Unit 11

# INTRODUCTION TO NUMERICAL METHODS

## NUMERICAL SOLUTION OF NON-LINEAR EQUATIONS

**Roots of an equation / zeros of a function:**

A functions f(x) of one variable x, has a number r as its zero if f(r)=0. The number r is also called a root of the equation f(x)=0.

The roots of f(x)=0 are often calculated using standard methods such as quadratic formula, Synthetic division etc, however these methods can't be applied to non-algebraic equations such as $f(x)=x^2+\sin x=0$. In such case approximate roots are calculated using numerical methods.

**Bisection method:**

If a function f(x) has values with opposite signs at $a_o$ and $b_o$, there lies a root in-between the interval $[a_o, b_o]$. A better approximation to the interval containing the root can be evaluating f(x) at the midpoint of $a_o$ and $b_o$, i.e.

$$c_o = \frac{a_o + b_o}{2}$$

Repeating this procedure leads to a better approximation of the actual root of f(x) in the interval $[a_o,b_o]$

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Replacing the endpoint a or b with c depending upon the sign of f(c) defines the new interval for the root. Repeating the above procedures leads to better approximation of the actual root of f(x).

**Regula-Falsi method:**

If a function values f(a) and f(b) have opposite signs, there lies a root in interval [a,b]. An approximation to the root can be obtained by

$$c = a - \frac{(a-b)f(a)}{f(a)-f(b)} \rightarrow (i)$$



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Repeating the above procedures leads to better approximation of the actual root of f(x). This procedure is called Regula-Falsi Method.

**Newton-Raphson Method:**

Newton-Raphson method is a recursive procedure for finding a successively better approximation of the actual root of f(x) from an initial estimate $x_o$, using formula.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad i=0,1,2.....$$

**Numerical Quadrature:**

Numerical quadrature is a primary tool to find the approximate value of a definite integral that can't be solved analytically, such as

$$I = \int_0^1 e^{x^2} dx$$

**Procedure:**

If a set of points in the interval [a,b] is $a=x_o < x_1 < x_2 ..... ..... < x_{n-1} < x_n = b$, then

$$Q[f(x)] = \sum_{j=0}^{n} w_j f(x_j) = w_o f(x_o) + w_1 (x_1) + ..... + w_n f(x_n)$$

With the property

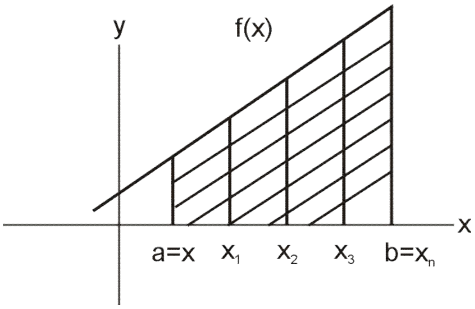
$$\int_a^b f(x)dx = Q[f(x)] + E[f(x)]$$

is called numerical integration or quadrature formula.

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**Approximation by rectangles:**

The definite integral of a function f(x) over [a,b] is the area under the curve of f(x) from a to b as shown in the figure.




The area can be approximated by



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the rectangles with width $\Delta x$ and height f(x) at each x at grid points

x: $a=x_o, x_1, x_2, x_3 \ldots\ldots x_n=b$

f(x): $f_o, f_1, f_2, f_3 \ldots\ldots f_n$

The approximate area is given by:

$$\Delta\int_a^b f(x)dx \approx f(x_o)\Delta x + f(x_1)\Delta x + \ldots\ldots\ldots\ldots + f(x_n)\Delta x \rightarrow (i)$$

As $\Delta x \rightarrow 0$ the approximate area becomes equal to the definite integral.

**Trapezoidal rule:**

The above area can be better approximated by using trapezoids instead of rectangles. The resultant area thus becomes.

$$\Delta = \int_a^b f(x)dx \approx \frac{\Delta x}{2}\left[fx_o + 2f_1 + 2f_2 + \ldots\ldots\ldots\ldots + f_{n-1} + f_n\right]\Delta x \rightarrow (ii)$$

**Simpson's rule:**

A parabolic strip instead of a linear strip can give a better approximation to the curve f(x). The equation (ii) thus becomes;

$$\Delta = \int_a^b f(x)dx \approx \frac{\Delta x}{3}\left[f_o + 4(f_1 + f_3 + f_5 \ldots + f_{2n-1}) + 2(f_2 + f_4 + f_6 \ldots .f_{2n-2}) + f_n\right]$$

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## EXERCISE 11.1

**Q.1 Find an interval $a \le x \le b$ at which f (a) and f (b) have opposite signs for the following functions.**

**a) $f(x)=e^x-2-x$**

To find the interval where f(x) takes values with opposite sign, we put different values of x in f(x).

Put $x=-1$, $f(-1)=e^{-1}-2-(1)=-0.63$ (negative)



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Put x=0, $f(0)=e^0-2-0=1-2=-1$ (negative)

Put x=1, $f(1)=e^1-2-1=-0.28=$ (negative)

Put x=2, $f(2)=e^2-2-2=7.38=$ (Positive)

**Hence the function takes values with opposite sign in the interval [1,2]**

**b) $f(x)=\cos x+1-x$ x is in radius.**

To find the interval where f(x) takes values with opposite sign, we put different values of x in f(x).

Put x=0, $f(0)=\cos 0+1-0=1+1-0=2$ (positive)

Put x=1, $f(1)=\cos 1+1-1=\cos 1=0.54$ (positive)

Put x=2, $f(2)=\cos 2+1-2=-1.4$ (negative)

**Hence the function takes values with opposite sign in the interval [1,2]**

**c) $f(x)=x^2-10x+23$**

To find the interval where f(x) takes values with opposite sign, we put different values of x in f(x).

Put x=3, $f(3)=(3)^2-10(3)+23=2$ (positive)

Put x=4, $f(4)=(4)^2-10(4)+23=-1$ (negative)

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**Hence the function takes values with opposite sign in the interval [3,4]**

**Q.2 Compute four iterations of the bisection method for the following function with indicated interval $[a_0, b_0]$.**

**a) $f(x)=e^x-2-x$ [1, 1.8]**

**Solution:**

Given function, $f(x)=e^x-2-x$



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Given interval $[a_0, b_0] = [1, 1.8]$

$a_0=1, b_0 = 1.8$

$f(a_0)=f(1)=e^1-2-1=-0.2817$ (negative)

$f(b_0)=f(1.8)=e^{1.8}-2-1.8=2.25$ (positive)

As the function has opposite sign at 1 and 1.8, there is a root in the interval [1, 1.8]

For first iteration, $C_0$ the midpoint of $a_0$ and $b_0$

$$C_0 = \frac{a_0 + b_0}{2} = \frac{1+1.80}{2} = 1.4$$

$f(C_0)=f(1.4)=e^{1.4}-2-1.4=0.6552$ (positive)

**As f(1.4) is positive, thus f(1) and f(1.4) are with opposite signs and the root lines in interval $[a_1, b_1]=[1, 1.4]$**

For second iteration, $C_1$ the midpoint of $a_1$ and $b_1$ is,

$$C_1 = \frac{a_1 + b_1}{2} = \frac{1+1.4}{2} = \mathbf{1.2}$$

$f(C_1)=f(1.2)=e^{1.2}-2-1.2=0.12$ (positive)

As f(1.2) is positive, thus f(1) and f(1.2) are with opposite signs and root lies in interval $[a_2, b_2]=[1, 1.2]$.

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For third iteration, $C_2$ the midpoint of $a_2$ and $b_2$ is,

$$C_2 = \frac{a_2 + b_2}{2} = \frac{1+1.2}{2} = \mathbf{1.1}$$

$f(C_2)=f(1.1)=e^{1.1}-2-1.1=-0.0958$ (negative)

As f(1.1) is negative, thus f(1.1) and f(1.2) are with opposite signs and root lies in interval $[a_3, b_3]=[1, 1.2]$

For fourth iteration, $C_4$ the midpoint of $a_3$ and $b_3$ is,



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$$C_3 = \frac{a_3 + b_3}{2} = \frac{1.1+1.2}{2} = \mathbf{1.15}$$

$f(C_3) = f(1.15) = e^{1.15} - 2 - 1.15 = 0.00938$ (positive)

**As f(1.5) is positive, thus f(1.1) and f(1.15) are with opposite signs and root lies in interval $[a_4, b_4] = [1.1, 1.15]$**

**b) $f(x) = \cos x + 1 - x$ [0.8, 1.6] x is in radians**

**Solution:**

Given function, $f(x) = \cos x + 1 - x$

Given interval $[a_0, b_0] = [0.8, 1.6]$

$a_0 = 0.8, b_0 = 1.6$

$f(a_0) = f(0.8) = \cos 0.8 + 1 - 0.8 = 0.897$ (positive)

$f(b_0) = f(1.6) = \cos 1.6 + 1 - 1.6 = -0.63$ (negative)

As the function has opposite sign at 0.8 and 1.6, there is a root in the interval [0.8, 1.6]

For first iteration, $C_0$ the midpoint of $a_0$ and $b_0$

$$C_0 = \frac{a_0 + b_0}{2} = \frac{0.8+1.6}{2} = \mathbf{1.2}$$

$f(C_0) = f(1.2) = \cos 1.2 + 1 - 1.2 = 0.1623$ (positive)

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As the function has opposite sign at 1.2 and 1.6, there is a root in the interval [1.2, 1.6]

For second iteration, the midpoint $C_1$ of $a_1$ and $b_1$ is,

$$C_1 = \frac{a_1 + b_1}{2} = \frac{1.2+1.6}{2} = \mathbf{1.4}$$

$f(C_1) = f(1.4) = \cos 1.4 + 1 - 1.4 = -0.23$ (negative)

