

CHAPTER 11

Chords and Arcs

Theorem: 11.1

If two arcs of a circle are congruent then the corresponding chords are equal.

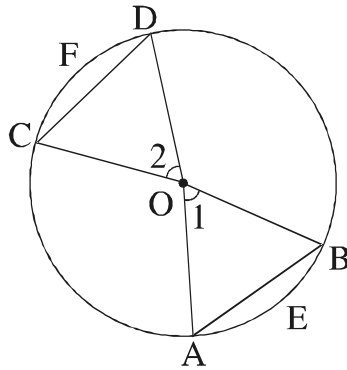
We will prove the theorem for two cases;

Case a: For one circle.

Case b: For two congruent circles.

Case a: For one circle

Figure:



Given:

A circle with centre O, $\widehat{ABC}$ and $\widehat{CFD}$ are congruent i.e $\widehat{ABC} \cong \widehat{CFD}$. $\overline{AB}$ and $\overline{CD}$ are the corresponding chords of the given congruent arcs.

To Prove:

$$\overline{AB} \cong \overline{CD}$$

Construction:

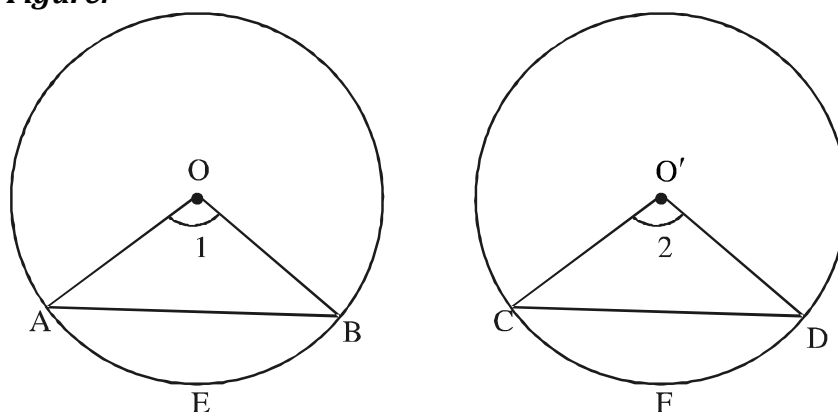
Join O to A ,B,C and D respectively and name central $\angle 1$ and $\angle 2$.

Proof:

Statements	Reasons
In $m\angle OAB \leftrightarrow 2\angle OCD$	
$\overline{OA} \cong \overline{OC}$	Radii of the same circle
$\overline{OB} \cong \overline{OD}$	Radii of the same circle
$\angle 1 = \angle 2$	Central angles of two congruent arcs
$\therefore \triangle OAB \cong \triangle OCD$	S.A.S postulate
$\therefore \overline{AB} \cong \overline{CD}$	Corresponding sides of two congruent triangles.
Similarly	
$m\angle AOB = 2m\angle ADB$ (ii)	

Case b: For two congruent sides:

Figure:



Given:

Two congruent circles with centres O and O' respectively. AEB and CFD are two congruent arcs, where $\overline{AB}$ and $\overline{CD}$ are congruent sides.

To Prove:

$$\overline{AB} \cong \overline{CD} \text{ or } m\overline{AB} \cong m\overline{CD}$$

Construction:

Join O to A and B and O' to C and D respectively.

Proof:

Statements	Reasons
In the $\triangle OAB \leftrightarrow \triangle O'CD$	
$\overline{OA} \cong \overline{O'C}$	Radii of two congruent circles.
$\overline{OB} \cong \overline{O'D}$	Radii of two congruent circles.
$\therefore \angle 1 \cong \angle 2$ or $m\angle 1 = m\angle 2$	Central angles of two congruent arcs.
$\therefore \triangle OAB \cong \triangle O'CD$	S.A.S Postulate
$\therefore \overline{AB} \cong \overline{CD}$	Corresponding sides of two congruent triangles.
Or $m\overline{AB} \cong m\overline{CD}$	

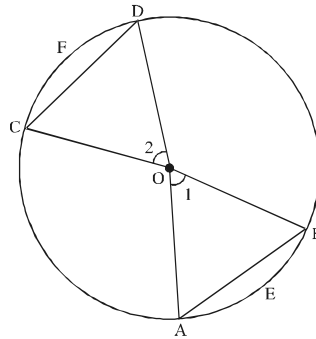
Theorem: 11.2

If two chords of a circle are equal then their corresponding arcs are congruent.

