construction:

(i). Take any point A on the circumference of the given circle.

(ii). With A as centre and $m\overline{OA}$ as radius, draw an arc which cuts the circumference of the circle at B. Similarly draw successive arcs which cut the circumference of the circle at D, E and F.



(iii). Draw $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}$ and $\overline{EF}$.

ABCDEF is the required hexagon which is inscribed in the given circle.

Exercise 13.2

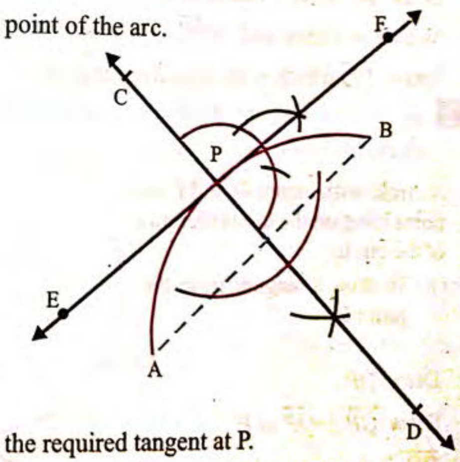
1. Circumscribe a square about a circle of radius 5cm.
2. Inscribe a square in a circle of radius 6 cm.
3. Draw a square of side 6 cm. Circumscribe a circle about that square and then inscribe a circle in the same square. Measure the radii of these two circles.
4. First draw a circle of suitable radius, so that the square circumscribed about that circle has sides of length 8 units.
5. Inscribe a square of side 10 cm in a circle. What will be the size of the radius?
6. Inscribe a regular hexagon in a circle of radius 4 cm.
7. Construct a circle of radius 4cm and draw a regular hexagon about the circle.
8. Draw a circle of radius 8 cm. Circumscribe a regular hexagon about that circle and also inscribe a regular hexagon in the same circle. Find the areas of these geometrical figures. Comment on the values of these areas.
9. Draw two regular hexagons of perimeters 6 cm and 30 cm respectively. Determine their centres. From their centers draw perpendicular to any of their sides respectively. What is the relation of these two perpendiculars?
10. Can you construct a square whose area equals the areas of a given circle? Discuss in detail.

13.3 Tangent to the circle

13.3.1 (i) Draw a tangent to a given arc, without using the centre, through a given point *P* when *P* is the middle point of the arc

Given: An arc such that P is the mid point of the arc.

Required: To draw a tangent at P.



Steps of construction:

(i). Draw $\overline{AB}$.

(ii). From P, draw $\overleftrightarrow{CD} \perp \overline{AB}$.

(iii). At P, draw $\overleftrightarrow{EF} \perp \overleftrightarrow{CD}$ which is the required tangent at P.

(ii) Draw a tangent to a given arc, without using the centre, through a given point P when P is at the end of the arc

Given: An arc $\overparen{AP}$ of a circle.

Required: To draw a tangent at the end point P.

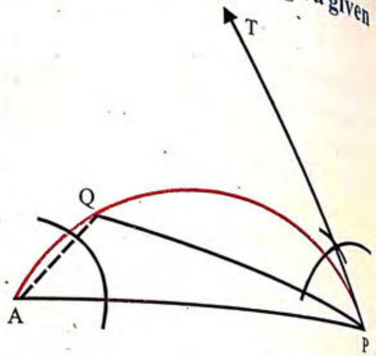
Steps of Construction:

(i). Take any point Q on the arc $\overparen{AP}$ and draw the chords $\overline{AP}$ and $\overline{PQ}$.

(ii). Join A and Q.

(iii). Now construct $\angle QPT \cong \angle PAQ$.

(iv). $\overrightarrow{PT}$ is the required tangent at P.



(iii) Draw a tangent to a given arc, without using the centre, through a given point P when P is outside the arc

Given: $\overparen{AT}$ is an arc and P is a point outside the arc.

Required: To draw a tangent from point P on the arc.

Steps of Construction:

(i). Join A and P. Which cuts the arc at B.

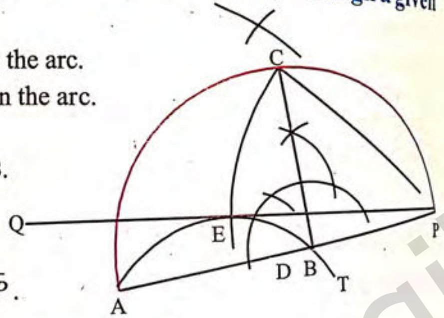
(ii). Bisect $\overline{AP}$ at D.

(iii). With D as centre and radius equal to $m\overline{DP}$, draw a semi - circle on $\overline{AP}$.

(iv). Draw $\overline{BC} \perp \overline{AP}$, which intersects the semi-circle at C.

(v). With P as centre and $m\overline{PC}$ as radius draw an arc to cut the given arc at E.

(vi). Draw $\overleftrightarrow{PE}$ which is the required tangent.



13.3.2 (i) Draw a tangent to a given circle from a point P when P lies on the circumference

Given: A circle with centre O and P is a point lying on the circumference of the circle.

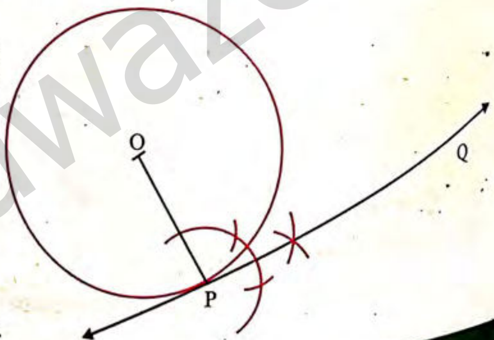
Required: To draw a tangent from the point P.

Steps of construction:

(i). Draw $\overline{OP}$.

(ii). Draw $\overline{QP} \perp \overline{OP}$ at P.

(iii). $\overleftrightarrow{PQ}$ is the required tangent at P.



(ii) Draw a tangent to a given circle from a point P when P lies outside the circle

Given: A circle with centre at O. P is a point lying outside the circle.

Required: To draw a tangent to the circle from the given point P.

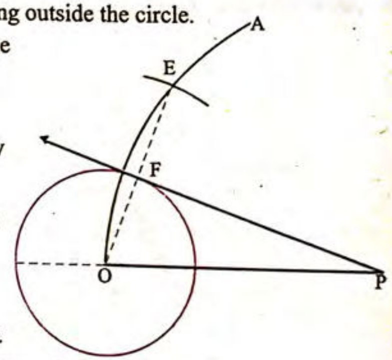
Steps of Construction:

(i). With P as centre and $m\overline{OP}$ as radius, draw an arc $\overparen{OA}$.

(ii). With O as center and the diameter of the given circle as radius, draw another arc which intersects arc $\overparen{OA}$ at E.

(iii). Draw $\overline{OE}$ which intersects the circle at F.

(iv). Through F draw $\overleftrightarrow{PF}$ which is the required tangent.



13.3.3 Draw two tangents to a circle meeting each other at a given angle

Given: A circle with centre O and an angle equal to angle θ. The vertex of the given angle is k.

Required: To draw two tangents to the given circle such that the tangents are inclined to each other at the angle θ.

Steps of Construction:

(i). Draw any diameter $\overline{AB}$.

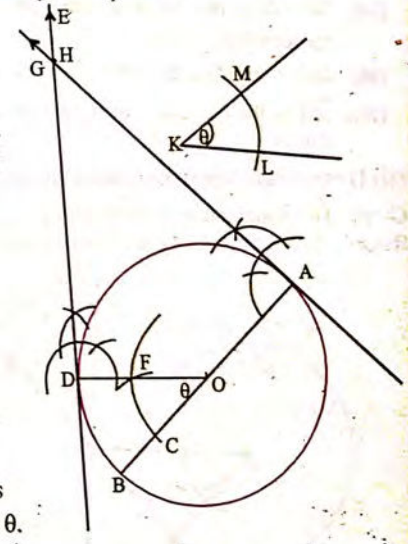
(ii). At O, construct $\angle BOD \cong \angle\theta$.

(iii). At D draw $\overline{DE} \perp \overline{OD}$.

(iv). At A draw $\overline{AG} \perp \overline{OA}$.

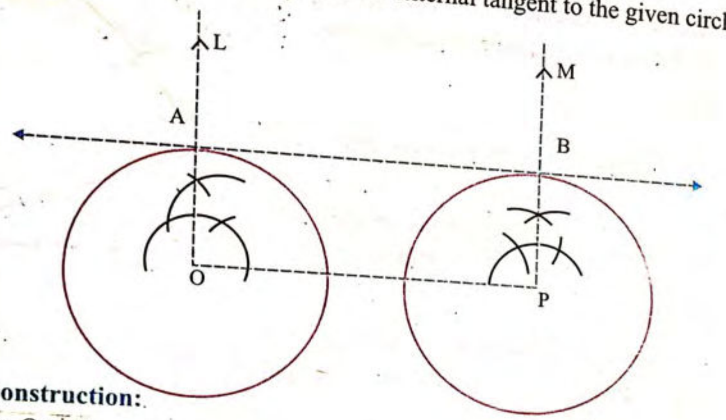
(v). These two perpendiculars intersect each other at H.

(vi). $\overleftrightarrow{DH}$ and $\overleftrightarrow{AH}$ are the required tangents inclined to each other at angle the angle θ.



13.3.4 (i) Draw direct common tangent or external tangent to two equal circles

Given: Two equal circles with centres at O and P respectively.
Required: To draw direct common tangent or external tangent to the given circles.

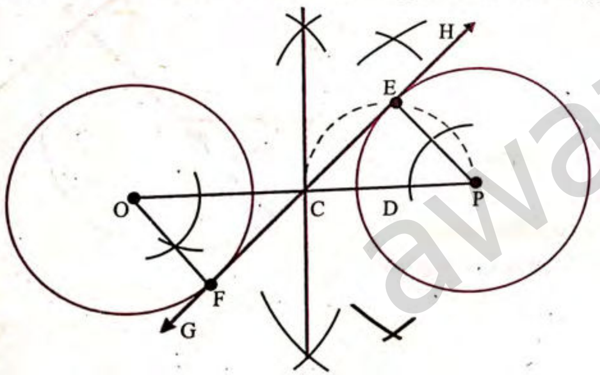


Steps of Construction:

(i). Join O to P.
(ii). Now draw two perpendiculars on $\overline{OP}$ at O and P which cut the circles at A and B respectively.
(iii). Join A and B and produce it towards both ends.
(iv). $\overleftrightarrow{AB}$ is the required direct common tangent or external tangent to two given equal circles.

(ii) Draw transverse common tangent or internal tangent to two equal circles

Given: Two equal circles with centres at O and P respectively.
Required: To draw transverse common tangent or internal tangent to the two given circles.