As the function has opposite sign at 1.2 and 1.4, there is a root in interval $[a_2, b_2] = [1.2, 1.4]$



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For third iteration, the midpoint $C_2$ of $a_2$ and $b_2$ is,

$$C_2 = \frac{a_2 + b_2}{2} = \frac{1.2 + 1.4}{2} = \mathbf{1.3}$$

$f(C_2) = f(1.3) = \cos 1.3 + 1 - 1.3 = -0.032$ (negative)

As the function has opposite sign at 1.2 and 1.3, there is a root in interval $[a_3, b_3] = [1.2, 1.3]$

For fourth iteration, the midpoint of $C_3$ of $a_3$ and $b_3$ is,

$$C_3 = \frac{a_3 + b_3}{2} = \frac{1.2 + 1.3}{2} = \mathbf{1.25}$$

$f(1.25) = \cos 1.25 + 1 - 1.25 = -0.01846$ (negative)

**As the function has opposite sign at 1.20 and 1.25, there is root in the interval, [1.2, 1.25]**

**The root lies in interval [1.2, 1.25]**

**c) $f(x) = x^2 - 10x + 23$ [3.2, 4]**

**Solution:**

Given function, $f(x) = x^2 - 10x + 23$

Given interval $[a_0, b_0] = [3.2, 4]$

$a_0 = 3.2$, $b_0 = 4$

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$f(a_0) = f(3.2) = (3.2)^2 - 10(3.2) + 23 = 1.24$ (positive)

$f(b_0) = f(4) = (4)^2 - 10(4) + 23 =$ **–1 (negative)**

As the function has opposite sign at 3.2 and 4, there is a root in the interval $[a_0, b_0] = [3.2, 4]$

For first iteration, the midpoint $C_0$ of $a_0$ and $b_0$ is

$$C_0 = \frac{a_0 + b_0}{2} = \frac{3.2 + 4}{2} = \mathbf{3.6}$$



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$f(C_0)=f(3.6)=(3.6)^2-10(3.6)+23=-11.7$ (negative)

As the function has opposite sign at 3.2 and 3.6, there is a root in the interval $[a_1, b_1] = [3.2, 3.6]$

For second iteration, the midpoint $C_1$ of $a_1$ and $b_1$ is,

$$C_1 = \frac{a_1 + b_1}{2} = \frac{3.2 + 3.6}{2} = \mathbf{3.4}$$

$f(C_1)=f(3.2)=(3.4)^2-10(3.4)+23=0.15$ (positive)

As the function has opposite sign at 3.4 and 3.6, there is a root in interval $[a_2, b_2]=[3.4,3.6]$

For third iteration, the midpoint $C_2$ of $a_2$ and $b_2$ is,

$$C_2 = \frac{a_2 + b_2}{2} = \frac{3.4 + 3.6}{2} = \mathbf{3.5}$$

$f(C_2)=f(3.5)=(3.5)^2-10(3.5)+23=0.25$ (positive)

As the function has opposite sign at 3.5 and 3.6, there is a root in interval $[a_3, b_3]=[3.5, 3.6]$

For fourth iteration, the midpoint of $C_3$ of $a_3$ and $b_3$ is,

$$C_3 = \frac{a_3 + b_3}{2} = \frac{3.5 + 3.6}{2} = \mathbf{3.55}$$

$f(C_3)=f(3.55)=(3.55)^2-10(3.55)+23=0.1025$ (positive)

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**As the function has opposite sign at 3.55 and 3.6, there is root in the interval, [3.55, 3.6]**

**So, there is a root of function is interval [3.55, 3.6]**

**Q.3 Compute four iterations of the Regula-falsi method for the following function with indicated interval $[a_0, b_0]$.**

**a) $f(x)=e^x-2-x$ [–2.4, –1.6]**

**Solution:**

Given function, $f(x) = e^x-2-x$



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Given interval $[a_0, b_0] = [-2.4, -1.6]$

$a_0 = -2.4, b_0 = -1.6$

$f(a_0) = f(-2.4) = e^{-2.4} - 2 - (-2.4) = e^{-2.4} - 2 + 2.4 = 0.49078$ (positive)

$f(b_0) = f(-1.6) = e^{-1.6} - 2 - (-1.6) = e^{-1.6} - 2 + 1.6 = -0.1981$ (negative)

As the $f(a_0)$ and $f(b_0)$ have opposite signs, there lies a root is the interval $[a_0, b_0] = [-2.4, -1.6]$

For first iteration, by Regula-falsi method

$$C_0 = a_0 - \frac{(a_0 - b_0)f(a_0)}{f(a_0) - f(b_0)}$$ Putting values

$$C_0 = -2.4 - \frac{(-2.4 - (-1.6))(0.49078)}{0.49078 - (-0.1981)} = \mathbf{-2.4 + 0.569922}$$

$C_0 = -1.830078$

Now, $f(C_0) = f(-1.830078) = e^{-1.830078} - 2 - (-1.830078)$

$f(C_0) = f(-1.830078) =$ **$-0.00952$ (negative)**

As $f(a_0)$ and $f(C_0)$ have opposite signs, there lies a root in the interval

$[a_0, C_0] = [-2.4, -1.830078]$

For second iteration of Regula-falsi method

$$C_1 = a_1 - \frac{(a_0 - C_0)f(a_0)}{f(a_0) - f(c_0)}$$ Putting values

$$C_1 = -2.4 - \frac{(-2.4 - (-1.830078))(0.490718)}{0.490718 - (-0.00952)}$$

$C_1 = -2.4 + 0.55907$

$\mathbf{C_1 = -1.8409}$

Now, $f(C_1) = f(-1.8409) = e^{-1.8409} - 2 - (-1.8409)$

$\mathbf{f(C_1) = -0.00043 < 0}$

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As, $f(a_0)$ and $f(C_1)$ have opposite signs, there lies a root in the interval

$[a_0, C_1] = [-2.4, -1.8409]$

For third iteration of Regula-falsi method

$$C_2 = a_0 + \frac{(a_0 - C_1)f(a_0)}{f(a_0) - f(c_1)}$$ Putting values

$$C_2 = -2.4 + \frac{(-2.4 - (-1.8409))(0.490718)}{0.490718 - (-0.00043)}$$

$$C_2 = -2.4 + 0.55861$$

$$\mathbf{C_2 = -1.84138}$$

Now, $f(C_2) = f(-1.84138) = e^{-1.84138} - 2 - (-1.84138)$

$f(C_2) = -0.000022$ (negative)

As, $f(a_0)$ and $f(C_2)$ have opposite signs, there lies a root in the interval

$[a_0, C_2] = [-2.4, -1.84138]$

For fourth iteration of Regula-falsi method

$$C_3 = a_0 + \frac{(a_0 - C_2)f(a_0)}{f(a_0) - f(c_2)}$$ Putting values

$$C_3 = -2.4 - \frac{(-2.4 - (-1.84138))(0.490718)}{0.490718 - (-0.000022)}$$

$$C_3 = -2.4 + 0.558594$$

$$\mathbf{C_3 = -1.841405}$$

**b)** **$f(x) = \cos x + 1 - x$** **[0.8, 1.6] x is in radians**

**Solution:**

Given function, **$f(x) = \cos x + 1 - x$**

Given interval $[a_0, b_0] = [0.8, 1.6]$

$a_0 = 0.8$, $b_0 = 1.6$

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$f(a_0)=\cos 0.8+1-0.8=0.6967+0.2=$**0.8967 (positive)**

$f(b_0)=\cos 1.6+1-1.6=-0.0292+1-1.6=-0.0292-0.6$

**$f(b_0)=-0.6292$ (negative)**

As $f(a_0)$ and $f(b_0)$ has opposite signs, there lies a root in interval $[a_0,b_0] = [0.8, 1.6]$

For first iteration of Regula-falsi method

$$C_0 = a_0 - \frac{(a_0 - b_0)f(a_0)}{f(a_0) - f(b_0)}$$ Putting values

$$C_0 = 0.8 - \frac{(0.8 - 1.6)(0.8967)}{0.8967 - (-0.6292)} = 0.8 + 0.47$$

**$C_0 = 1.27$**

$f(C_0)=f(1.27)= \cos 1.27+1-1.27=0.29628-0.27$

$f(C_0)=0.02628$ (positive)

As $f(C_0)$ and $f(b_0)$ has opposite signs, these lies a root in the interval

$[C_0,b_0] = [1.27, 1.6]$

For second iteration of Regula-falsi method

$$C_1 = C_0 - \frac{(C_0 - b_0)f(C_0)}{f(C_0) - f(b_0)}$$ Putting values

$$C_1 = 1.27 - \frac{(1.27 - 1.6)(0.02628)}{0.02628 - (-0.6292)} = 1.27 + 0.013$$

**$C_1 = 1.283$**

$f(C_1)=f(1.283)= \cos 1.283+1-1.283=0.28383-0.283$

$f(C_1)=0.00083$ (positive)

As $f(C_1)$ and $f(b_0)$ has opposite signs, there lies a root in the interval

$[C_1,b_0] = [1.283, 1.6]$

For third iteration of Regula-falsi method



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$$C_2 = C_1 - \frac{(C_1 - b_0)f(C_1)}{f(C_1) - f(b_0)} = 1.283 - \frac{(1.283 - 1.6)(0.00083)}{0.00083 - (-0.6292)}$$

$$C_2 = 1.283 + 0.000412 = \mathbf{1.283412}$$

$$f(C_2) = f(1.283412) = \cos 1.283412 + 1 - 1.283412 = 0.000056 \text{ (Positive)}$$