- For one circle.
- For two congruent circle.

a. For One circle:

Figure:



Given:

A circle with center O having two chords $\overline{AB}$ and $\overline{CD}$ such that $\overline{AB} \cong \overline{CD}$.

To Prove:

$$\angle AEB = \angle CFD \text{ and } \angle AFB = \angle CED$$

Construction:

Join O with A, B, C and D respectively.

Proof:

Statements	Reasons
In the $\triangle AOB \leftrightarrow \triangle COD$	
$\overline{OA} \cong \overline{OC}$	
$\overline{OB} \cong \overline{OD}$	Radii of the same circle
$\overline{AB} \cong \overline{CD}$	Radii of the same circle
$\therefore \triangle AOB \cong \triangle COD$	Radii of the same circle
$\therefore \angle 1 \cong \angle 2$	
But $\angle 1$ and $\angle 2$ are central angles	Given
$\therefore \angle AEB \cong \angle CFD$	S.S.S $\cong$ S.S.S
Thus the corresponding	Corresponding angles of two

minor arcs of two equal chords $\overline{AB}$ and $\overline{CD}$ of a circle are congruent.	congruent triangles. Definition of equal arcs.
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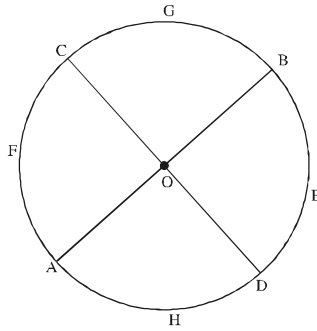
Now prove the same result for major arcs.

Proof:

Statements	Reason
Since measure of the major arc corresponding to the chord $\overline{AB} = 360^\circ$ minus the measures of the major arc of $\overline{AB}$.	Definition of the measure of the major arc corresponding to a chord of circle.
i.e. $m[\text{major arc of } \overline{AB}] = 360^\circ - m\angle 1$ -----(1)	$\therefore m[\text{minor arc of } \overline{AB}] = m\angle 1$
Similarly $m[\text{minor arc of } \overline{CD}] = 360^\circ - m\angle 2$ -----(2)	$m[\text{minor arc of } \overline{CD}] = m\angle 2$
but $m\angle 1 \cong m\angle 2$	proved above
So (1) and (2) becomes $m\widehat{AFB} \cong 360^\circ - m\angle 1$ ---(3)	major arc of $\overline{AB} = \widehat{AFB}$
And $m\widehat{CED} \cong 360^\circ - m\angle 2$ --(4)	and major arc of $\overline{CD} = \widehat{CED}$
From (3) and (4) $m\widehat{AFB} \cong m\widehat{CED}$ $\therefore \widehat{AFB} = \widehat{CED}$	$\therefore L.H.S = R.H.S$

Now we consider the case when the two chords of a circle are diameter.

Figure:



Proof:

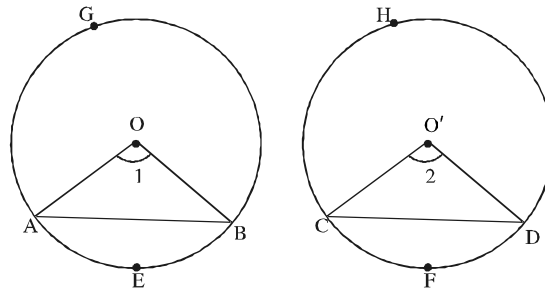
Statements	Reasons
Since $m\overline{AB} = m\overline{CD}$ $\therefore$ The measure of the minor arc of $\overline{AB}$ = measure of the major arc of $\overline{AB}$ or $m[\text{minor arc of AB}] = m[\text{major arc of AB}] = 180^\circ$ or $m\overline{AEB} = m\overline{AFB} = 180^\circ \dots\dots (5)$ similarly $m\overline{CGD} = m\overline{CHD} = 180^\circ \dots\dots (6)$ $\therefore m\overline{AEB} = m\overline{AFB} = m\overline{CGD} = m\overline{CHD} = 180^\circ$ or $m\overline{AEB} \cong m\overline{AFB} \cong m\overline{CGD} \cong m\overline{CHD}$ Similarly arcs of a circle are	$\therefore \overline{AB}$ and $\overline{CD}$ are the diameter of the same circle. $\therefore \overline{AB}$ is the diameter so it divides the circle into two halves. i.e. semicircle arc. Comparing equation 5 and 6.

congruent.