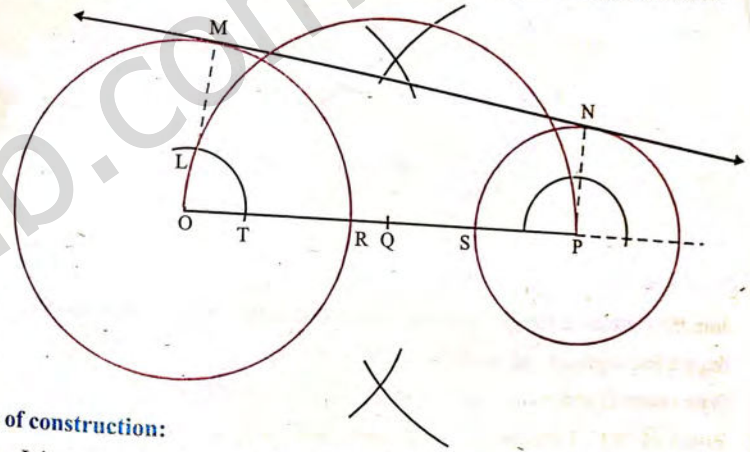


Steps of construction:

(i). Join O and P, the centers of the given circles.
(ii). Bisect $\overline{OP}$ at C.
(iii). Bisect $\overline{CP}$ at D.
(iv). Draw a semi−circle on $\overline{CP}$ diameter which cuts the given circle at E.
(v). Join E and P.
(vi). Draw $\overline{OF} \parallel \overline{EP}$.
(vii). Join E and F and extend the line to both directions.
(viii). $\overleftrightarrow{GH}$ is the required tangent to both the circles.

13.3.5 (i) Draw direct common tangent or external tangent to two un –equal circles

Given: Two unequal circles with centres at O and P respectively.
Required: To draw a direct common tangent or external tangent to the given circles.



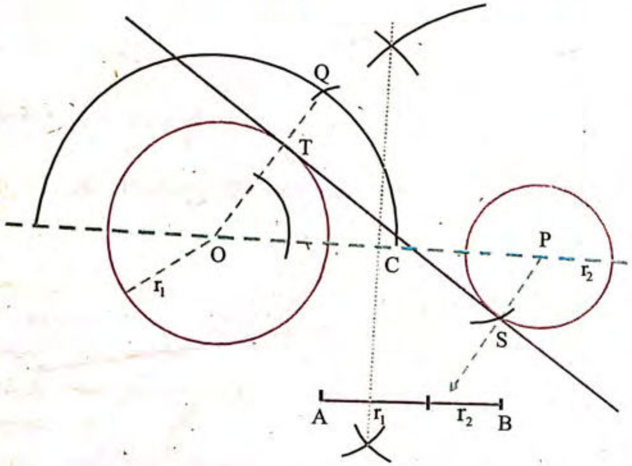
Steps of construction:

(i). Join centres of the circles O and P. $\overline{OP}$ cuts the circles at *R* and *S* respectively.
(ii). Bisect $\overline{OP}$ at Q. With Q as centre and $m\overline{QP}$ as radius, draw a semi circle.
(iii). Take a point T on $\overline{OP}$ in the bigger circle such that $m\overline{RT} = m\overline{SP}$ =length of the radius of the smaller circle.
(iv). With center O and $m\overline{OT}$ as radius, describe an arc cutting the semicircle at *L*.
(v). Draw $\overrightarrow{OL}$ to meet the bigger circle at *M*.
(vi). Draw a $\overline{PN} \parallel \overline{OM}$. Join M and N and produce it to both sides to get the common tangent or external tangent $\overline{MN}$ to the two given circles.

(ii) Draw transverse common tangent or internal tangent to two unequal circles

Given: Two unequal circles with centres O and P having radii r_1 and r_2, where $r_1 > r_2$

Required: To draw a transverse common tangent or internal tangent to the given circles.



Steps of Construction:

(i). Join the centres of the given circles and produce the line to both directions. Also draw a line segment $\overline{AB}$ such that $m\overline{AB} = r_1 + r_2$.

(ii). With centre O and radius equal to $m\overline{AB}$, draw a semicircle.

(iii). Bisect $\overline{OP}$ at C. Then choosing C as centre and $m\overline{OC}$ as radius, draw an arc which cuts the semicircle at Q.

(iv). Draw $\overline{OQ}$ cutting the bigger circle at T. Now from P, draw $\overrightarrow{PS} \parallel \overline{OQ}$ in the opposite sense cutting the smaller circle at S.

(v). Join $\overline{OQ}$ cutting the given circle at T. Now draw a line parallel to $\overline{OQ}$ but passing through P which is the centre of the smaller circle. This line cuts the smaller circle at S.

(vi). Join T and S and produce it in both directions. The line $\overleftrightarrow{TS}$ so obtained is the required transverse tangent or internal tangent to the given unequal circles.

13.3.6 (a) Draw a tangent to two unequal touching circles

Given: Two unequal circles with centers O and P touching each other at T.

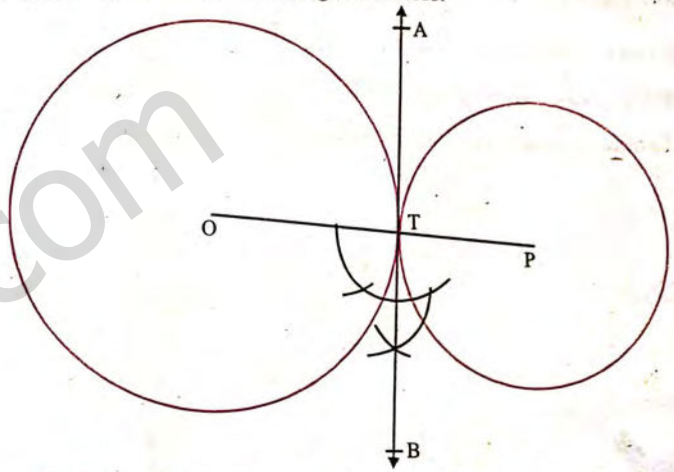
Required: To draw a tangent to the circles.

Steps of Construction:

(i). Join the centers O and P of the given circles. $\overline{OP}$ passes through T, the point contact.

(ii). At T draw a line $\overleftrightarrow{ATB} \perp \overline{OP}$.

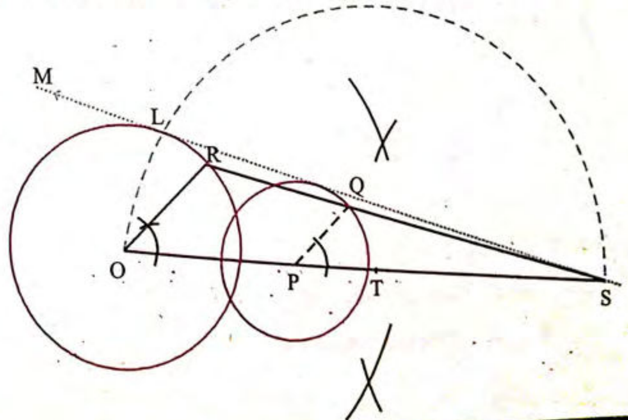
(iii). $\overleftrightarrow{ATB}$ is the required tangent to the given circles.



(b) Draw a tangent to two unequal intersecting circles

Given: Two unequal circles with centers O and P and intersecting each other.

Required: To draw a tangent to the given circles.



Steps of Construction:

(i). Join the centers of the given circles and produce it to the right.

(ii). Draw a line segment $\overline{PQ}$ as a radius of the smaller circle such that $\overline{PQ}$ is not along $\overline{OP}$. Also draw $\overline{OR}$, the radius of the bigger circle such that $\overline{OR}$ is parallel to $\overline{PQ}$.

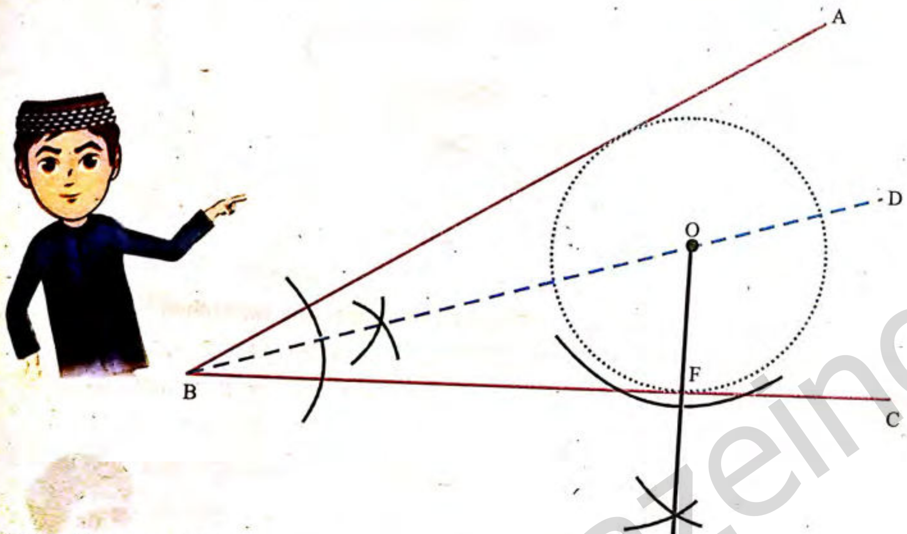
(iii). Join R and Q and produce it to meet the line $\overline{OP}$ produced at S. Bisect the line segment $\overline{OS}$ at T and draw a semicircle on $\overline{OS}$ which cuts the bigger circle at L.

(iv). Finally join S and L and produce it to M. $\overline{SM}$ is the required tangent.

13.3.7 (i) Draw a circle which touches both the arms of a given angle

Given: An angle $\angle ABC$ whose vertex is at B. $\overline{AB}$ and $\overline{BC}$ are its arms.

Required: To draw a circle which touches both the arms $\overline{AB}$ and $\overline{BC}$ of the given angle.



Steps of construction:

(i). Draw $\overline{BD}$, the bisector of $\angle ABC$.

(ii). Take any point O on $\overline{BD}$ and draw a perpendicular $\overline{OF}$ on $\overline{BC}$.

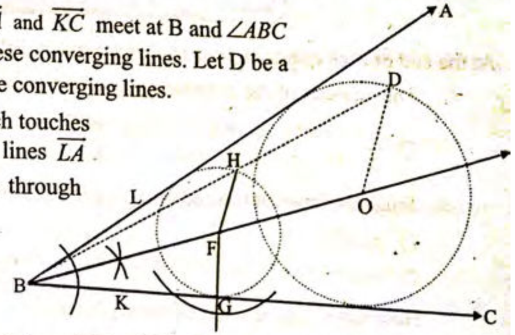
(iii). Taking O as centre and $m\overline{OF}$ as radius draw a circle.

(iv). This is the required circle.

(ii) Draw a circle which touches two converging lines and passes through given point between them

Given: Two converging lines $\overline{LA}$ and $\overline{KC}$ meet at B and $\angle ABC$ is an angle formed by these converging lines. Let D be a point lying between these converging lines.