As $f(C_2)$ and $(b_0)$ has opposite signs, there lies a root in the interval

$[C_2, b_0] = [1.283412, 1.6]$

For fourth iteration of Regula-falsi method

$$C_3 = C_2 - \frac{(C_1 - b_0)f(C_2)}{f(C_2) - f(b_0)}$$

$$C_3 = 1.283412 - \frac{(1.283412 - 1.6)(0.000056)}{0.000056 - (-0.6292)}$$

$\mathbf{C_3 = 1.283402}$

**c)** **$f(x) = x^2 - 10x + 23$, [–2.4, –1.6] x is in radians**

**Solution:**

Given function, $f(x) = x^2 - 10x + 23$

Given interval $[a_0, b_0] = [-2.4, -1.6]$

$a_0 = -2.4$, $b_0 = -1.6$

$f(a_0) = f(-2.4) = (-2.4)^2 - 10(-2.4) + 23 = 52.76$ (positive)

$f(b_0) = f(-1.6) = (-1.6)^2 - 10(-1.6) + 23 = 41.56$ (positive)

**As both $f(a_0)$ and $f(b_0)$ are positive, there is no root is the given interval.**

**Q.4** **Let $f(x) = x^2 - 2x - 1$. The actual roots of $f(x)$ are $r_1 = 2.414214$ and $r_2 = 1.4142$. Use Newton-Raphson method.**

**a)** **With initial start $x_0 = 2.5$ to compute the iterates $x_1$, $x_2$ & $x_3$ that will converge to actual root $r_1$ of $f(x)$.**



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**Solution:**

Newton-Raphso Method

$$\mathbf{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}} \rightarrow \text{(i)}, \quad i=0,1,2,3$$

Here $f(x) = x^2-2x-1$

$f'(x) = 2x-2$

Putting in (i)

$$x_{i+1}=x_i - \frac{(x_i^{\,2} - 2x_i - 1)}{2x_i - 2} \quad \rightarrow \quad \text{(ii)}$$

Given $x_0=2.5$, to find $x_1$ put i=0 in (ii)

$$x_{0+1}=x_0 - \frac{(x_0^{\,2} - 2x_0 - 1)}{(2x_0 - 2)}$$

$$x_1=2.5 - \frac{(2.5)^2 - 2(2.5) - 1}{(2(2.5) - 2)}$$

$$x_1=2.5 - \frac{0.25}{3} = 2.4166667$$

$\mathbf{x_1 = 2.4166667}$

To find $x_2$ put i=1 in (ii)

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$$x_{1+1}=x_1 - \frac{(x_1^{\,2} - 2x_1 - 1)}{(2x_1 - 2)}$$

$$x_2=2.41666667 - \frac{\left[(2.41666667)^2 - 2(2.41666667) - 1\right]}{\left[2(2.41666667) - 2\right]}$$

$$x_2=2.41666667 - 0.002450986$$

$\mathbf{x_2 = 2.41421569}$

To find $x_3$, put i=2 in (ii)



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$$x_{2+1}=x_2-\frac{\left(x_2^{\,2}-2x_2-1\right)}{\left(2x_2-2\right)}$$

$$x_3=2.41421569-\frac{\left[(2.41421569)^2-2(2.41421569)-1\right]}{\left[2(2.41421569)-2\right]}$$

**$x_3$ = 2.41421356**

**Which converges to actual root $r_1$=2.414214**

**b) With initial start $x_0$=–0.5 to compute iterates $x_1$, $x_2$ & $x_3$ that will converges to actual root $r_2$ of f(x).**

**Solution:**

Here $f(x) = x^2-2x-1$

$f'(x) = 2x-2$

For

$$x_{i+1}=x_i-\frac{f(x)}{f'(x)}=x_i-\frac{\left(x_i^{\,2}-2x_i-1\right)}{\left(2x_i-2\right)}\rightarrow (i)$$

For $x_1$, putting i=0 in (i)

$$x_{0+1}=x_0-\frac{(x_0^{\,2}-2x_0-1)}{(2x_0-2)}$$

$$x_1=-0.5-\frac{\left((-0.5)^2-2(-0.5)-1\right)}{\left(2(-0.5)-2\right)}=-0.5+0.0833333$$

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**$x_1$ = –0.41666667**

To find $x_2$ put i=1 in (i)

$$x_{1+1}=x_1-\frac{\left(x_1^{\,2}-2x_1-1\right)}{\left(2x_1-2\right)}$$

$$x_2=0.41666667-\frac{\left[(0.41666667)^2-2(0.41666667)-1\right]}{\left[2(0.41666667)-2\right]}$$



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$x_2 = -0.41666667 + 0.002450982$

**$x_2 = -0.41421569$**

To find $x_3$, put i=2 in (i)

$$x_{2+1} = x_2 - \frac{\left(x_2^{\,2} - 2x_2 - 1\right)}{\left(2x_2 - 2\right)}$$

$$x_3 = -0.41421569 - \frac{\left[(0.41421569)^2 - 2(-0.41421569) - 1\right]}{2(-0.41421569) - 2}$$

$x_3 = -0.41421356$

**$x_3 = -0.414214$ (approx)**

**Q.5 Find iterate $x_3$ of Newton-Rampson method for the following function with initial start $x_0$.**

**a) $f(x) = x^3 - 3$, $x_0 = 1$.**

**Solution:**

By Newton-Rampson iterative method,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \rightarrow (i)$$

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Here $f(x) = x^3 - 3$, $f'(x) = 3x^2$

Putting these values in (i)

$$x_{i+1} = x_i - \frac{(x_i^{\,3} - 3)}{3x_i^{\,2}} \rightarrow (ii)$$

For $x_1$, putting i=0 in (i)

$$x_{0+1} = x_0 - \frac{(x_0^{\,3} - 3)}{3x_0^{\,2}} = 1 - \frac{\left((1^3) - 3\right)}{3(1)^2}$$



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$x_1 = 1 + \frac{2}{3} \quad \Rightarrow \quad \mathbf{x_1 = 1.6667}$

For $x_2$, putting i=1 in (ii)

$$x_{1+1} = x_1 - \frac{(x_1^3 - 3)}{3x_1^2}$$

$$x_2 = 1.6667 - \frac{\left((1.6667)^3 - 3\right)}{3(1.6667)^2} = 1.6667 - 0.19581$$

$\mathbf{x_2 = 1.471}$

For $x_3$, putting i=2 in (ii)

$$x_{2+1} = x_2 - \frac{(x_2^3 - 3)}{3x_2^2}$$

$$x_{2+1} = 1.471 - \frac{\left((1.471)^3 - 3\right)}{3(1.471)^2} = 1.471 - 0.0282$$

$\mathbf{x_3 = 1.443}$

**b)** $\mathbf{f(x) = \sin x, \quad x_0 = 1.}$

**Solution:**

By Newton-Raphson iteration method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \rightarrow \quad \text{(i)}$$

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Here $f(x) = \sin x$

$f'(x) = \cos x$

Putting in (i)

$$x_{i+1} = x_i - \frac{\sin x_i}{\cos x_i} = x_i - \tan x_i \rightarrow \quad \text{(ii)}$$

For $x_1$, putting i=0 in (ii)

$x_{0+1} = x_0 - \tan x_0$

$x_1 = 1 - \tan 1 = 1 - 1.557$



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$\mathbf{x_1 = -0.557}$

For $x_2$, putting i=1 in (ii)

$x_{1+1}=x_1-\tan x_1$

$x_2=-0.557-\tan(-0.557)$

$x_2=-0.557+0.6228$

$\mathbf{x_2 = -0.0458}$

For $x_3$, putting i=2 in (ii)

$x_{2+1}=x_2-\tan x_2$

$x_3=0.0458-\tan(0.0458)=0.0458-0.0458$

$\mathbf{x_3 = 0}$

**c)** $\mathbf{f(x)=x^3+2x-1,}$ $\mathbf{x_0=0.}$

**Solution:**

By Newton-Raphson iteration Method

$$\mathbf{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}} \rightarrow \text{(i)}$$

Here $f(x) = x^3+2x-1$

$f'(x)=3x^2+2,$

Putting in (i)

$$x_{i+1}=x_i-\frac{\left(x_i^3+2x_i-1\right)}{\left(3x_i^2+2\right)} \rightarrow \text{(ii)}$$

For $x_1$, putting i=0 in (ii)

$$x_{0+1}=x_0-\frac{\left(x_0^3+2x_0-1\right)}{\left(3x_0^2+2\right)}$$

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$$x_1 = 0 - \frac{\left((0)^3 + 2(0) - 1\right)}{\left(3(0)^2 + 2\right)} = 0 - \frac{-1}{2}$$

**$x_1 = 0.5$**

For $x_2$, putting i=1 in (ii)

$$x_{1+1} = x_1 - \frac{\left(x_1^{\,3} + 2x_1 - 1\right)}{\left(3x_1^{\,2} + 2\right)}$$

$$x_2 = 0.5 - \frac{\left((0.5)^3 + 2(0.5) - 1\right)}{\left(3(0.5)^2 + 2\right)} = 0.5 - \frac{0.125}{2.75}$$

$$x_2 = 0.5 - 0.04545$$

**$x_2 = 0.4545$**

For $x_3$, putting i=2 in (ii)

$$x_{2+1} = x_2 - \frac{\left(x_2^{\,3} + 2x_2 - 1\right)}{\left(3x_2^{\,2} + 2\right)}$$

$$x_3 = 0.4545 - \frac{\left((0.4545)^3 + 2(0.4545) - 1\right)}{\left(3(0.4545)^2 + 2\right)}$$

$$x_3 = 0.454545 - 0.001147$$

**$x_3 = 0.453398$**

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**Q.6 Use Newton-Raphson iterative Method to approximate the actual r of the following non-linear equations with indicated interval.**