Case B:

Now we prove the same result for two congruent circles.

Figure:



Given:

The congruent circles with centre O and O' with two chords; $\overline{AB}$ and $\overline{CD}$ respectively which are not the diameters such that

$$m\overline{AB} = m\overline{CD} \text{ or } m\overline{AB} \cong m\overline{CD}.$$

To Prove:

$$\overline{AE} \cong \overline{CF} \text{ and } \overline{AG} \cong \overline{CH}$$

Construction:

Join O with A and B and O' with C and D. This gives central angles labeled 1 and 2.

Proof:

Statements	Reasons
In the $\triangle OAB \leftrightarrow \triangle O'CD$	Radii of two circles

$\overline{OA} \cong \overline{OC}$ $\overline{OB} \cong \overline{OD}$ $\overline{AB} \cong \overline{CD}$ $\therefore \triangle OAB \cong \triangle OCD$ $\therefore \angle 1 \cong \angle 2$ or $m\angle 1 = m\angle 2$ $\therefore m\widehat{AEB} = m\widehat{CFD}$ or $\widehat{AEB} = \widehat{CFD}$ Thus the corresponding major arcs of two equal chords $\overline{AB}$ and $\overline{CD}$ of two congruent circles are congruent.	Given S.S.S $\cong$ S.S.S Corresponding angles of two sides. Condition of equality of two arcs.
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Now we prove the same result for major arcs.

Proof:

Statements	Reasons
Since the measure of major arc are corresponding to the chord $\overline{AB} = 360^\circ$ minus the measure of the minor arc of the same chord $\overline{AB}$. $\therefore m[\text{major arc of } \overline{AB}] = 360^\circ - m\angle 1 \dots\dots (7)$ $m[\text{major arc of } \overline{CD}] = 360^\circ - m\angle 2 \dots\dots (8)$ But $m\angle 1 = m\angle 2$	Definition of the measures of the major arc corresponding to a chord of circle. $\therefore m[\text{major arc of } \overline{AB}] = m\angle 1$ $\therefore m[\text{minor arc of } \overline{CD}] = m\angle 2$

So 7 and 8 becomes $m\angle AGB = 360^\circ - m\angle 1 \dots\dots (9)$	Proved
$m\angle CHD = 360^\circ - m\angle 1 \dots\dots (10)$	
From (9) and (10) $m\angle AGB \cong m\angle CHD$	Major arc corresponding $\overline{AB}$ is AGB
	Major arc corresponding $\overline{CD}$ is CHD
	R.H.S of 9 and 10 are equal so are the LHS.

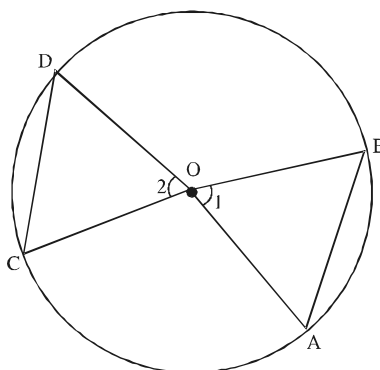
Theorem: 11.3

Equal chords of a circle subtend equal angles of the centre.

- For one circle.
- For two congruent circles.

a. For one circle:

Figure:



Given:

A circle with centre O. $\overline{AB}$ and $\overline{CD}$ are two chords of the circle. Such that $\overline{AB} \cong \overline{CD}$ or $m\overline{AB} \cong m\overline{CD}$ are subtended $\angle 1$ and $\angle 2$ at the centre.