Required: To draw a circle which touches the two converging lines $\overline{LA}$ and $\overline{KC}$ and passes through given point D.



Steps of construction:

(i). Draw the bisector $\overline{BE}$ of the angle $\angle ABC$.

(ii). Take a point F on $\overline{BE}$ and draw $\overline{FG} \perp \overline{BC}$. With F as centre and $m\overline{FG}$ as radius draw a circle. This circle touches the given converging lines tangentially.

(iii). Join D and B $\overline{DB}$ cuts the circle at H. Join F to H and draw $\overline{DO} \parallel \overline{FH}$.

(vi). With O as centre and $m\overline{DO}$ as radius, draw a circle C which as required.

Exercise 13.3

1. Draw an arc of length 7 cm. Without using the center draw a tangent through a given point P when P is
   (i). The middle point of the arc.
   (ii). End point of the arc.
   (iii). Outside the arc.
2. Draw a circle passing through a point D and touching a given line $\overleftrightarrow{BC}$ at point D.
3. Describe a circle of radius 4cm, passing through a given point C and touching a given straight line $\overleftrightarrow{AB}$.
4. Radius of a circle is 2.5 cm. A point Q is at a distance of 5cm from the centre. Draw tangents to the circle from the point Q.
5. Radii of two circles are 2 cm and 3cm and their centres are 8 cm apart. Draw direct common tangents to the circles.
6. Two congruent circles are of radius 4 cm each. Their centres are 10 cm apart. Draw transverse common tangents to these circles.
7. Radii of two circles are 2cm and 2.5 cm respectively. Distance between their centres is 5.5 cm. Draw transverse common tangents to the circles.
8. Draw $\angle ABC$ of measure $60°$. Construct a circle having radius 2.5 cm and touching the arms of the angle.

Review Exercise 13

1. At the end of each question, four circles are given. Fill in the correct circle only.

 (i). The measure of the external angle of a regular hexagon is

 ○ $\frac{\pi}{3}$ ○ $\frac{\pi}{4}$ ○ $\frac{\pi}{6}$ ○ none of these

 (ii). Tangents drawn at the end points of the diameter of circle are

 ○ parallel ○ perpendicular
 ○ intersecting ○ none of these

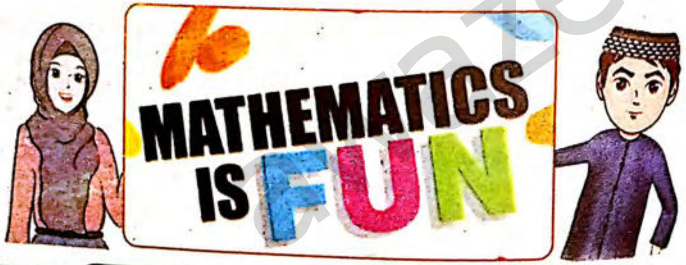
 (iii). How many tangents can be drawn from a point outside the circle?

 ○ 1 ○ 2 ○ 3 ○ 4

 (iv). If the distance between the centers of two circles is equal to the sum of their radii, then the circles will

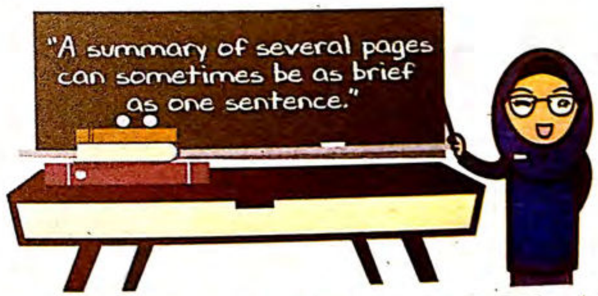
 ○ intersect ○ do not intersect
 ○ touch each other externally ○ touch each other internally

2. Practically find the centre of an arc ABC.
3. Escribe a circle opposite to vertex A of a triangle ABC with sides $|AB| = 5cm$, $|BC| = 4cm$, $|CA| = 3cm$. Find its radius also.
4. Circumscribe a circle about an equilateral triangle ABC with each side of length 5cm.
5. Circumscribe a regular hexagon about a circle of radius 4cm.
6. Construct a circle of radius 3 cm. Draw two tangents making an angle of 60° with each other.
7. Draw two equal circles each of radius 3.5cm. If the distance between their centres is 7cm, then draw their transverse tangents.
8. Draw two comon tangents to two intersecting circles of radii 2.5cm and 3.5cm.
9. Draw two common tangents to two touching circle of radii 3 cm and 4cm.



Summary

- The perpendicular bisectors of two non-parallel chords of a circle intersect at a point which is known as centre of the circle.
- A circle of any radius can be traced by rotating a compass about fixed point.
- A circle can be drawn through given three non-collinear points.
- If a triangle, the circumscribed circle, inscribed circle and escribed circle opposite to each vertex can be constructed.
- When a part of circumference of a circle is given, the circle can be completed.
- If a circle is given, then the circumscribed and inscribed equilateral triangles can be constructed.
- For a given circle, the circumscribed an inscribed squares can be drawn.
- For a given circle, the circumscribed and inscribed regular hexagon can be constructed.
- Tangents can be drawn to a given circle, when a point is an its circumference and from a point outside the circle.
- We can draw tangents to a given arc as its mid point, its any end point, and a point not on the arc.
- Tangents to two unequal touching circles can be traced.
- We can construct a circle touching the arms of a given angle.
- Direct or transverse common tangents of two equal circles or two unequal circles can be drawn.
- A circle passing through a given point between two converging lines and touching each of them, can be traced.



ANSWERS

Exercise 1.1

1.

i. $\{-1, -4\}$ ii. $\{1, 5\}$ iii. $\{-5, 2\}$

iv. $\left\{\frac{1}{2}, \frac{5}{3}\right\}$ v. $\left\{-1, \frac{7}{2}\right\}$ vi. $\{3, -7\}$

2.

i. $\{-10, 4\}$ ii. $\{5+\sqrt{14}, 5-\sqrt{14}\}$ iii. $\{0, -2\}$

iv. $\{-12, 14\}$ v. $\left\{\frac{1}{9}, \frac{5}{9}\right\}$ vi. $\{-2, 5\}$

3.

i. $\{3, 5\}$ ii. $\{1+\sqrt{5}, 1-\sqrt{5}\}$ iii. $\left\{0, \frac{-3}{4}\right\}$

iv. $\left\{1+\frac{\sqrt{6}}{3}, 1-\frac{\sqrt{6}}{3}\right\}$ v. $\left\{\frac{3}{2}, \frac{4}{3}\right\}$ vi. $\left\{\frac{1}{2}, -\frac{1}{4}\right\}$

4.

i. $\{1, 7\}$ ii. $\{12, -6\}$ iii. $\{-2+\sqrt{3}, -2-\sqrt{3}\}$

iv. $\{-10\}$

5. $\{-13\}$ **6.** $\left\{1, \frac{3}{2}\right\}$ **7.** $\{\sqrt{85}\}$

Exercise 1.2

1.

(i) $\{\pm 1, \pm 2\}$ (ii) $\{\pm 2, \pm\sqrt{3}\}$ (iii) $\left\{\pm\frac{1}{\sqrt{2}}, \pm\sqrt{\frac{5}{3}}\right\}$ (iv) $\left\{0, \frac{-5}{2}\right\}$

(v) $\{1\pm\sqrt{5}\}$ (vi) $\left\{10, \frac{-2}{5}\right\}$ (vii) $\left\{1, 3, \frac{1}{3}\right\}$

(viii) $\{1, 4\pm\sqrt{15}\}$ (ix) $\left\{\frac{-1\pm\sqrt{5}}{2}, 1\pm\sqrt{2}\right\}$ (x) $\{2, 0\}$

(xi) $\{0, -2\}$ (xii) $\{-1, 2\}$ (xiii) $\{-1\pm\sqrt{71}, 2, -4\}$

(xiv) $\left\{\frac{-5\pm\sqrt{5}}{2}\right\}$ (xv) $\{-4\pm\sqrt{5}\}$

2. $\{\pm 1, 1\pm\sqrt{2}\}$

Exercise 1.3

1.

(i) $\{3\}$ (ii) $\{5\}$ (iii) $\{5\}$ (iv) $\{1\}$

(v) $\{2\}$ (vi) $\{0\}$ (vii) $\{4\}$ (viii) $\{6\}$

(ix) $\{0, -2\}$ (x) $\left\{-\frac{1}{2}, -1\right\}$

2. -13

Review Exercise 1

1.

i. $x = -1, 5$ ii. $\frac{1\pm\sqrt{5}}{2}$ iii. cannot be simplified

iv. $a = 2, b = -1, c = -3$ v. $x = 4, -1$ vi. $x = \frac{-2\pm\sqrt{22}}{2}$

vii. $x = \pm\frac{1}{2}$ viii. 2 or -9 ix. $x = 3$ or $x = 5$

2. $\left\{\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}\right\}$ **3.** $a = -2, b = 0$ **4.** $x = -2$

5. $\left\{6, \frac{22}{49}\right\}$

Exercise 2.1

1. (i) -36 (ii) 9 (iii) -3
2. (i) Real (irrational) and unequal (ii) Real (rational) and unequal
   (iii) Complex and unequal
3. (i) ± 6 (ii) ± 12 (iii) $\frac{25}{4}$
4. (i) $\left\{\frac{-5\pm\sqrt{5}}{2}\right\}$ (ii) $\left\{\frac{-3}{2}, \frac{-3}{2}\right\}$ (iii) $\left\{\frac{1}{2}, \frac{-2}{3}\right\}$
5. (i) The roots are real (rational) and unequal: $3, \frac{1}{3}$
   (ii) The roots are real (irrational) and unequal $: 3+\sqrt{5}, 3-\sqrt{5}$
   (iii) The roots are real (irrational) and unequal $: \sqrt{3}\ , -\sqrt{3}$
6. (i) (a) $k \le \frac{9}{8}$ (b) $k > \frac{9}{8}$ (ii) (a) $k \le 1$ (b) $k > 1$
   (iii) (a) $k \le \frac{25}{4}$ (b) $k > \frac{25}{4}$

Exercise 2.2

1. (i) $-1, -\omega, -\omega^2$ (ii) $2, 2\omega, 2\omega^2$ (iii) $-3, -3\omega, -3\omega^2$
2. (i) 0 (ii) $-128\omega^2$ $6(\omega-1)$

Exercise 2.3

1. (i) Sum of the roots $= 1$, Product of the roots $= \frac{-3}{4}$
   (ii) Sum of the roots $= \frac{-5}{2}$, Product of the roots $= 3$
   (iii) Sum of the roots $= \frac{-2}{3}$, Product of the roots $= \frac{-5}{3}$