**a) $f(x) = x^3 + 3x - 1 = 0$. (0,1)**

**Solution:**

By Newton-Raphson iterative method,

$$x_{i+1} = x_i - \frac{f(x)}{f'(x)} \qquad \rightarrow \qquad (i)$$



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Here $f(x) = x^3+3x-1$, $f'(x)=3x^2+3$

Putting these values in (i)

$$x_{i+1}=x_i-\frac{(x_i^3+3x_i-1)}{(3x^2_i+3)} \quad \rightarrow \quad \text{(ii)}$$

Here the interval is (0, 1), so let take $x_0$ as the midpoint of the interval i.e.

$$x_0=\frac{0+1}{2}=0.5$$

**$x_o = 0.5$**

For $x_1$, putting i=0 in (ii)

$$x_{0+1}=x_0-\frac{(x_0^3+3x_0-1)}{(3x_0^2+3)}, \text{putting } x_0=0.5$$

$$x_1=0.5-\frac{\left((0.5)^3+3(0.5)-1\right)}{(3(0.5)^2+3)}=0.5-\frac{0.625}{3.75}$$

$$x_1=0.5-0.1666=0.3333$$

**$x_1 = 0.3333$**

For $x_2$, putting i=1 in (ii)

$$x_{1+1}=x_1-\frac{\left(x_1^3+3x_1-1\right)}{\left(3x_1^2+3\right)} \quad \text{put } x_1=0.333$$

$$x_2=0.333-\frac{\left((0.333)^3+3(0.333)-1\right)}{\left(3(0.333)^2+3\right)}$$

$$x_2=0.333-0.109$$

**$x_2 = 0.225$**

For $x_3$, putting i=2 in (ii)

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$$x_3 = x_2 - \frac{\left(x_2^{\,3} + 3x_2 - 1\right)}{\left(3x_2^{\,2} + 3\right)} \qquad \text{put } x_2 = 0.225$$

$$x_3 = 0.225 - \frac{\left((0.225)^3 + 3(0.225) - 1\right)}{\left(3(0.225)^2 + 3\right)} = 0.225 + 0.099$$

$$\mathbf{x_3 = 0.324}$$

$$x_4 = 0.324 - \frac{\left((0.324)^3 + 3(0.324) - 1\right)}{\left(3(0.324)^2 + 3\right)}$$

$$x_4 = 0.324 - \frac{(0.00601224)}{(3.314928)}$$

$$x_4 = 0.324 - 0.001813$$

$$\mathbf{x_4 = 0.324}$$

**b)** $\mathbf{f(x) = x^3 + 2x^2 - x + 1 = 0.}$ **(–3,–2)**

**Solution:**

By Newton-Raphson iterative method,

$$x_{i+1} = x_i - \frac{f(x)}{f'(x)} \qquad \rightarrow \qquad \text{(i)}$$

Here $f(x) = x^3 + 2x^2 - x + 1$,

$f'(x) = 3x^2 + 4x - 1$

Putting these values in (i)

$$x_{i+1} = x_i - \frac{(x_i^{\,3} + 2x_i^{\,2} - x_i + 1)}{(3x_i^{\,2} + 4x_i - 1)} \qquad \rightarrow \qquad \text{(ii)}$$

Here the interval is (–3, –2), so let take $x_0$ as the midpoint of the interval i.e.

$$x_0 = \frac{-3 + (-2)}{2} = -2.5$$

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For $x_1$, putting i=0 in (ii)

$$x_{0+1}=x_0-\frac{(x_0^3+2x_0^2-x+1)}{(3x_0^2+4x_0-1)} \qquad \text{putting } \mathbf{x_0=-2.5}$$

$$x_1=-2.5-\frac{\left((-2.5)^3-2(-2.5)^2-(-2.5)+1\right)}{3(-2.5)^2+4(-2.5)-1}$$

$$x_1=-2.5-0.04838$$

$$\mathbf{x_1 = -2.54838}$$

For $x_2$, putting i=1 in (ii)

$$x_{1+1}=x_1-\frac{\left(x_1^3+2x_1^2-x+1\right)}{\left(3x_1^2+4x-1\right)} \qquad \text{put } x_1=-2.54838$$

$$x_2=-2.54838-\frac{\left((-2.54838)^3-2(-2.54838)^2-(-2.54838)+1\right)}{\left(3(-2.54838)^2+4(-2.54838)-1\right)}$$

$$\mathbf{x_2 = -2.5474}$$

For $x_3$, putting i=2 in (ii)

$$x_{2+1}=x_2-\frac{\left(x_2^3+2x_2^2-x_2+1\right)}{\left(3x_2^2+4x_2,-1\right)} \qquad \text{put } x_2=-2.5474$$

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$$x_3=-2.5474-\frac{\left((-2.5474)^3-2(-2.5474)^2-(-2.5474)+1\right)}{\left(3(-2.5474)^2+4(-2.5474)-1\right)}$$

$$\mathbf{x_3 = -2.5468}$$

**c)** $\mathbf{\sqrt[3]{x-3}=x+1}$ **on [–3,–2]**

**Solution:**

Here $f(x) = \sqrt[3]{x-3}-x-1=0$



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$$f(x)=(x-3)^{\frac{1}{3}}-x-1=0$$

$$f'(x)=\frac{1}{3}(x-3)^{\frac{-2}{3}}-1$$

By Newton-Raphson Method

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)} \quad \rightarrow \quad \text{(i)}$$

Substituting f(x) and f'(x) in (i)

$$x_{i+1}=x_i-\frac{\left[(x_i-3)^{\frac{1}{3}}-x_i-1\right]}{\left[\frac{1}{3}(x_i-3)^{\frac{-2}{3}}-1\right]} \quad \rightarrow \quad \text{(ii)}$$

Here the interval is (–3, –2), so taking $x_0$ as the midpoint of the interval i.e.

$$x_0=\frac{-2+(-3)}{2}=-2.5$$

**$x_o$ = –2.5**

For $x_1$, putting i=0 in (ii)

$$x_{1+1}=x_0-\frac{\left[(x_0-3)^{\frac{1}{3}}-x_0-1\right]}{\left[\frac{1}{3}(x_0-3)^{\frac{-2}{3}}-1\right]} \quad \text{putting } x_1=-2.5$$

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$$x_1=-2.5-\frac{\left[(-2.5-3)^{\frac{1}{3}}-(-2.5)-1\right]}{\left[\frac{1}{3}(-2.5-3)^{\frac{-2}{3}}-1\right]}=-2.5-\frac{-0.265174}{-0.8930197}$$

$$x_1=-2.5+0.29694$$

**$x_1$ = –2.20306**



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For $x_2$, putting i=1 in (ii)

$$x_{1+1}=x_1-\frac{\left[(x_1-3)^{\frac{1}{3}}-x_1-1\right]}{\left[\frac{1}{3}(x_1-3)^{\frac{-2}{3}}-1\right]} \qquad \text{putting } x_1=-2.20306$$

$$x_2=-2.20306-\frac{\left[(-2.20306-3)^{\frac{1}{3}}-(-2.20306)-1\right]}{\left[\frac{1}{3(-2.20306-3)^{\frac{2}{3}}}-1\right]}$$

$$x_2=-2.20306-\frac{-0.529757}{-0.888987}$$

$$x_2=-2.20306-0.595911$$

$$\mathbf{x_2 = -2.798971}$$

For $x_3$, putting i=2 in (ii)

$$x_{2+1}=x_2-\frac{\left[(x_2-3)^{\frac{1}{3}}-x_2-1\right]}{\left[\frac{1}{3}(x_2-3)^{\frac{2}{3}}-1\right]}$$

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$$x_3=-2.798971-\frac{\left[(-2.798971-3)^{\frac{1}{3}}-(-2.798971)-1\right]}{\left[\frac{1}{3(-2.798971-3)^{\frac{2}{3}}}-1\right]}$$

$$x_3=-2.798971-\frac{-0.002375}{-0.69018}$$

$$\mathbf{x_3 = -2.79553}$$



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**Q.7 Can Newton-Raphson Method be used for the following functions?**

**a)** $f(x)=9x^4-16x^3-36x^2+96x-60=0,\ x_0=\frac{4}{3}$

**Solution:**

By Newton-Raphson method,

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)} \quad \rightarrow \quad (i)$$

Here $f(x)=9x^4-16x^3-36x^2+96x-60$

$$f'(x)=36x^3-48x^2-72x+96$$

$$f'\left(\frac{4}{3}\right)=36\left(\frac{4}{3}\right)^3-48\left(\frac{4}{3}\right)^2-72\left(\frac{4}{3}\right)+96$$

$$f'\left(\frac{4}{3}\right)=36\left(\frac{64}{27}\right)-48\left(\frac{16}{9}\right)-\cancel{96}+\cancel{96}$$

$$f'\left(\frac{4}{3}\right)=\frac{\cancel{256}}{\cancel{3}}-\frac{\cancel{256}}{\cancel{3}}=0$$

**As $f'(x)=0$, so (i) can't be calculated and therefore Newton-Raphson Method can't be used.**

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**b)** $f(x)=1-\frac{1}{x}, x_0=2$

**Solution:**

By Newton-Raphson method,

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)} \quad \rightarrow \quad (i)$$

Here $f(x)=1-\frac{1}{x}=1-x^{-1},\ f'(x)=-(-1)x^{-2}=\frac{1}{x^2}$



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$$f'(2)=\frac{1}{(2)^2}=\frac{1}{4}$$

$$f(2)=1-\frac{1}{2}=\frac{1}{2}$$

putting i=0 in (i)

$$x_{0+1}=x_0-\frac{f(x_0)}{f'(x_0)}$$

Putting $x_0=2$

$$f(x_0)=f(2)=\frac{1}{2} \text{ and } f'(x_0)=f'(2)=\frac{1}{4}$$

$$x_1=2-\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{4}\right)}=2-2 \qquad \Rightarrow \qquad x_1=0$$

Now, for $x_2$, putting i=1 in (i)

$$x_{1+1}=x_1-\frac{f(x_1)}{f'(x_1)}; \quad x_1=0$$

Now, $f'(x_1)=f'(0)=\frac{1}{0^2}=\infty$

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**As $f'(x)=f'(0)$ undefined, Network-Raphson Method can't be proceed further.**



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## EXERCISE 11.2

**Q.1 Use the trapezoidal rule to approximate the value of definite integral. Round the answer to the nearest hundredth and compare your result with the exact value of the definite integral.**

**a)** $\mathbf{I=\int_1^3 x^2dx \text{ and } n=4}$

**Solution:**

Given that, $I=\int_1^3 x^2dx,\ n=4$.