To Prove:

$$\angle 1 \cong \angle 2$$

Construction:

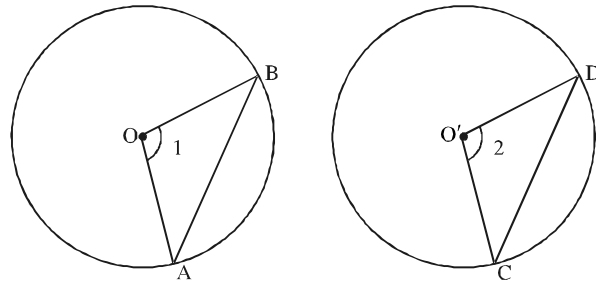
We join O to A, B, C and D respectively so that $m\overline{OA} = m\overline{OB} = m\overline{OC} = m\overline{OD}$ = radii of a circle.

Proof:

Statemets	Reasons
$In \triangle OAB \leftrightarrow \triangle OCD$ $\overline{OA} \cong \overline{OC}$ $\overline{OB} \cong \overline{OD}$ $\overline{AB} \cong \overline{CD}$ $\therefore \triangle OAB \cong \triangle OCD$ $\therefore \angle 1 \cong \angle 2$	Radii of the same circle Radii of the same circle Radii of the same circle Given Corresponding angles of congruent triangles.

b. For two Congruent Circles:

Figure:



Given:

Two congruent circles with centre O and O' having two equal chords $\overline{AB}$ and $\overline{CD}$ i.e. $\overline{AB} \cong \overline{CD}$.

To Prove:

These chords subtended equal angles at the center i.e. $\angle 1 \cong \angle 2$.

Constructions:

We join O to A and B and O' with C and D.

Proof:

Statements	Reasons
In the $\triangle OAB \leftrightarrow \triangle O'CD$	Radii of two circles
$\overline{OA} \cong \overline{O'C}$	
$\overline{OB} \cong \overline{O'D}$	Given
$\overline{AB} \cong \overline{CD}$	
$\therefore \triangle OAB \cong \triangle O'CD$	
$\therefore \angle 1 \cong \angle 2$	S.S.S $\cong$ S.S.S Corresponding angles of two sides.

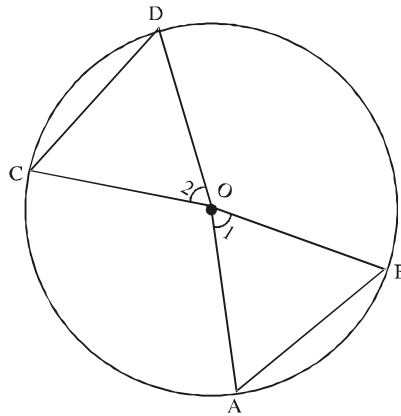
Theorem: 11.4

If the angle subtended by two chords of a circle at the center are equal in measures, then the chords are equal in measures.

- a. For one circle
- b. For two congruent circles.

a. For one circle

Figure:



Given:

A circle with centre O. $\overline{AB}$ and $\overline{CD}$ are two chords of the circle and $\angle 1 \cong \angle 2$

To Prove:

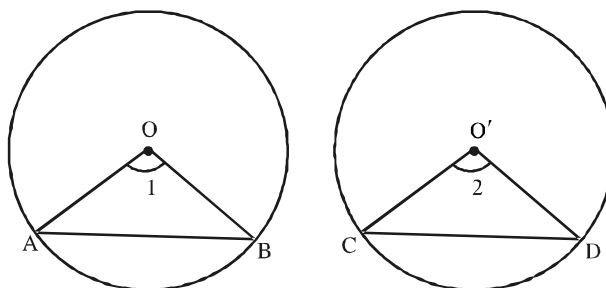
$$\overline{AB} \cong \overline{CD}$$

Statement	Reasons
In $\triangle OAB \leftrightarrow \triangle OCD$	
$\overline{OA} \cong \overline{OC}$	Radii of same circle
$\overline{OB} \cong \overline{OD}$	Radii of same circle
$\angle 1 \cong \angle 2$	Given
$\therefore \triangle AOB \cong \triangle COD$	SAS postulate

$\overline{AB} \cong \overline{CD}$	corresponding side of congruent Δ s.
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b. For two congruent circles