2. $k=-6$ 3. $k=\pm 1$
4. $k=6$ 5. $k=16$
6. $k=6$ 7. $m=5, n=-3$

Exercise 2.4

1. (i) $\frac{c(b^2-2ac)}{a^3}$ (ii) $\frac{b^2-4ac}{a^2}$
2. (i) $2x^2-3x+1=0$ (ii) $x^2-x-12=0$ (iii) $x^2-6x+7=0$
   (iv) $x^2+ax-2a^2=0$
3. $a^2x^2-(b^2-2ac)x+c^2=0$
4. (i) $\frac{5}{2}$ (ii) 5 (iii) $-\frac{9}{4}$
5. $15x^2+26x+15=0$ 6. $2x^2-12x+17=0$

Exercise 2.5

1. (i) $Q(x)=3x^2-7x+20,\ R=-61$
   (ii) $Q(x)=2x^2-x+9,\ R=0$
   (iii) $Q(x)=2x^3-4x^2+5x-5,\ R=3$
2. k = 8 3. $p=-4, q=1$ 4. $a=-2$, b = −1
5. $-1, -2$ 6. $2\pm\sqrt{3}$

Exercise 2.6

1. (i) $(1, -1), \left(\frac{7}{5}, -\frac{1}{5}\right)$ (ii) $(-4, 2), (4, -2)$ (iii) $(2, 12), (-4, 0)$
   (iv) $(1, 2), \left(\frac{13}{5}, -\frac{6}{5}\right)$ (v) $(1, 0), (-1, 0)$

vi. $(3, 6), (-3, 6), (3, -6), (-3, -6)$

vii. $\left(\frac{1}{\sqrt{2}}, 1\right), \left(-\frac{1}{\sqrt{2}}, 1\right), \left(\frac{1}{\sqrt{2}}, -1\right), \left(-\frac{1}{\sqrt{2}}, -1\right)$

2. i. $(18, -9)$ ii. $(-2, 2), (-1, 3)$

Exercise 2.7

1. 8, 9
2. 3, 4, 5 or −3, −4, −5,
3. 9 m, 4m
4. 4, 7
5. 6 and 8, −6 and −8
6. 21m, 12m
7. 9 cm, 6 cm
8. 9 cm, 12 cm
9. 3 cm, 5 cm, 4 cm
10. 15 goats.

Review Exercise 2

1. i. 2 ii. $x^2 - 2x + 15 = 0$ iii. none of these

   iv. $\frac{3}{7}$

2. $k = \frac{25}{12}$
3. −128
4. (i) Sum of roots = 0 , Product of roots = $-\frac{1}{4}$

   (ii) Sum of roots = $-\frac{4}{3}$, Product of roots = 0
5. $k = \frac{4}{3}$
6. $k = 1$
7. $6x^2 - 17x + 12 = 0$
8. The other root is 1, $k = -6$
9. $-1, -2$
10. (i) $(-1,4),(4,-1)$

    (ii) $(\sqrt{2},\sqrt{2}),(-\sqrt{2},\sqrt{2}),(\sqrt{2},-\sqrt{2}),(-\sqrt{2},-\sqrt{2})$
11. 8cm, 6cm

Exercise 3.1

1. 151:208
2. $x = 63$
3. First Partner = Rs. 8,000, Second Partner = Rs. 12,800, Third Partner = Rs. 19,200
4. 14.5
5. (i). $y = 81$ (ii). $x = 10, -10$
6. (i) $y = 2$ (ii). $x = 1$
7. (i). $r = 54$ (ii). $p = 18$
8.

x	4	6	7	15
y	2	3	3.5	7.5

Exercise 3.2

1. (i) Continued Proportion

   (ii) Not in continued proportion (iii) Continued Proportion
2. ±6
3. 45
4. $\frac{4}{7}$
5. ±(a + b)
7. (i) $x = -9$ (ii) $-\frac{1}{8}, 5.5$ (iii) $\sqrt{\frac{5}{3}}a$

Exercise 3.3

1. $y = \frac{968}{9}$
2. $f = 405$
3. $a = 2$
4. $p = 2400$
5. $a = \frac{21}{2}$
6. $k = \frac{3}{2}$
7. $y = kx^2z, 32$
8. $p = \frac{12}{5}, \frac{qr^2}{st^2}, -\frac{128}{5}$

Exercise 3.5

1. (i) 18cm (ii). 3
2. $119 N/cm^2$
3. $100 N/cm^2$
4. 8 days
5. $400 N/m^2$
6. 6cm.
7. $V = 2816\ cm^3$

Review Exercise 3

1.

i. $a \propto b$ ii. $mn = k$ iii. 3 to 2

iv. $\frac{a}{c} = \frac{b}{d}$ v. $x = 21$ vi. $\frac{y^2}{x}$

vii. $x \propto z$ viii. $a = \frac{11}{2}$ ix. $\frac{\ell a + mc + ne}{\ell b + md + nf}$

x. $x = \frac{7}{16} y$

2. $k = \frac{1}{20}$
3. $x = 2$
4. 300 units
5. 60amp
6. $a = 126$
7. 7
8. 5
9. 32, 12
10. 8, 4, 2
11. 4, 9
12. 45°, 60°, 75°

Exercise 4.1

1. $\frac{2}{x} - \frac{1}{2x-1}$
2. $\frac{3}{2(x+5)} - \frac{1}{2(x+1)}$
3. $-\frac{1}{2(x+1)} + \frac{1}{2(x-1)}$
4. $\frac{5}{6(x+5)} + \frac{1}{6(x-1)}$
5. $\frac{6}{5(x+2)} + \frac{8}{5(2x-1)}$
6. $\frac{9}{4(x-1)} - \frac{5}{4(x+2)} + \frac{1}{2(x+1)^2}$
7. $\frac{11}{x+3} - \frac{10}{x+2} + \frac{6}{(x+2)^2}$
8. $-\frac{1}{9x} + \frac{1}{9(x-3)} + \frac{5}{3(x-3)^2}$
9. $1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}$
10. $\frac{1}{4(x+1)} + \frac{1}{2(x-1)^2} + \frac{3}{4(x-1)}$

Exercise 4.2

1. $\frac{1}{x} - \frac{x}{x^2+1}$
2. $\frac{5}{4(x-1)} - \frac{x-11}{4(x^2+3)}$
3. $\frac{-3x+1}{x^2+1} + \frac{3}{2(x-1)}$
4. $-\frac{3}{5x^2} + \frac{3}{5(x^2+5)}$
5. $-\frac{2}{7(x+4)} + \frac{1}{7}\frac{6x-3}{3x^2+1}$
6. $\frac{-5}{x+1} + \frac{5x-5}{x^2-2} - \frac{5x-10}{(x^2-2)^2}$
7. $\frac{4}{5(x-2)} - \frac{4x+8}{5(x^2+1)} + \frac{x-2}{(x^2+1)^2}$
8. $\frac{4x-5}{(x^2+4)^2}$
9. $-\frac{4}{(x^2+1)^2} + \frac{2}{x^2+1} + \frac{1}{x+1} - \frac{1}{x-1}$
10. $\frac{-1-x}{(x^2+1)^2} + \frac{1}{2}\frac{-3x-3}{x^2+1} + \frac{3}{2(x-1)}$

Review Exercise 4

1. (i) $\dfrac{1}{2(x-1)}-\dfrac{1}{2(x+1)}$ (ii) Rational fraction (iii) Improper fraction

(iv) x^2-3x+1

2. i. $2+\dfrac{1}{x-1}-\dfrac{1}{x+1}$ ii. $2x+3+\dfrac{30}{x-2}-\dfrac{16}{x-1}$

iii. $-\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{2}{(x-1)^2}$ iv. $\dfrac{1}{x-1}+\dfrac{2}{(x-1)^2}$

v. $\dfrac{1}{x^2-2}+\dfrac{1}{x^2+2}$ vi. $\dfrac{2}{x-1}-\dfrac{1}{2(x+1)}-\dfrac{3x-1}{2(x^2+1)}$

vii. $\dfrac{x+3}{x^2+1}-\dfrac{x+2}{(x^2+1)^2}$ viii. $2-\dfrac{1}{x^2}-\dfrac{3}{x+1}+\dfrac{1}{x}$

ix. $\dfrac{x-1}{x^2+2x+4}+\dfrac{3}{x-2}$

3. $\dfrac{x}{(x^2+1)^2}+\dfrac{1}{x+1}$

Exercise 5.1

1. (i) {0, 1, 2, 3} (ii) { 1 } (iii) {1, 2, 3, 4} (iv) {1, 3}
(v) {0, 1, 3, 4} (vi) {1, 2, 3}

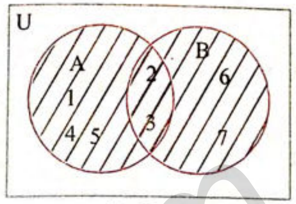
2. (i) {1}, {4, 6, 8} (ii) {0, 1, 2, 3}, { } (iii) {2, 4, 6, … }, { }

3. (i) {1, 3, 5, 7, … 19} (ii) {2, 4, 6, … 20}
(iii) {1, 2, 3, … 20} (iv) {1, 2, 3, … 20}
(v) { } (vi) {1, 3, 5, … 19}
(vii) {1, 2, 3, … 20} (viii) {2, 4, 6, … 20}
(ix) { } (x) {1, 2, 3, … 20}

4. (i) {1, 2, 3, … 15} (ii) { }
(iii) { } (iv) {1, 2, 3, … 15}
(v) {1, 3, … 15} (vi) {1, 2, 3, … 15}
(vii) { } (viii) {2, 4, 6, … 14}

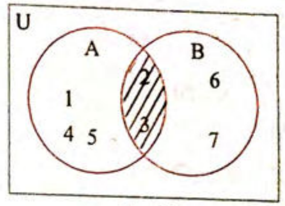
Exercise 5.3

1. (i)



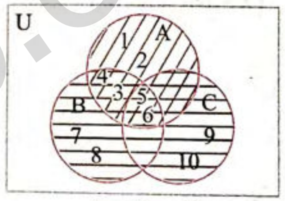
A∪B

(ii)



A∩B

2. (i)



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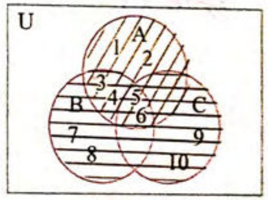
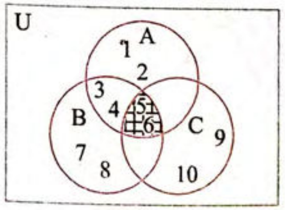


A∪(B∪C) (A∪B)∪C

(ii)



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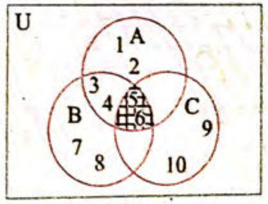
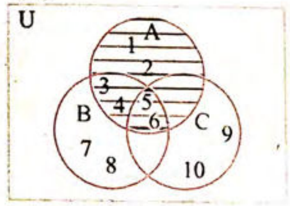


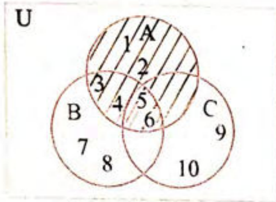
A∩(B∩C) (A∩B)∩C

(iii)



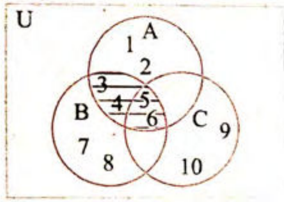
A∪(B∩C)

=



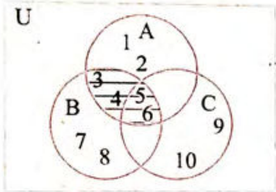
(A∪B)∩(A∪C)

(iv)



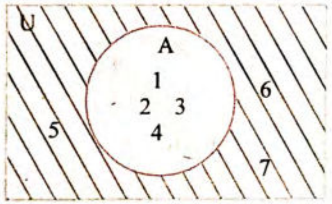
A∩(B∪C)

=

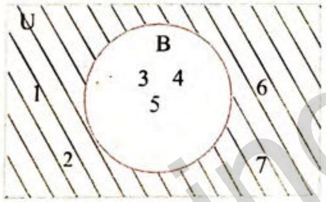


(A∩B)∪(A∩C)

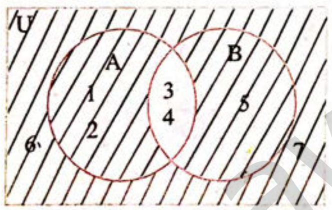
3.