Here a=1, b=3

$$\Delta x=\frac{b-a}{n}=\frac{3-1}{4}=\frac{2}{4}=\frac{1}{2}$$

Now, the grid points are

$$x_0=a=1, f(x_0)=f(1)=1^2=1$$

$$x_1=a+\Delta x=1+\frac{1}{2}=\frac{3}{2}, f\left(\frac{3}{2}\right)=\left(\frac{3}{2}\right)^2=\frac{9}{4}\Rightarrow f_1=\frac{9}{4}$$

$$x_2=a+2\Delta x=1+2\left(\frac{1}{2}\right)=2, f(x_2)=f(2)=(2)^2=4\Rightarrow f_2=4$$

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$$x_3=a+3\Delta x=1+3\left(\frac{1}{2}\right)=\frac{5}{2}, f(x_3)=f\left(\frac{5}{2}\right)=\left(\frac{5}{2}\right)^2=\frac{25}{4}\Rightarrow f_3=\frac{25}{4}$$

$$x_4=a+4\Delta x=1+4\left(\frac{1}{2}\right)=3, f(x_4)=f(3)=(3)^2=9\Rightarrow f_4=9$$

Now, by trapezoidal rule

$$\int_1^3 x^2dx=\frac{\Delta x}{2}\left[f_0+2(f_1+f_2+f_3)+f_4\right]$$



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$$\int_1^3 x^2 dx = \frac{\left(\frac{1}{2}\right)}{2}\left[1+2\left(\frac{9}{4}+4+\frac{25}{4}\right)+9\right]$$

$$\int_1^3 x^2 dx = \frac{1}{4}\left[10+2\left(\frac{9+16+25}{4}\right)\right] = \frac{1}{4}\left[10+2\times\frac{50}{4}\right]$$

$$\int_1^3 x^2 dx = \frac{35}{4} = 8.75 \quad \rightarrow \qquad \text{(i)}$$

Now, the actual value

$$\int_1^3 x^2 dx = \left|\frac{x^3}{3}\right|_1^3 = \frac{(3)^3}{3} - \frac{(1)^3}{3} = \frac{26}{3} = 8.67$$

Error=actual value – approximate value

Error= 8.67 – 8.75

**Error= –0.08**

**b)** $\mathbf{I = \int_0^1 \left(\frac{x^2}{2}+1\right) dx, \quad n = 4}$

**Solution:**

Given that, $I = \int_0^1 \left(\frac{x^2}{2}+1\right) dx, \ n = 4$.

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Here a=0, b=1

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$$

$$\Delta x = \frac{1}{4}$$

The grid points are thus

$x_0 = a = 0,$



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$$x_1 = a + \Delta x = 0 + \frac{1}{4} = \frac{1}{4}$$

$$x_2 = a + 2\Delta x = 0 + 2\left(\frac{1}{4}\right) = \frac{1}{2}$$

$$x_3 = a + 3\Delta x = 0 + 3\left(\frac{1}{4}\right) = \frac{3}{4}$$

$$x_4 = a + 4\Delta x = 0 + 4\left(\frac{1}{4}\right) = 1$$

And $f(x_0) = f(0) = \left(\frac{0^2}{2} + 1\right) = 1 \Rightarrow f_o = 1$

$$f(x_1) = f\left(\frac{1}{4}\right) = \left(\frac{\left(\frac{1}{4}\right)^2}{2} + 1\right) = \frac{33}{22} \Rightarrow f_1 = \frac{33}{22}$$

$$f(x_2) = f\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^2}{2} + 1 = \frac{9}{8} \Rightarrow f_2 = \frac{9}{8}$$

$$f(x_3) = f\left(\frac{3}{4}\right) = \frac{\left(\frac{3}{4}\right)^2}{2} + 1 = \frac{41}{32} \Rightarrow f_3 = \frac{41}{32}$$

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$$f(x_4) = f(1) = \frac{(1)^2}{2} + 1 = \frac{3}{2} \Rightarrow f_4 = \frac{3}{2}$$

Now by Trapezoid Rule

$$\int_0^1 \left(\frac{x^2}{2} + 1\right) dx = \frac{\Delta x}{2}\left[f_0 + 2(f_1 + f_2 + f_3) + f_4\right]$$

$$= \frac{\left(\frac{1}{4}\right)}{2}\left[1 + 2\left(\frac{33}{22} + \frac{9}{8} + \frac{41}{32}\right) + \frac{3}{2}\right]$$



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$$= \frac{1}{8}\left[\frac{5}{2} + 2\left(\frac{33+36+41}{32}\right)\right]$$

$$= \frac{1}{8}\left(\frac{5}{2} + \frac{110}{16}\right) = \frac{150}{128} = 1.17$$

Now the actual value

$$\int_0^1 \left(\frac{x^2}{2} + 1\right) dx = \left|\frac{x^3}{6} + x\right|_0^1 = \left(\frac{(1)^3}{6} + 1\right) - \left(\frac{0^3}{6} + 0\right)$$

$$\int_0^1 \left(\frac{x^2}{2} + 1\right) dx = \frac{1}{6} + 1 = \mathbf{1.17}$$

**Error= actual value – approximate value**

**Error= 1.17 – 1.17**

**Error= 0**

**c)** $\mathbf{I = \int_1^3 \frac{dx}{x},\ n = 6}$

**Solution:**

Given that, $I = \int_1^3 \frac{dx}{x},\ n = 6$.

Here a=1, b=3

$$\Delta x = \frac{b-a}{6} = \frac{3-1}{6} = \frac{2}{6} = \frac{1}{3}$$

Thus the grid points are

$x_0 = a = 1,$

$$x_1 = a + \Delta x = 1 + \frac{1}{3} = \frac{4}{3}$$

$$x_2 = a + 2\Delta x = 1 + 2\left(\frac{1}{3}\right) = \frac{5}{3},$$

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$$x_3 = a + 3\Delta x = 1 + 3\left(\frac{1}{3}\right) = 2$$

$$x_4 = a + 4\Delta x = 1 + 4\left(\frac{1}{3}\right) = \frac{7}{3},$$

$$x_5 = a + 5\Delta x = 1 + 5\left(\frac{1}{3}\right) = \frac{8}{3},$$

$$x_6 = a + 6\Delta x = 1 + 6\left(\frac{1}{3}\right) = 3,$$

And the values of function at the grid points are

$$f(x_0) = f(1) = \frac{1}{1} = 1 \Rightarrow f_o = 1$$

$$f(x_1) = f\left(\frac{4}{3}\right) = \frac{1}{4/3} = \frac{3}{4} \Rightarrow f_1 = \frac{3}{4}$$

$$f(x_2) = f\left(\frac{5}{3}\right) = \frac{1}{5/3} = \frac{3}{5} \Rightarrow f_2 = \frac{3}{5}$$

$$f(x_3) = f(2) = \frac{1}{2} \Rightarrow f_3 = \frac{1}{2}$$

$$f(x_4) = f\left(7/3\right) = \frac{1}{7/3} = \frac{3}{7} \Rightarrow f_4 = \frac{3}{7}$$

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$$f(x_5) = f\left(8/3\right) = \frac{1}{8/3} = \frac{3}{8} \Rightarrow f_5 = \frac{3}{8}$$

$$f(x_6) = f(3) = \frac{1}{3} \Rightarrow f_6 = \frac{1}{3}$$

By trapezoidal rule

$$\int_1^3 \frac{dx}{x} = \frac{\Delta x}{2}\left[f_0 + 2(f_1 + f_2 + f_3 + f_4 + f_5) + f_6\right]$$



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$$\int_1^3 \frac{dx}{x} = \frac{(1/3)}{2}\left[1+2\left(\frac{3}{4}+\frac{3}{5}+\frac{1}{2}+\frac{3}{7}+\frac{3}{8}\right)+\frac{1}{3}\right]$$

$$\int_1^3 \frac{dx}{x} = \mathbf{1.11}$$

Now, the actual value

$$\int_1^3 \frac{dx}{x} = |\ln x|_1^3 = \ln(3)-\ln(1) = 1.099-0$$

**Error=actual value – approximate value**

**Error= 1.099 – 1.11**

**Error= –0.011**

**d)** $\mathbf{I = \int_0^2 \sqrt{1+x^2}\, dx, \quad n = 6}$

**Solution:**

Given that, $I=\int_0^2 \sqrt{1+x^2}dx,\ n=6$.