Figure:



Given:

Two congruent circles with center O and O'.
 $\overline{AB}$ and $\overline{CD}$ are two chords $\angle 1 \cong \angle 2$

To Prove: $\overline{AB} \cong \overline{CD}$

Statement	Reasons
In the $\Delta OAB \leftrightarrow \Delta O'CD$	
$\overline{OA} \cong \overline{O'C}$	Radii of two congruent circle
$\overline{OB} \cong \overline{O'D}$	Radii of two congruent circle
$\angle 1 \cong \angle 2$	Given
$\Delta OAB \cong \Delta O'CD$	SAS postulate
$\overline{AB} \cong \overline{CD}$	Corresponding sides of two congruent Δ s

Exercise

Q1. An equilateral triangle is inscribed in a circle. Prove that the circle is divided into three equal parts.

Proof: Given equilateral triangle $\triangle ABC$

Prove: Arc of a triangle are equal

Statement	Reason
$OA = OB = OC$	Centre of circle and radius distance
$AB = BC$	Definition of equilateral triangle
$AC = AB$	Equilateral triangle
$\alpha = 60^\circ$ $\angle \beta = 60^\circ$ $\angle C = 60^\circ$	Sum of interior angle of triangle is equal to $60^\circ + 60^\circ + 60^\circ = 180^\circ$
$\overline{AB} = \overline{BC} = \overline{AC}$	Arc of triangle are equal. Three circle in three equal part.

Q2. Prove that in two equal circle equal central angle have equal arc.

Diagram

Given: Two congruent circle with centre O and O', $\overline{AB}$ and $\overline{CD}$ are two chords of these circles such that they subtend equal angles at the circles

$\angle 1 \cong \angle 2$

To prove

$\overline{AB} = \overline{CD}$

Proof:

Statement	Reason
In the $\triangle OAB \leftrightarrow \triangle O'CD$	Radii of two congruent circles
$\overline{OA} = \overline{O'C}$	Radii of two congruent circle
$\overline{OB} = \overline{O'D}$	Given
$\angle 1 \cong \angle 2$	S.A.S postulate
$\triangle OAB = \triangle O'CD$	
$\overline{AB} = \overline{CD}$	

3. In a circle with centre O. $\overline{AB}$ is a chord (which is not diameter, and radius $\overline{OD}$ bisects the chord.
 Prove that $m\overline{AD} = m\overline{BD}$

Diagram

Proof:

Statements	Reasons
O	Centre of circle
$\overline{AB}$	Given chord of a circle.
$\angle ADO = 90^\circ$	Perpendicular angles
$\angle BDO = 90^\circ$	Perpendicular angles
$\angle ADO = \angle BDO$	Perpendicular bisector
$\overline{AD} = \overline{BD}$	

Q4. Prove that in two equal circle equal arcs have equal central angles

Proof:

Diagram

Given: Two congruent circles with centers O and O' having two equal chords $\overline{AB}$ and $\overline{CD}$: $\overline{AB} = \overline{CD}$

To Prove:

These chords subtend equal angles at the centre $\angle 1 \cong \angle 2$

Statement	Reasons
In $\triangle OAB \leftrightarrow \triangle O'CD$	Radii of two congruent circle.
$\overline{OB} = \overline{O'C}$	Radii of two congruent circle.
$\overline{OA} = \overline{O'C}$	Given
$\triangle OAB = \triangle O'CD$	S.S.S = S.S.S
$\angle 1 \cong \angle 2$	Corresponding angles of congruent triangles.

Q5. $\triangle ABC$ is a triangle which is inscribed in a circle. If $m\angle A = m\angle B$ then prove that $m\overline{AC} = m\overline{BC}$

Proof:
Diagram

Given: Circle with centre O. and AB is chord of circle.

$$m\angle A = m\angle B$$

To Prove: $m\angle C = m\angle B$

Proof:

Statement	Reason
In triangle $\triangle ABC$	
AB	Chord given
$m\angle 1 = m\angle A$	Given
$m\angle 2 = m\angle B$	Given
$m\angle A = m\angle B$	Corresponding angle are same.
$\angle C = m\angle C$	