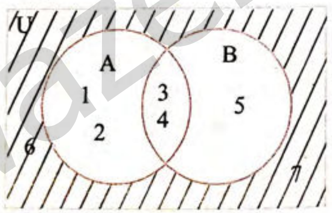
A′



B′

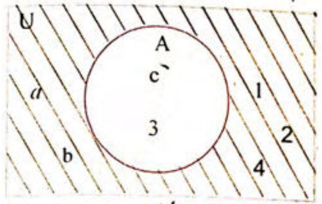


A′∪B′

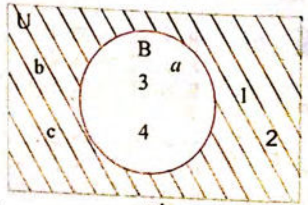


A′∩B′

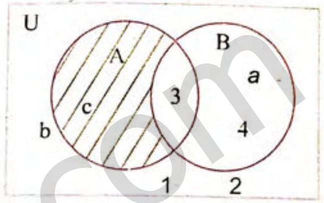
4.



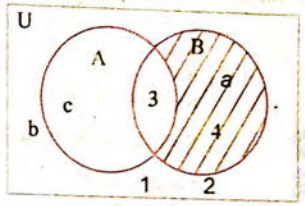
A′



B′

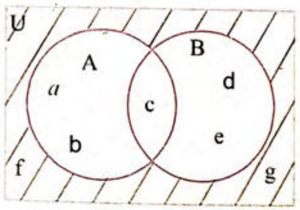


A\B



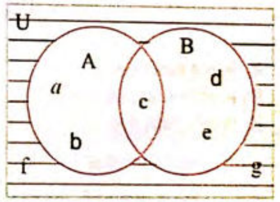
B\A

5.

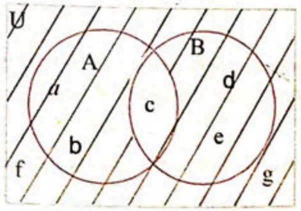


(A∪B)′

=

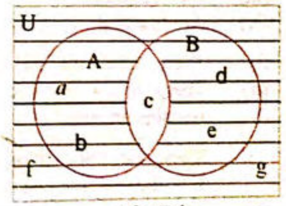


A′∩B′



(A∩B)′

=



A′∪B′

Exercise 5.4

1. (i) $R_1 = \{(1, 4), (2, 4)\}$
$R_2 = \{(2, 5), (3, 4)\}$
$R_3 = \{(1, 4), (2, 5), (3, 5)\}$

(ii) $R_1 = \{(4, 1)\}$
$R_2 = \{(5, 1), (5, 2)\}$
$R_3 = \{(4, 1), (4, 2), (4, 3)\}$
$R_4 = \{(4, 2), (4, 3), (5, 1), (5, 2)\}$

(iii) $R_1 = \{(1, 1), (1, 2)\}$
$R_2 = \{(1, 3), (2, 2)\}$
$R_3 = \{(1, 1), (2, 2), (3, 3)\}$
$R_4 = \{(3, 1), (3, 2)\}$

(iv) $R_1 = \{(4, 4)\}$
$R_2 = \{(4, 4), (4, 5)\}$

2. $R = \{(2, 1), (3, 1), (4, 1), (4, 3)\}$

3. Range = $\{0, 8, 16\}$

4. Range = $\{1, 7, 17, 31\}$

Exercise 5.5

1. R_1 is not function.
R_2 is onto function.
R_3 is into function.

2. (i) into function
(ii) not a function, because both conditions of being a function are not satisfied.
(iii) not a function
(iv) into function

3. (i) There exists one–one correspondence between set A and set B.
(ii) There is not one-one correspondence because $P \in B$ remains unpaired.

4. (i) There does not exist one-one correspondence between set A and set B. It is one-one function.
(ii) There does not exist one-one correspondence between set A and set B. It is into function.

5. (i) $\{(1, 5), (2, 6), (3, 7), (4, 8)\}$
(ii) $\{(1, 6), (2, 5), (3, 8), (4, 7)\}$
(iii) $\{(1, 6), (2, 5), (3, 7), (4, 8)\}$
(iv) $\{(5, 1), (6, 2), (7, 3), (8, 4)\}$
(v) $\{(5, 4), (6, 3), (7, 2), (8, 1)\}$
(vi) $\{(1, 5), (2, 6), (3, 5), (4, 6)\}$

6. (i) It is a function.
Range = $\{1, 2, 3, 5\} \neq A$, not onto
(ii) It is a function.
Range = $\{1, 2, 3, 4\} \neq A$, not onto
(iii) It is a function.
Range = $\{1, 2, 4, 5\} \neq A$, not onto

Review Exercise 5

1. (i) An onto function from A to B (ii) 2×3 (iii) $\{0,4,8,12\}$ and $\{6,10,14,18\}$
(iv) 13 (v) Range $f = B$ (vi) $\{0,8,10,14\}$

2. (i) $\{1, 2, 3, \ldots 100\}$ (ii) $\{\ \}$ (iii) $\{2, 4, 6, .. 100\}$
(iv) $\{1, 3, 5, \ldots 99\}$

7. (i) Function (ii) Bijective function
(iii) Bijective function (iv) Not a function

8. (i) Function, Range = $\{1, 2, 3, 5\}$, not onto
(ii) Function, Range = $\{1, 2, 3, 4\}$, not onto
(iii) Function, Range = $\{1, 2, 3, 5\}$, not onto
(iv) Not a Function

9. (i) $\{(-5, 1), (-6, 2), (-4, 3), (-3, 4)\}$
(ii) $\{(-5, 2), (-6, 1), (-4, 4), (-3, 3)\}$
(iii) $\{(-5, 4), (-6, 3), (-4, 2), (-3, 1)\}$
(iv) $\{(-5, 2), (-6, 2), (-4, 1), (-3, 3)\}$

Note: Several answers are possible.

Exercise 6.1

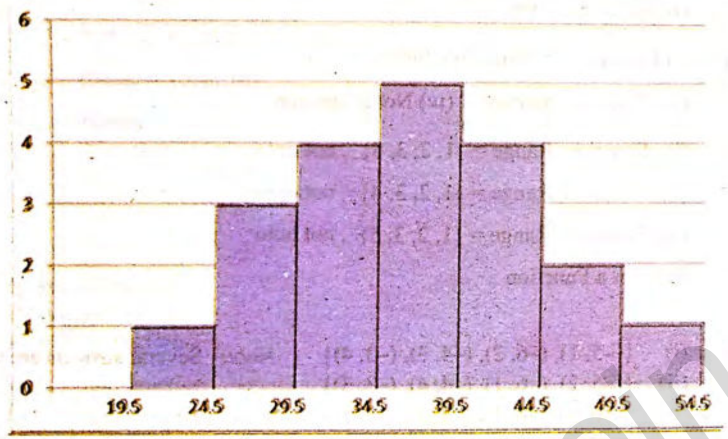
1.

Class-Limits	Tally Marks	Frequency
35 — 45	II	2
46 — 55	IIII	4
56 — 65	卌I	6
66 — 75	卌IIII	9
76 — 85	卌I	6
86 — 90	III	3

2.

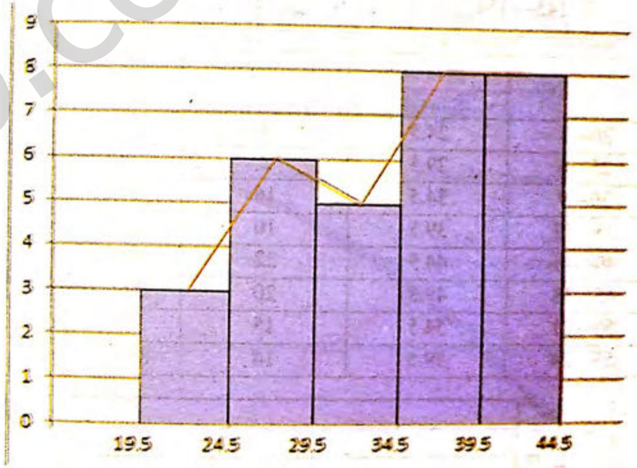
S. No	Class interval	Frequency
1	1—3	01
2	4—6	04
3	7—9	04
4	10—12	03
5	13—15	04
6	16—18	04
7	19—21	03
8	22—24	02
		Σf = 25

3.



4.

Class-Limits	Tally Marks	Frequency
20 — 24	|||	3
25 — 29	卌 |	6
30 — 34	卌	5
35 — 39	卌 |||	8
40 — 44	卌 |||	8



5.

Class interval	Frequency
1—2	15
3—4	11
5—6	4
	Σf=30

Exercise 6.2

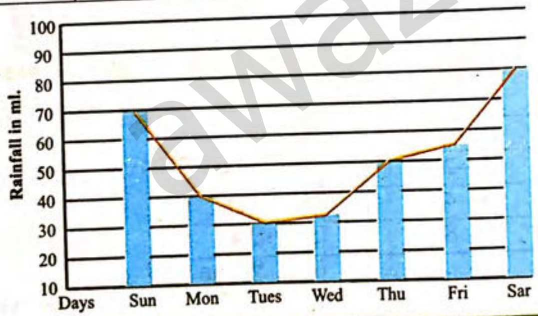
1.