Here a=0, b=2

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$$

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The grid points are thus,

$x_0 = a = 0,$

$$x_1 = a+\Delta x = 0+\frac{1}{3} = \frac{1}{3}$$

$$x_2 = a+2\Delta x = 0+2\left(\frac{1}{3}\right) = \frac{2}{3},$$

$$x_3 = a+3\Delta x = 0+3\left(\frac{1}{3}\right) = 1$$



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$$x_4 = a + 4\Delta x = 0 + 4\left(\frac{1}{3}\right) = \frac{4}{3},$$

$$x_5 = a + 5\Delta x = 0 + 5\left(\frac{1}{3}\right) = \frac{5}{3},$$

$$x_6 = a + 6\Delta x = 0 + 6\left(\frac{1}{3}\right) = 2,$$

And the values of function at the grid points are

$$f(x_o) = f(0) = \sqrt{1+0^2} = 1 \Rightarrow f_o = 1$$

$$f(x_1) = f\left(\frac{1}{3}\right) = \sqrt{1+\left(\frac{1}{3}\right)^2} = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3} \Rightarrow f_1 = \frac{\sqrt{10}}{3}$$

$$f(x_2) = f\left(\frac{2}{3}\right) = \sqrt{1+\left(\frac{2}{3}\right)^2} = \frac{\sqrt{13}}{3} \Rightarrow f_2 = \frac{\sqrt{13}}{3}$$

$$f(x_3) = f(1) = \sqrt{1+1^2} = \sqrt{2} \Rightarrow f_3 = \sqrt{2}$$

$$f(x_4) = f\left(4/3\right) = \sqrt{1+\left(4/3\right)^2} = \frac{5}{3} \Rightarrow f_4 = \frac{5}{3}$$

$$f(x_5) = f\left(5/3\right) = \sqrt{1+\left(5/3\right)^2} = \frac{\sqrt{34}}{3} \Rightarrow f_5 = \frac{\sqrt{34}}{3}$$

$$f(x_6) = f(2) = \sqrt{1+2^2} = \sqrt{5} \Rightarrow f_6 = \sqrt{5}$$

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By trapezoidal rule

$$I = \int_0^2 \sqrt{1+x^2}\,dx = \frac{\Delta x}{2}\left[f_0 + 2(f_1 + f_2 + f_3 + f_4 + f_5) + f_6\right]$$

$$\int_0^2 \sqrt{1+x^2}\,dx = \frac{\left(1/3\right)}{2}\left[1 + 2\left(\frac{\sqrt{10}}{3} + \frac{\sqrt{13}}{3} + \sqrt{2} + \frac{5}{3} + \frac{\sqrt{34}}{3}\right) + \sqrt{5}\right]$$



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$$\int_0^2 \sqrt{1+x^2}\,dx = \frac{1}{6}\left[\frac{3+6.32+7.21+8.49+10+11.66+6.71}{18}\right]$$

$$\int_0^2 \sqrt{1+x^2}\,dx = \frac{53.39}{18}$$

$$\int_0^2 \sqrt{1+x^2}\,dx = 2.97$$

Now, actual value,

$$\int_0^2 \sqrt{1+x^2}\,dx \quad \rightarrow \quad \text{(i)}$$

As, $\sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| \rightarrow$ (ii)

Comparing (ii) with (i) $a=1$

$$\int_0^2 \sqrt{1+x^2}\,dx = \left|\frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right|\right|_0^2$$

$$= \left(\frac{2\sqrt{1+2^2}}{2} + \frac{1}{2}\ln\left|2+\sqrt{1+2^2}\right|\right) - \left(\frac{0\sqrt{1+0}}{2} + \frac{1}{2}\ln\left|0+\sqrt{1+0^2}\right|\right)$$

$$= \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right) - 0 - 0$$

$= 2.96$

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**Error = actual value – approximate value**

**Error = 2.96 – 2.97**

**Error = –0.01**

**e)** $\mathbf{I = \int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx}$, **n = 5**

**Solution:**

Given that, $I = \int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$, $n = 5$.



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Here a=0, b=1

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5} = 0.2$$

Thus the grid points are,

$$x_0 = a = 0,$$

$$x_1 = a + \Delta x = 0 + 0.2 = 0.2$$

$$x_2 = a + 2\Delta x = 0 + 2(0.2) = 0.4,$$

$$x_3 = a + 3\Delta x = 0 + 3(0.2) = 0.6$$

$$x_4 = a + 4\Delta x = 0 + 4(0.2) = 0.8,$$

$$x_5 = a + 5\Delta x = 0 + 5(0.2) = 1,$$

And the values of function at the grid points are

$$f(x_0) = f(0) = \frac{1}{\sqrt{1+0^2}} = \frac{1}{1} = 1 \Rightarrow f_0 = 1$$

$$f(x_1) = f(0.2) = \frac{1}{\sqrt{1+(0.2)^2}} = \frac{1}{\sqrt{1+\frac{1}{25}}} = \frac{\sqrt{25}}{\sqrt{26}} = \frac{5}{\sqrt{26}} \Rightarrow f_1 = \frac{5}{\sqrt{26}}$$

$$f(x_2) = f(0.4) = \frac{1}{\sqrt{1+(0.4)^2}} = \frac{1}{\sqrt{1+\frac{4}{25}}} = \frac{\sqrt{25}}{\sqrt{29}} = \frac{5}{\sqrt{29}} \Rightarrow f_2 = \frac{5}{\sqrt{29}}$$

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$$f(x_3) = f(0.6) = \frac{1}{\sqrt{1+(0.6)^2}} = \frac{1}{\sqrt{1+\frac{9}{25}}} = \frac{\sqrt{25}}{\sqrt{34}} = \frac{5}{\sqrt{34}} \Rightarrow f_3 = \frac{5}{\sqrt{34}}$$

$$f(x_4) = f(0.8) = \frac{1}{\sqrt{1+(0.8)^2}} = \frac{1}{\sqrt{1+\frac{16}{25}}} = \frac{\sqrt{25}}{\sqrt{41}} = \frac{5}{\sqrt{41}} \Rightarrow f_4 = \frac{5}{\sqrt{41}}$$

$$f(x_5) = f(1) = \frac{1}{\sqrt{1+1^2}} = \frac{1}{\sqrt{2}} \Rightarrow f_5 = \frac{1}{\sqrt{2}}$$



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By trapezoidal method

$$I=\int_0^1 \frac{1}{\sqrt{1+x^2}}dx=\frac{\Delta x}{2}\left[f_0+2\left(f_1+f_2+f_3+f_4\right)+f_5\right]$$

$$I=\frac{0.2}{2}\left[1+2\left(\frac{5}{\sqrt{26}}+\frac{5}{\sqrt{29}}+\frac{5}{\sqrt{34}}+\frac{5}{\sqrt{41}}\right)+\frac{1}{\sqrt{2}}\right]$$

$$I=\frac{0.2}{2}[1+2(0.981+2(0.93)+2(0.86)+2(0.78)+0.71]$$

$$I=0.1[1+1.96+1.86+1.72+1.56+0.71]$$

$$I=\frac{1}{10}(8.801945)=\mathbf{0.8802}$$

Now, actual value

$$I=\int_0^1 \frac{1}{\sqrt{1+x^2}}dx \quad \rightarrow \quad \text{(i)}$$

As, $$\frac{1}{\sqrt{a^2+x^2}}dx=\ln\left|\frac{x+\sqrt{a^2+x^2}}{a}\right| \quad \rightarrow \quad \text{(ii)}$$

Comparing (ii) with (i) a=1

$$I=\int_0^1 \frac{1}{\sqrt{1+x^2}}dx=\ln\left|\frac{x+\sqrt{1+x^2}}{1}\right|_0^1$$

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$$=\ln\left(1+\sqrt{1+1^2}\right)-\ln\left(0+\sqrt{1+0^2}\right)$$

$$=\ln\left(1+\sqrt{2}\right)-\ln(1)=0.8814-0 \ =0.8814$$

**Error=actual value – approximate value**

**Error= 0.8814 – 0.8802**

**Error= 0.0012**



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**Q.2 Use Simpson's rule to approximate the value of each definite integral. Round the answer to nearest hundredth and compare your results with the exact value of the definite integral.**

**a)** $\mathbf{I = \int_2^4 x^2 dx, \quad n = 3}$

**Solution:**

Given that, $I = \int_2^4 x^2 dx, \ n = 3$.

Here a=2, b=4

$$\Delta x = \frac{b-a}{2n} = \frac{4-2}{2(3)} = \frac{1}{3}$$

Let the grid points are

$x_0 = a = 2$

$$x_1 = a + \Delta x = 2 + \frac{1}{3} = \frac{7}{3}$$

$$x_2 = a + 2\Delta x = 2 + 2\left(\frac{1}{3}\right) = \frac{8}{3}$$

$$x_3 = a + 3\Delta x = 2 + 3\left(\frac{1}{3}\right) = 3$$

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$$x_4 = a + 4\Delta x = 2 + 4\left(\frac{1}{3}\right) = \frac{10}{3}$$

$$x_5 = a + 5\Delta x = 2 + 5\left(\frac{1}{3}\right) = \frac{11}{3}$$

$$x_6 = a + 6\Delta x = 2 + 6\left(\frac{1}{3}\right) = 4$$

and the values of functions at the grid points

$$f(x_0) = f(2) = (2)^2 = 4 \Rightarrow f_0 = 4$$



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$$f(x_1) = f\left(\frac{7}{3}\right) = \left(\frac{7}{3}\right)^2 = \frac{49}{9} \Rightarrow f_1 = \frac{49}{9}$$

$$f(x_2) = f\left(\frac{8}{3}\right) = \left(\frac{8}{3}\right)^2 = \frac{64}{9} \Rightarrow f_2 = \frac{64}{9}$$

$$f(x_3) = f(3) = (3)^2 = 9 \Rightarrow f_3 = 9$$

$$f(x_4) = f\left(\frac{10}{3}\right) = \left(\frac{10}{3}\right)^2 = \frac{100}{9} \Rightarrow f_4 = \frac{100}{9}$$

$$f(x_5) = f\left(\frac{11}{3}\right) = \left(\frac{11}{3}\right)^2 = \frac{121}{9} \Rightarrow f_5 = \frac{121}{9}$$

$$f(x_6) = f(4) = (4)^2 = 16 \Rightarrow f_6 = 16$$

By Simpson's rule;

$$I = \int_2^4 x^2 dx = \frac{\Delta x}{3}\left[f_0 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) + f_6\right]$$

Putting values

$$I = \int_2^4 x^2 dx = \frac{\left(\frac{1}{3}\right)}{3}\left[4 + 4\left(\frac{49}{9} + 9 + \frac{121}{9}\right) + 2\left(\frac{64}{9} + \frac{100}{9}\right) + 16\right]$$

$$= 18.6666$$

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Actual value

$$I = \int_2^4 x^2 dx = \left|\frac{x^3}{3}\right|_2^4 = \frac{(4)^3}{3} - \frac{(2)^3}{3} = \frac{64 - 8}{3}$$

$$I = \frac{56}{3} = \mathbf{18.6666}$$

**Error=actual value – approximate value**

**Error= 18.6666 – 18.6666**

**Error= 0.0000**



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**b)** $\mathbf{I = \int_2^3 \left( \frac{x^2}{3} - 1 \right) dx, \quad n = 4}$

**Solution:**

Here a=2, b=3

$$\Delta x = \frac{b-a}{2n} = \frac{3-2}{2(4)} = \frac{1}{8}$$

The grid points are thus,

$x_0 = a = 2$

$$x_1 = a + \Delta x = 2 + \frac{1}{8} = \frac{17}{8}$$

$$x_2 = a + 2\Delta x = 2 + 2\left(\frac{1}{8}\right) = \frac{9}{4}$$

$$x_3 = a + 3\Delta x = 2 + 3\left(\frac{1}{8}\right) = \frac{19}{8}$$

$$x_4 = a + 4\Delta x = 2 + 4\left(\frac{1}{8}\right) = \frac{5}{2}$$

$$x_5 = a + 5\Delta x = 2 + 5\left(\frac{1}{8}\right) = \frac{21}{8}$$

$$x_6 = a + 6\Delta x = 2 + 6\left(\frac{1}{8}\right) = \frac{11}{4}$$

$$x_7 = a + 7\Delta x = 2 + 7\left(\frac{1}{8}\right) = \frac{23}{8}$$

$$x_8 = a + 8\Delta x = 2 + 8\left(\frac{1}{8}\right) = 3$$

and the values of functions at the grid points

$$f(x_0) = f(2) = \frac{(2)^2}{3} - 1 = \frac{1}{3} \Rightarrow f_0 = \frac{1}{3}$$

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$$f(x_1)=f\left(17/8\right)=\left(17/8\right)^2-1=\frac{289}{192}-1=\frac{97}{192}\Rightarrow f_1=\frac{97}{192}$$

$$f(x_2)=f\left(\frac{9}{4}\right)=\frac{\left(9/4\right)^2}{3}-1=\frac{81}{48}-1=\frac{33}{48}\Rightarrow f_2=\frac{33}{48}$$

$$f(x_3)=f\left(\frac{19}{8}\right)=\frac{\left(19/8\right)^2}{3}-1=\frac{361}{192}-1=\frac{169}{192}\Rightarrow f_3=\frac{169}{192}$$

$$f(x_4)=f\left(\frac{5}{2}\right)=\frac{\left(5/2\right)^2}{3}-1=\frac{25}{12}-1=\frac{13}{12}\Rightarrow f_4=\frac{13}{12}$$

$$f(x_5)=f\left(\frac{21}{8}\right)=\frac{\left(21/8\right)^2}{3}-1=\frac{441}{192}-1=\frac{249}{192}\Rightarrow f_5=\frac{249}{192}$$

$$f(x_6)=f\left(\frac{11}{4}\right)=\frac{\left(11/4\right)^2}{3}-1=\frac{121}{48}-1=\frac{78}{48}\Rightarrow f_6=\frac{78}{48}$$

$$f(x_7)=f\left(\frac{23}{8}\right)=\frac{\left(23/8\right)^2}{3}-1=\frac{529}{192}-1=\frac{337}{192}\Rightarrow f_7=\frac{337}{192}$$

$$f(x_8)=f(3)=\frac{(3)^2}{3}-1=3-1=2$$

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By Simpson's rule;

$$I=\int_2^3\left(\frac{x^2}{3}-1\right)dx=\frac{\Delta x}{3}\left[f_0+4\left(f_1+f_3+f_5+f_7\right)+2\left(f_2+f_4+f_6\right)+f_8\right]$$

Putting values

$$I==\frac{\left(1/8\right)}{3}\left[\frac{1}{3}+4\left(\frac{97}{192}+\frac{169}{192}+\frac{249}{192}+\frac{337}{192}\right)+2\left(\frac{33}{48}+\frac{13}{12}+\frac{73}{48}\right)+2\right]$$

$I=\mathbf{1.111}$



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Now, Actual value

$$I=\int_2^3\left(\frac{x^2}{3}-1\right)dx=\left|\frac{x^3}{9}-x\right|_2^3=\left(\frac{(3)^3}{9}-3\right)-\left(\frac{(2)^3}{9}-2\right)$$

$$I=(3-3)-\left(\frac{8}{9}-2\right)=0+\frac{10}{9}=1.111$$

**Error=actual value – approximate value**

**Error= 1.111 – 1.111**

**Error= 0**

**c)** $\mathbf{I=\int_1^3\frac{dx}{x},\quad n=3}$

**Solution:**

Here $a=1, b=3, \quad 2n=2(3)=6$

$$\Delta x=\frac{b-a}{2n}=\frac{3-1}{6}=\frac{2}{6}=\frac{1}{3}$$

The grid points are thus,

$x_0=a=1$

$$x_1=a+\Delta x=1+\frac{1}{3}=\frac{4}{3}$$

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$$x_2=a+2\Delta x=1+2\left(\frac{1}{3}\right)=\frac{5}{3}$$

$$x_3=a+3\Delta x=1+3\left(\frac{1}{3}\right)=2$$

$$x_4=a+4\Delta x=1+4\left(\frac{1}{3}\right)=\frac{7}{3}$$

$$x_5=a+5\Delta x=1+5\left(\frac{1}{3}\right)=\frac{8}{3}$$



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$x_6 = a + 6\Delta x = 1 + 6\left(\frac{1}{3}\right) = 3$

The values of functions at the grid points

$f(x_0) = f(1) = \frac{1}{3} = 1 \Rightarrow f_0 = 1$

$f(x_1) = f\left(\frac{4}{3}\right) = \left(\frac{1}{\frac{4}{3}}\right) = \frac{3}{4} \Rightarrow f_1 = \frac{3}{4}$

$f(x_2) = f\left(\frac{5}{3}\right) = \frac{1}{\left(\frac{5}{3}\right)} = \frac{3}{5} \Rightarrow f_2 = \frac{3}{5}$

$f(x_3) = f(2) = \frac{1}{2} \Rightarrow f_3 = \frac{1}{2}$

$f(x_4) = f\left(\frac{7}{3}\right) = \frac{1}{\left(\frac{7}{3}\right)} = \frac{3}{7} \Rightarrow f_4 = \frac{3}{7}$

$f(x_5) = f\left(\frac{8}{3}\right) = \frac{1}{\left(\frac{8}{3}\right)} = \frac{3}{8} \Rightarrow f_5 = \frac{3}{8}$

$f(x_6) = f(3) = \frac{1}{3} \Rightarrow f_6 = \frac{1}{3}$

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By Trapezoidal rule;

$I = \int_1^3 \frac{dx}{x} = \frac{\Delta x}{3}\left[f_0 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) + f_6\right]$

$I = \frac{\left(\frac{1}{3}\right)}{3}\left[1 + 4\left(\frac{3}{4} + \frac{1}{2} + \frac{3}{8}\right) + 2\left(\frac{3}{5} + \frac{3}{7}\right) + \frac{1}{3}\right]$