Class-Limits	Tally Marks	Frequency
55— 64	||	2
65 — 74	|||	3
75 — 84	~~||||~~ |	6
85 —94	||||	4
95— 104	|||	3
105 — 114		0
115— 124	|	1
125 — 134		0
135 — 144		0
145— 154	|	1

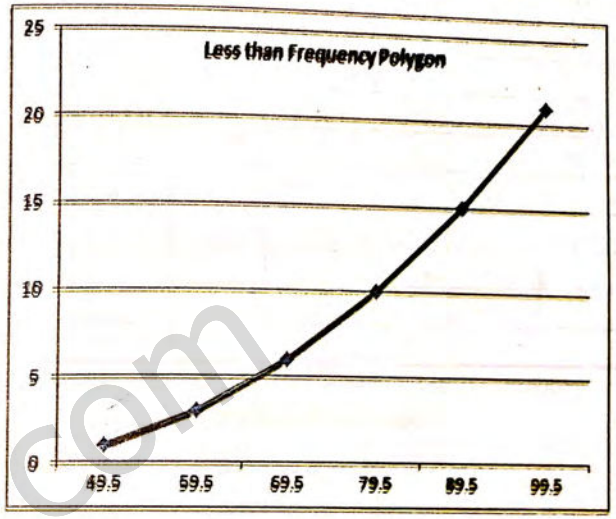
2.

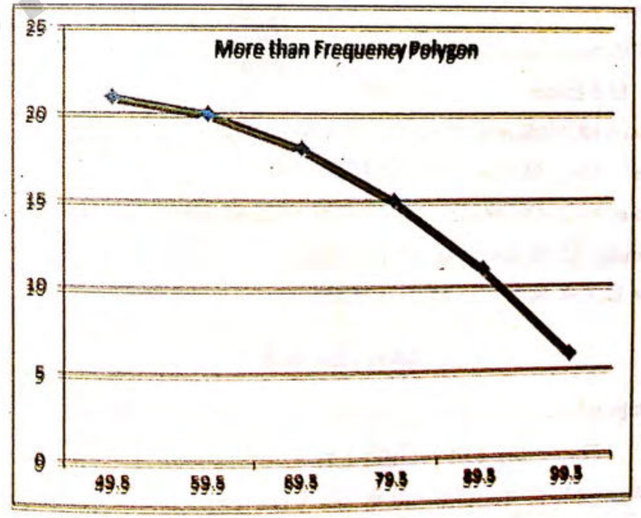
Age in years	Upper class Boundary	Frequency	Cumulative frequency
20—3	24.5	1	1
25—6	29.5	2	1+2=3
30—9	34.5	16	3+16=19
35—12	39.5	10	19+10=29
40—15	44.5	22	29+22=51
45—18	49.5	20	51+20=71
50—21	54.5	15	71+15=86
55—24	59.5	14	86+14=100

3.



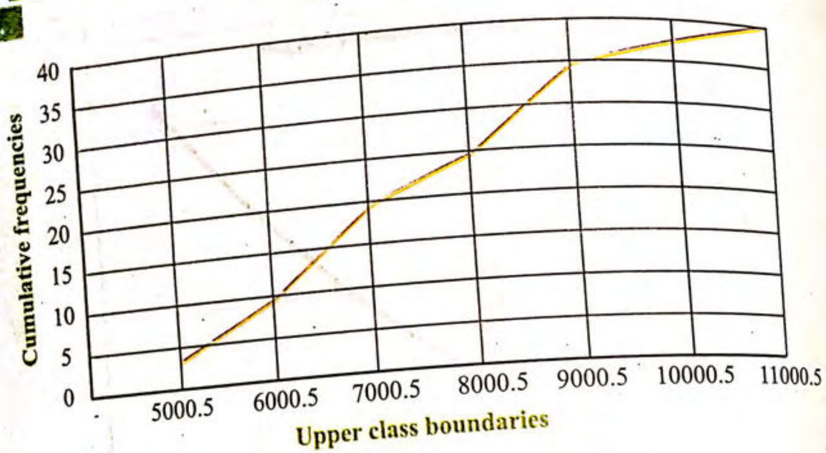
4.





5. (i). 20 (ii). 6 (iii). 5 (iv). 10 (v). 79.5

6.



Exercise 6.3

1. 37.8
2. 37.8
3. 1247(for assumed mean of 1244)
4. 53.5 marks
5. (i) 65.5 inches (i) 7500
6. Mean = 66, Median = 67, Mode = 71, GM = 64.44
7. Mean = 154.3 Median = 156.5, Mode = 160
8. Median = 121.125, Mode = 119.62, G.M = 122.49, HM = 119.44
9. Median = 12.5th item Q_1 = 4.25th item
   Q_3 = 12.75th item Mode = 26.166

Exercise 6.4

1. Range = 12
2. (i) 7.72 (ii) 8.01 (iii) 1.71
3. Range = 65, Variance = 10.38 and S.D = 3.22
4. (a) AM: Section A = 7, Section B = 7
   (b) Variance: Section A = 2.4, Section B = 6.4

5. S.D: Maths = 9.96, Physics = 3.15, in Physics students are more consistent.
6. Variance = 2.88, S.D = 1.7

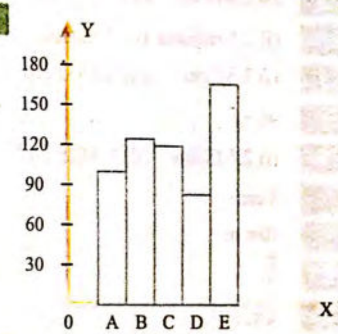
Review Exercise 6

1. (i) Class interval (ii) Ogive (iii) Frequency
   (iv) 1 (v) 1, 2, 3, 3, 2, 1, 2 (vi) Geometric mean
   (vii) 66 (viii) 51 (ix) 7
   (x) 0 (xi) Rs. 600/ (xii) None of these
   (xiii) Range (xiv) Arithmetic Mean (xv) 0.5
   (xvi) Variance (xvii) Mode

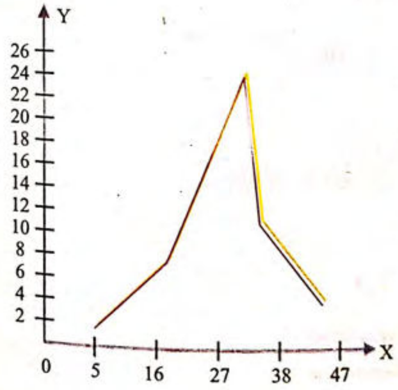
2.

Classes	f
15 – 16	12
17 – 18	12
19 – 20	3
Total	27

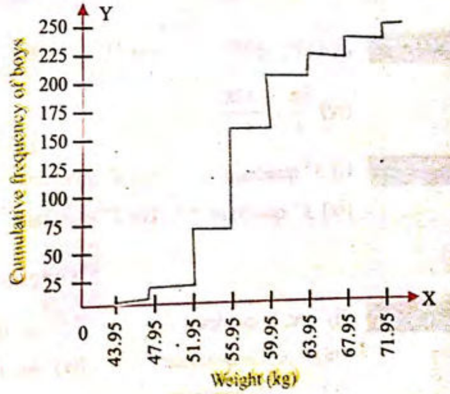
3.



4.



5.



Exercise 7.1

1. (i) 8.2597° (ii) 39.8152° (iii) 84.3194° (iv) 18.1058°
2. (i) 42°15′ (ii) 57°19′30″ (iii) 12° 59′44″ (iv) 32°37′30″
3. (i) 114.6° (ii) 300° (iii) 30° (iv) −135°
4. (i) $\frac{\pi}{4}$ radians (ii) $\frac{2\pi}{3}$ radians (iii) $\frac{-7\pi}{6}$ radians (iv) 1.0576 radians

Exercise 7.2

1. (i) 1.047 cm (ii) 3.1415 cm (iii) 12.57 cm
2. (i) 2.5 radians (ii) 5 radians (iii) 2.09 radians
3. (i) 3.82 cm (ii) 0.1632 cm (iii) 6.366 cm
4. 96 m^2
5. (i) 2.625 cm (ii) 6.5625 cm^2
6. 8 cm
7. 70π m
8. $\frac{\pi}{2}$
9. 4πcm
10. $6\pi cm^2$

Exercise 7.3

1. (i) 415°, −305° (ii) 315°, −405° (iii) $\frac{13\pi}{6}, -\frac{11\pi}{6}$ (iv) $\frac{5\pi}{4}, \frac{-11\pi}{4}$
2. (i) 3rd quadrant (ii) 1st quadrant (iii) 3rd quadrant (iv) 3rd quadrant (v) 2nd quadrant

Exercise 7.4

1. (i) +ve, II quadrant (ii) +ve, II-quadrant (iii) +ve, III-quadrant (iv) −ve, II-quadrant (v) −ve, III-quadrant (vi) +ve, II-quadrant

2. (i) $\sin(-180°)=0$, $\cos(-180°)=-1$, $\tan(180°)=0$, $\text{cosec}(-180°)=$ undefined, $\sec(-180°)=-1$, $\cot(-180°)=$ undefined

   (ii) $\sin(-270°)=1$, $\cos(-270°)=0$, $\tan(-270°)=$ undefined, $\text{cosec}(-270°)=1$, $\sec(-270°)=$ undefined, $\cot(-270°)=0$

   (iii) $\sin 720°=0$, $\cos 720°=1$, $\tan 720°=0$, $\text{cosec} 720°=$ *undefined*, $\sec 720°=1$, $\cot 720°=$ *undefined*

   (iv) $\sin(1470°)=\frac{1}{2}$, $\cos(1470°)=\frac{\sqrt{3}}{2}$, $\tan(1470°)=\frac{1}{\sqrt{3}}$, $\text{cosec}(1470°)=2$, $\sec(1470°)=\frac{2}{\sqrt{3}}$, $\cot(1470°)=\sqrt{3}$

3. $\cos\theta=\frac{1}{2}$, $\sin\theta=-\frac{\sqrt{3}}{2}$, $\tan\theta=-\sqrt{3}$, $\text{cosec}\theta=-\frac{2}{\sqrt{3}}$, $\cot\theta=-\frac{1}{\sqrt{3}}$
4. $\text{cosec}\theta=\frac{5}{4}$, $\cos\theta=-\frac{3}{5}$, $\tan\theta=\frac{4}{3}$, $\sec\theta=\frac{5}{3}$, $\cot\theta=\frac{3}{4}$
5. (i) 1 (ii) $\frac{1}{\sqrt{3}}$ (iii) $\frac{1}{1+\sqrt{2}}$ (iv) 2 (v) 0
6. (i) 1st quadrant (ii) 3rd quadrant (iii) 2nd quadrant (iv) 4th quadrant (v) 4th quadrant (vi) 2nd quadrant
7. (i) 25.56 (ii) 26.16 (iii) 7.79
8. 333.92yd

Exercise 7.6

1. 40°
2. $50\sqrt{3}\,m$
3. 26.56°
4. 160m
5. 260 feet
6. $300(\sqrt{3}+1)m$
7. $\frac{210}{\sqrt{3}-1}m$

Review Exercise 7

1. (i) angle of depression (ii) $\frac{\cos\theta}{\sin\theta}$ (iii) $\sec^2\theta$ (iv) $-\frac{1}{2}$
(v) 1^{st} quadrant (vi) $\frac{\pi}{4}$ (vii) 2.6496 (viii) 0.4
(ix) $2\sqrt{2}$ (x) $45°$
2. $45.4916°$ 3. $216°40'12''$
4. (i) $\frac{3\pi}{2}$ *radian* (ii) 2π *radian*
5. (i) $550°, -170°$ (ii) $110°, -610°$
6. (i) $\sin 390° = \frac{1}{2}, \cos 390° = \frac{\sqrt{3}}{2}, \tan 390° = \frac{1}{\sqrt{3}}$
$\text{cosec}390° = 2, \sec 390° = \frac{2}{\sqrt{3}}, \cot 390° = \sqrt{3}$
(ii) $\sin(-240°) = \frac{\sqrt{3}}{2}, \cos(-240°) = \frac{-1}{2}, \tan(-240°) = -\sqrt{3}$
$\text{cosec}(-240°) = \frac{2}{\sqrt{3}}, \sec(-240°) = -2, \cot(-240°) = \frac{-1}{\sqrt{3}}$
8. $\cos\theta = \frac{1}{2}, \sin\theta = \frac{-\sqrt{3}}{2}, \tan\theta = -\sqrt{3}, \text{cosec}\theta = \frac{-2}{\sqrt{3}}, \cot\theta = -\frac{1}{\sqrt{3}}$
9. 44 m (approximately). 10. 194 feet (approximately).

Exercise 8.1

1. 14 cm 2. 1.5 cm 3. 6.1
4. $\angle B$ and $\angle D$ are obtuse.

Exercise 8.2

1. $5\sqrt{7}$ *cm*, $\sqrt{142}$ cm, $\sqrt{58}$*cm*
2. $\frac{\sqrt{79}}{2}$ cm
3. $m\overline{CD} = 6$ *cm*, $m\overline{AD} = 6\sqrt{3}$ cm, $m\overline{AB} = 4\sqrt{13}$ *cm*
4. $m\overline{AD} = \frac{36}{\sqrt{61}}$ cm, $m\overline{CD} = \frac{25}{\sqrt{61}}$ *cm*

Review Exercise 8

1. (i) Greek (ii) Perga (iii) Mathematician
2. 1.53 units.
3. $m\overline{BC} = \sqrt{89}$, $m\overline{AD} = \frac{40}{\sqrt{89}}$ cm, $m\overline{BD} = \frac{25}{\sqrt{89}}$ *cm*
4. $m\overline{BE} = \sqrt{41}$ cm
5. $x = \frac{86}{19} = 4.53$ (*approx.*)
$y = \frac{275}{19} = 14.47$ (*approx.*)
$h = \frac{4\sqrt{1794}}{19} = 8.92$ (*approx.*)

Exercise 9.1

1. $40\sqrt{2}$ cm 2. 60 cm 3. $\frac{3}{2} = 1.5$ units
4. $\sqrt{41}$ cm 5. $2\sqrt{39}$ cm 6. 24

Exercise 9.2

1. 1 cm or 7 cm 2. 8.39 cm
3. (a) QT>PR
(b) $\overline{PR}$ is farther from C than $\overline{QT}$.

Review Exercise 9

1. (i) congruent (ii) *CD* is closer to O (iii) 24 centimeters (iv) 15 units
(v) 14.4 units (vi) 6 cm (vii) 8
2. (i) AC = 12 m, AB = 24 m (ii) Radius = 10 cm, Diameter = 20 cm
(iii) 8 cm (iv) 48 Units (v) $5\sqrt{2}$ m
3. 17 cm

Exercise 10.1

1. $8\,cm$ 2. $\sqrt{17}cm$

4. (i) 2 (ii) 15.65 (iii) 10

5. 16 cm

6. (i) $x=49, y=14$ (ii) $x=58, y=15$
(iii) $x=34, y=14.8$ (iv) $x=35, y=55$

7. (i) 6 cm (ii) 3 cm
(iii) $3\sqrt{5}$ cm (iv) $3(\sqrt{5}-1)$cm

Exercise 10.2

1. 11 cm 2. 12cm 3. 24 cm
4. 12 cm 5. 16cm 6. 10 cm
7. $2\sqrt{7}$ cm 8. 6 cm

Review Exercise 10

1. (i) an arc (ii) $\overline{OT} \perp \overleftrightarrow{PQ}$ (iii) equal
2. 39 ft
3. 24^0
4. (i) 26^0 (ii) 122^0
5. (a) $i=8, j=67.4$
(b) $k=12.6, l=50.0$

Exercise 11

1. (i) $x=12$, $y=90°$ (ii) $x=11$, $y=90°$
(iii) $x=12$, $y=67.4°$ (iv) $x=11.0$, $y=61.9°$
(v) $x=16$, $y=53.1°$ (vi) $x=6$, $y=50.2°$
2. $x=\frac{3}{2}$

Review Exercise 11

1. (i) $3\sqrt{2}$ (ii) $\overline{AB} \cong \overline{CD}$
3. $\angle ADB = 65°, \angle BDC = 25°, m\widehat{BC} = 50°$

Exercise 12

1. $x=35^0, y=145^0$
2. 10 units
3. (i) $c=62, x=10$ (ii) $a=45, b=30$

Review Exercise 12

1. (i) 160° (ii) 60° (iii) 110° (iv) 40° (v) $37\frac{1°}{2}, 37\frac{1°}{2}$
(vi) 50° (vii) 25° (viii) 125°
2. (i) $x=98°, y=60°$ (ii) $x=38°, y=25\frac{1°}{3}$
4. (i) $x=80^0$ (ii) $x=125^0$ (iii) $x=50^0$
(iv) $x=45^0$ (v) $y=12^0$ (vi) $x=70^0$

Review Exercise 13

1. (i) none of these (ii) none of these (iii) 4
(iv) touch each other internally

GLOSSARY

UNIT 1

Quadratic equation

An equation that can be written in the form

$$ax^2+bx+c=0$$

where a, b, and c are real numbers with $a \neq 0$, is a quadratic equation.

Standard form

A quadratic equation written in the form $ax^2+bx+c=0$ is in standard form.

Zero-factor property of real numbers

If a and b are real numbers, with $ab=0$, then either $a=0$ or $b=0$

Square root property

The solution set of $x^2=k$ is

$$\{\sqrt{k}, -\sqrt{k}\}$$

Quadratic formula

The solutions of the quadratic equation $ax^2+bx+c=0$, where $a \neq 0$, are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Radical equation

An equation in which the variable appears in one or more radicands.

UNIT 2

The discriminant

The expression b^2-4ac that appears under the radical sign in the quadratic formula is called the discriminant.

Cube roots of unity

$1, \omega, \omega^2$ are the cube roots of unity, where $\omega=\frac{-1+i\sqrt{3}}{2}$ and $\omega^2=\frac{-1-i\sqrt{3}}{2}$.

Properties of the cube roots of unity

1. The sum of the cube roots of unity is zero, *i.e.* $1+\omega+\omega^2=0$
2. The product of the cube roots of unity is 1, *i.e.* $1\times\omega\times\omega^2=\omega^3=1$.
3. Each complex cube root of unity is reciprocal of the other, i.e. $\omega=\frac{1}{\omega^2}$ and $\omega^2=\frac{1}{\omega}$.

Relation between the roots and the coefficients of a quadratic equation

If α, β are the roots of the quadratic equation $ax^2+bx+c=0$, then

Sum of the roots $=\frac{-b}{a}$

and product of the roots $=\frac{c}{a}$

Formation of quadratic equation

The quadratic equation whose roots are α, β is given by

x^2- (sum of the roots) $x+$ product of the roots $=0$.

Simultaneous equations

More than one equation which are satisfied by the same values of t[he] variables involved, are called simultaneous equations or a system of equations.

UNIT 3

Ratio

A relation between two quantities of the same kind is called ratio.

Proportion

A proportion is a statement, which is expressed as equivalence of two ratios.

If two ratios a : b and c : d are equal, then we can write a : b = c : d

Direct variation

If two quantities are related in such a way that when one change[s] in any ratio so does the other is called direct variation.

Inverse variation

If two quantities are related in such a way that when one quantity increases, the othe[r] decreases is called inverse variation.

Joint Variation

Joint variation is the same as direct variation with two or more quantities, i.e. Joi[nt] variation is a variation where a quantity varies directly as the product of two or mor[e] other quantities. If x is jointly proportional to y and z, we can write *xkyz*=for som[e] constant k. We can also write this relationship as

$$\frac{x}{yz}=k.$$

UNIT 4

Rational fraction

If $P(x)$ and $Q(x)$ are two polynomials and $Q(x)$ is non zero polymomials than the fraction $\frac{P(x)}{Q(x)}$ is called rational fraction.

Proper rational fraction

A rational fraction, $Q(x)\neq 0$ is a proper rational fraction, if the degree of numerator $P(x)$ is less than the degree of denominator $Q(x)$.

Improper rational fraction

A rational fraction $\frac{P(x)}{Q(x)}$, $Q(x)\neq 0$ is an improper rational fraction, if the degree of numerator $P(x)$ is equal to or greater than the degree of denominator $Q(x)$.

Partial fractions

Splitting up a single rational fraction into two or more rational fraction with single factor in denominator, such a procedure is called partial fractions

UNIT 5

A set is a "collection of well-defined distinct objects. Sets are represented by capital English alphabets, A, B, C .... Z and elements of sets are represented by small English alphabets, *a*, b, c, ... z.

Union of two sets.

If A and B are two sets, then the union of set A and set B consists of all elements in set A or in set B or in both A and B, and it is denoted by $A\cup B$. Symbolically.

$$A\cup B=\{x\mid x\in A \text{ or } x\in B\}$$

Intersection of two sets

If A and B are two sets, then the intersection of set A and set B consists of all those elements which are common to both A and B, and it is denoted by $A\cap B$, Symbolically,

$$A\cap B=\{x\mid x\in A \wedge .x\in B\}$$

Disjoint sets

Two sets A and B are disjoint if $A\cap B=\varphi$.

Complement of a set

If U is universal set and A is a subset of U, then U \ A is called complement of set A, and is denoted by A' or A^c.

Difference of two sets

If A and B are two sets, then their difference consists of all those elements of A which are not in B, and it is denoted by A \ B or A – B symbolically

$$A\setminus B=\{x\mid x\in A \wedge x\notin B\}$$

Commutative property of union

For any two sets A and B

$$A\cup B=B\cup A$$

Commutative property of intersection

For any two sets A and B

$$A\cap B=B\cap A$$

Associative property of union

For any three sets A, B and C

$$A\cup(B\cup C)=(A\cup B)\cup C$$

Associative property of intersection

For any three sets

$$A\cap(B\cap C)=(A\cap B)\cap C$$

Distributive property of union over intersection

For any three sets A, B and C

$$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$$

Distributive property of intersection over union

For any three sets A, B and C

$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$$

De. Morgan's laws

For any two sets A and B

(i) $(A\cup B)'=A'\cap B'$ (ii) $(A\cap B)'=A'\cup B'$

Venn diagrams

Concept of sets i.e. union, intersection, complement, difference of sets, can be explained easily with the help of Venn diagrams. In these diagrams, a set is usually represented by a circle and universal set is represented by a rectangle.

Ordered pairs

(*a*, b) is called an ordered pair of two elements *a* and b of a set or of different sets, where a is the first element and b is the second element.

$$(a, b)\neq(b, a)$$

Cartesian product

If A and B are two non- empty sets then A × B is called Cartesian product, which is set of all ordered pairs such that the first element of each ordered pair belongs to set A and second element of each ordered pair belongs to set B symbolically.

$$A\times B=\{(x, y)\mid x\in A \text{ and } y\in B\}$$

If $A\neq B$, then $A\times B\neq B\times A$

Binary relation

Any subset of A × B is called a binary relation, where A and B are two non-empty sets.

Domain and range of binary relation

If R is a binary relation from set A to set B, i.e.

$R=\{(x, y)\mid x\in A\wedge y\in B\}$, then domain of R is set of first elements of all ordered pairs in R and is denoted by "Dom R".

The range of R is set of second elements of all ordered pairs in R, and is denoted by "Range R".

Function

If A and B are two non-empty sets, then a binary relation f is said to be a function from A to B, if

(i) Dom f = A

(ii) There should be no repetition in the first elements of all ordered pairs contained in *f*.

Symbolically

$f : A \longrightarrow B$

Domain, co-domain and range of a function

Let $f : A \longrightarrow B$ be a function, then set A is called domain of f, set B is called co-domain of f and the set of second elements of all ordered pairs contained in f is called range of f. The range is subset of co-domain

i.e. Range $f \subseteq B$

Into function

Let f be a function from A to B, then f is into function, if

Range $f \neq B$

One – one or 1–1 function

Let f be a function from A to B, then f is one-one function, if for each $x \in A$ there exist unique $y \in B$, i.e. there is no repetition in the second element of all ordered pairs contained in f.

Into and one-one function (injective function)

Let f be a function from A to B, then f is injective (into and one - one) function if

(i) Range $f \neq B$

(ii) There is no repetition in the second elements of all ordered pairs contained in f.

Onto function (surjective function)

Let f be a function from A to B, then f is onto function if

Range f = B

One-one and onto function (bijective function)

Let f be function from A to B, then f is one-one and onto (bijective) function if it is both one-one and onto.

One-one correspondence

If A and B are two non-empty sets then one-one correspondence between A and B is a rule for which each element of set A is paired with one and only one element of B and each element of B is paired with one and only one element of A, and non of the members of any set remains unpaired. It is also known as one-to-one function. In one-one correspondence both sets A and B have same number of elements.

UNIT 6

Histogram

A graphical representation of data in the form of rectangles is called Histogram.

Frequency

A number showing the repetition of a value in a given set of data.

Frequency Polygons

A curve on the graph showing the frequency of values.

Median

A value lying in the middle of arranged data is called median.

Mode

A value repeated maximum times in a given set of data is called mode. It shows the trend of a data and hence is usually used to find public opinion.

Range

The difference between maximum value and minimum value in given set of data.

UNIT 7

An angle is a union of two rays which have a common point (vertex) one of the ray is called "initial side" and other ray is called "terminal side".

Sexagesimal system (degrees, minutes, seconds)

It is the system of measurement of an angle in which one complete rotation is divided into 360 parts called degrees, written as 360°. One degree is divided into 60 parts called minutes, written as $60'$ and one minute is again divided into 60 parts called seconds, written as $60''$.

Circular system (radians)

It is another system of measurement of an angle. In this system unit of measure of angle is radian. One radian is an angle subtended at the centre of a circle an arc whose length is equal to radius of the circle.

Length of an arc is measured by the formula $\ell = r\theta$ where ℓ is length of an arc, θ is central angle of a circle measured in radians and r is radius of circle.

Area sector of circle

Area of sector of a circle is given by $A = \frac{1}{2}r^2\theta$. Where r is radius of circle, θ is central angle of sector measured in radians.

Coterminal angles

Angles having the same initial and terminal sides are called co-terminal angles and differ by a multiple of 2π radians or 360° they are also called general angles.

Angle in standard position

In XY-plane (co-ordinate plane), if the vertex of an angle lies at origin and initial side lies on positive x-axis, then such an angle is said to be in standard position.

rants and quadrantal angles

XY-plane divided into four equal parts each part is called quadrant. Being in the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} quadrants means measure of angle from 0° to 90°, 90° to 180°, 180° to 270° and 270° to 360° respectively quadrant angles are 0°, 90°, 180°, 90, 360°.

Trigonometric ratios of unit circle

These are $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \frac{y}{x}$

and $\cos = \frac{1}{y}$, $\sec\theta = \frac{1}{x}$, $\cot\theta = \frac{x}{y}$

Trigonometric identities are

(i) $\cos^2\theta + \sin^2\theta = 1$

(ii) $1 + \tan^2\theta = \sec^2\theta$

(iii) $1 + \cot^2\theta = \cos^2\theta$

Angle of elevation

If an object is above the level of observer's sight then the angle between the horizontal line and observer's line of sight is called angle of elevation.

Angle of depression

If an object is below the level of observer's sight then the angle between the horizontal line and observer's line of sight is called angle of depression.

UNIT 8-13

Acute angle

An angle is acute if and only if it has a measure greater than 0° and less than 90°.

Acute triangle

A triangle is acute if and only if it has three acute angles.

Adjacent angles

Two coplanar angles are adjacent if and only if they have a common arm but no points in the interior of one angle are in the interior of the other.

Altitude of a triangle

An altitude of a triangle is the perpendicular segment joining a vertex of the triangle to the line that contains the opposite side.

Angle

An angle is the union of two non-collinear rays with the same end point.

Arc of a circle

Any portion of the circumference of a circle is called an arc of the circle.

Bisector of an angle

A ray that divides an angle into two equal adjacent angles is the bisector of the angle.

Central angle

A central angle of a given circle is an angle whose vertex is at the centre of the circle.

Chord

A chord is a segment whose end points are on the circle.

Circle

A circle is the set of all coplanar points equidistant from a given point.

Circular region

A circular region is the union of a circle and its interior

Circumference of a circle

The perimeter of a circle is called its circumference.

Collinear points

Points are collinear if and only if there is a line that contains all of them.

Complementary angles

Two angles are complementary if and only if the sum of their measures is 90°.

Concentric circles

Two or more circles (in the same plane) are concentric if they share the same centre point.

Concurrent lines

Two or more lines are concurrent if and only if there is a single point that lies on all of them.

Congruent angles

Angles of the same degree measure.

Congruent arcs

Two arcs of a circle are said to be congruent if and only if they have the same degree measures.

Congruent circles

Circles with congruent radii are called congruent circles.

Congruent figures

Geometric figures are congruent if and only if they have the same size and shape.

Congruent segments

Two or more line segments are congruent if and only if they have the same length.

Congruent triangles

Given a correspondence ABC « DEF between the vertices of two triangles, if the corresponding sides are congruent, and the corresponding angles are congruent then the correspondence ABC « DEF is called a congruence between the two triangles.

Coplaner points

points are coplaner if and only if there is a plane that contains all of them.

Corollary

A statement that can easily be proved by applying a theorem.

Diameter

A chord passing through the centre of a circle or in the same circle, two collinear radii form a diameter.

Externally tangent

Two or more circles are externally tangent if they intersect in exactly one point and if their interiors do not intersect.

Half – plane

Given a line and the plane containing it, the two sets separated by the line are called half – planes.

Internally tangent

Two or more circles are internally tangent if they intersect in exactly one point and if the interior of one contains the interior of the other.

Linear pair

Two angles form a linear pair if and only if they are adjacent angles and the noncommon sides are opposite rays.

Major and Minor arcs

The larger of the two arcs is called the major arc and the smaller one is called the minor arc.

Measure of an arc

The degree measure of a minor arc is the measure of the corresponding centre angle.

The degree measure of a major arc is 360° minus the degree measure of the corresponding minor arc.

The degree measure of a semi-circle is 180°. The degree measure of a circle is taken to be 360°.

Median of a triangle

A segment is a median of a triangle if and only if its end points are a vertex of the triangle and the midpoint of the opposite side.

Midpoint

A point B is called the midpoint of $\overline{AC}$ if and only if (i) B is between A and C and (ii) $m\overline{AB} = m\overline{BC}$.

Obtuse angle

An angles is obtuse if and only if it has a measure greater than 90° but less than 180°.

Obtuse triangle

A triangle is obtuse if and only if it has one obtuse angle.

Opposite rays

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called opposite rays if and only if A is between B and C.

Parallel lines

Two lines are parallel if they do not intersect.

Perpendicular bisector

In a given plane, the perpendicular bisector of a segment is the line that is perpendicular to the segment at its midpoint.

Radius of a circle

A radius of a circle is

(i) Any segment with one endpoint at the centre and the other end point on the circle and

(ii) The distance from the centre to the circle.

Ray

Ray AB $\left(\overrightarrow{AB}\right)$ is the figure that contains A and every point on the same side of A as B.

Right angle

An angle of measure 90°.

Secant

A secant is a line that intersects a circle in exactly two points.

Sector

If $\widehat{AB}$ is an arc of a circle with centre O and radius r, then the union of all segments $\overline{OP}$, where P is any point of $\widehat{AB}$, is a sector.

Segment of a circle

A segment of a circle is the region bounded by a chord and an arc of the circle.

Supplementary angles

Two angles are supplementary if and only if the sum of their measures is 180°.

Tangent

A line that intersects a circle at exactly one point is called a tangent, and the point of intersection is called the point of tangency (or point of contact).

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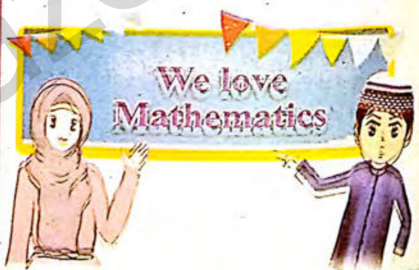

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مومن تو آپس میں بھائی بھائی ہیں

تو اپنے دو بھائیوں میں صلح کرا دیا کرو

اور اللہ سے ڈرتے رہو

تاکہ تم پر رحم کیا جائے۔

(سورۃ الحجرات : ۱۰)