$I = \int_1^3 \frac{dx}{x} = \mathbf{1.10}$

Now, Actual value



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$$I = \int_1^3 \frac{dx}{x} = |\ln x|_1^3 = \ln 3 - \ln 1$$

$$I = 1.098$$

**Error=actual value – approximate value**

**Error= 1.098 – 1.100**

**Error= 0.002**

**d)** $\mathbf{I = \int_0^1 e^{2x} dx, \quad n = 4}$

**Solution:**

Here a=0, b=1, 2n=2(4)=8

$$\Delta x = \frac{b-a}{2n} = \frac{1-0}{8} = \frac{1}{8}$$

The grid points are thus,

$x_0 = a = 0$

$$x_1 = a + \Delta x = 0 + \frac{1}{8} = \frac{1}{8}$$

$$x_2 = a + 2\Delta x = 0 + 2\left(\frac{1}{8}\right) = \frac{1}{4}$$

$$x_3 = a + 3\Delta x = 0 + 3\left(\frac{1}{8}\right) = \frac{3}{8}$$

$$x_4 = a + 4\Delta x = 0 + 4\left(\frac{1}{8}\right) = \frac{1}{2}$$

$$x_5 = a + 5\Delta x = 0 + 5\left(\frac{1}{8}\right) = \frac{5}{8}$$

$$x_6 = a + 6\Delta x = 0 + 6\left(\frac{1}{8}\right) = \frac{3}{4}$$

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$$x_7 = a + 7\Delta x = 0 + 7\left(\frac{1}{8}\right) = \frac{7}{8}$$

$$x_8 = a + 8\Delta x = 0 + 8\left(\frac{1}{8}\right) = 1$$

The values of functions at the grid points are

$$f(x_0) = f(0) = e^{2(0)} = e^0 = 1 \Rightarrow f_0 = 1$$

$$f(x_1) = f\left(\frac{1}{8}\right) = e^{2\left(\frac{1}{8}\right)} = e^{\frac{1}{4}} = 1.28 \Rightarrow f_1 = 1.28$$

$$f(x_2) = f\left(\frac{1}{4}\right) = e^{2\left(\frac{1}{4}\right)} = e^{\frac{1}{2}} = 1.65 \Rightarrow f_2 = 1.65$$

$$f(x_3) = f\left(\frac{3}{8}\right) = e^{2\left(\frac{3}{8}\right)} = e^{\frac{3}{4}} = 2.11 \Rightarrow f_3 = 2.11$$

$$f(x_4) = f\left(\frac{1}{2}\right) = e^{2\left(\frac{1}{2}\right)} = e^1 = 2.72 \Rightarrow f_4 = 2.72$$

$$f(x_5) = f\left(\frac{5}{8}\right) = e^{2\left(\frac{5}{8}\right)} = e^{\frac{5}{4}} = 3.49 \Rightarrow f_5 = 3.49$$

$$f(x_6) = f\left(3/4\right) = e^{2\left(\frac{5}{8}\right)} = e^{\left(\frac{3}{2}\right)} = 4.48 \Rightarrow f_6 = 4.48$$

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$$f(x_7) = f\left(7/8\right) = e^{2\left(\frac{7}{8}\right)} = e^{\left(\frac{7}{4}\right)} = 5.75 \Rightarrow f_7 = 5.75$$

$$f(x_8) = f(1) = e^{2(1)} = 7.39 \Rightarrow f_8 = 7.39$$

By Simpson's rule;

$$I = \int_0^1 e^{2x} dx = \frac{\Delta x}{3}\left[f_0 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6) + f_8\right]$$

$$I = \frac{\left(1/8\right)}{3}\left[1 + 4(1.28 + 2.11 + 3.49 + 5.75) + 2(1.65 + 2.72 + 4.48) + 7.39\right]$$



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$I = 3.19$

Now, Actual value

$$I = \int_0^1 e^{2x}dx = \left|\frac{e^{2x}}{2}\right|_0^1 = \frac{e^{2(1)}}{2} - \frac{e^{2(0)}}{2} = \frac{1}{2}\left(e^2 - e^0\right)$$

$I = \mathbf{3.195}$

**Error=actual value – approximate value**

**Error= 3.19 – 3.19 = 0**

**e)** $\mathbf{I = \int_0^1 \frac{1}{2+x+x^2}dx, \quad n = 2}$

**Solution:**

Here a=0, b=1, 2n=2(2)=4

$$\Delta x = \frac{b-a}{2n} = \frac{1-0}{4} = \frac{1}{4}$$

The grid points are thus,

$x_0 = a = 0$

$$x_1 = a + \Delta x = 0 + \frac{1}{4} = \frac{1}{4}$$

$$x_2 = a + 2\Delta x = 0 + 2\left(\frac{1}{4}\right) = \frac{1}{2}$$

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$$x_3 = a + 3\Delta x = 0 + 3\left(\frac{1}{4}\right) = \frac{3}{4}$$

$$x_4 = a + 4\Delta x = 0 + 4\left(\frac{1}{4}\right) = 1$$

The values of functions at the grid points are

$$f(x_0) = f(0) = \frac{1}{2+0+0^2} = \frac{1}{2} \Rightarrow f_0 = \frac{1}{2}$$



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$$f(x_1)=f\left(\frac{1}{4}\right)=\frac{1}{2+\frac{1}{4}+\left(\frac{1}{4}\right)^2}=\frac{16}{37}\Rightarrow f_1=\frac{16}{37}$$

$$f(x_2)=f\left(\frac{1}{2}\right)=\frac{1}{2+\frac{1}{2}+\left(\frac{1}{2}\right)^2}=\frac{4}{11}\Rightarrow f_2=\frac{4}{11}$$

$$f(x_3)=f\left(\frac{3}{4}\right)=\frac{1}{2+\frac{3}{4}+\left(\frac{3}{4}\right)^2}=\frac{16}{53}\Rightarrow f_3=\frac{16}{53}$$

$$f(x_4)=f(1)=\frac{1}{2+1+1^2}=\frac{1}{4}\Rightarrow f_4=\frac{1}{4}$$

By Simpson's rule;

$$I=\int_0^1\frac{1}{2+x+x^2}dx=\frac{\Delta x}{3}\left[f_0+4(f_1+f_3)+2(f_2)+f_4\right]$$

$$I=\frac{\left(1/4\right)}{3}\left[\frac{1}{2}+4\left(\frac{16}{37}+\frac{16}{53}\right)+2\left(\frac{4}{11}\right)+\frac{1}{4}\right]$$

$$I=0.3679$$

Now exact value of the given integral is

$$\int_0^1\frac{dx}{2+x+x^2}=\int_0^1\frac{dx}{\frac{1}{4}+\frac{7}{4}+x+x^2}$$

$$\int_0^1\frac{dx}{2+x+x^2}=\int_0^1\frac{dx}{x^2+x+\frac{1}{4}+\frac{7}{4}}$$

$$\int_0^1\frac{dx}{2+x+x^2}=\int_0^1\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}$$

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$$\int_0^1 \frac{dx}{2+x+x^2} = \int_0^1 \frac{dx}{\frac{7}{4}\left(\frac{2x+1/2}{7/4}\right)^2+1}$$

$$\int_0^1 \frac{dx}{2+x+x^2} = \int_0^1 \frac{dx}{\frac{7}{4}\left(\frac{2x+1}{7}\right)^2+1}$$

$$\int_0^1 \frac{dx}{2+x+x^2} = \int_0^1 \frac{dx}{\left(\frac{2x+1}{\sqrt{7}}\right)^2+(1)^2}$$

Now integration by substitution method $u = \frac{2x+1}{\sqrt{7}}$

and then $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x$

$$\int_0^1 \frac{dx}{2+x+x^2} = \frac{4}{7}\times\frac{\sqrt{7}}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)_0^1$$

$$\int_0^1 \frac{dx}{2+x+x^2} = \frac{2}{\sqrt{7}}\left[\tan^{-1}\left(\frac{3}{\sqrt{7}}\right)-\tan^{-1}\left(\frac{1}{\sqrt{7}}\right)\right]$$

$$\int_0^1 \frac{dx}{2+x+x^2} = \frac{2}{\sqrt{7}}(0.85-0.36) = 0.37$$

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**Error = Actual value – Approximate value**

**Error = 0.37 – 0.35**

**Error = 0.02**

**Q.3 A quarter circle of radius 1 has the equation $y = \sqrt{1-x^2}$ for $0 \le x \le 1$, which means that**

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{4}, \quad n = 4$$



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**Approximate the definite integral on the left by trapezoidal rule that equals the right hand rule, when π =3.1.**

**Solution:**

Given that, $\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4},\ n=4$

Here a=0, b=1,

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$$

The grid points are thus,

$x_0 = a = 0$

$$x_1 = a+\Delta x = 0+\frac{1}{4} = \frac{1}{4}$$

$$x_2 = a+2\Delta x = 0+2\left(\frac{1}{4}\right) = \frac{1}{2}$$

$$x_3 = a+3\Delta x = 0+3\left(\frac{1}{4}\right) = \frac{3}{4}$$

$$x_4 = a+4\Delta x = 0+4\left(\frac{1}{4}\right) = 1$$

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The values of functions at the grid points are

$$f(x_0) = f(0) = \sqrt{1-0^2} = 1$$

$$f(x_1) = f\left(\frac{1}{4}\right) = \sqrt{1-\left(\frac{1}{4}\right)^2} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4} \Rightarrow f_1 = \frac{\sqrt{15}}{4}$$

$$f(x_2) = f\left(\frac{1}{2}\right) = \sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \Rightarrow f_2 = \frac{\sqrt{3}}{2}$$



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$$f(x_3) = f\left(\frac{3}{4}\right) = \sqrt{1-\left(\frac{3}{4}\right)^2} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4} \Rightarrow f_3 = \frac{\sqrt{7}}{4}$$

$$f(x_4) = f(1) = \sqrt{1-1^2} = 0 \Rightarrow f_4 = 0$$

By Trapezoidal rule;

$$\text{L.H.S } \int_0^1 \sqrt{1-x^2}\,dx = \frac{\Delta x}{2}\left[f_0 + 2(f_1 + f_2 + f_3) + f_4\right]$$

$$= \frac{(1/4)}{2}\left[1 + 2\left(\frac{\sqrt{15}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{7}}{4}\right) + 0\right]$$

L.H.S $= 0.749 \quad \rightarrow \quad$ (ii)

and R.H.S $= \frac{\pi}{4} = \frac{3.10}{4} = 0.775 \quad \rightarrow \quad$ (iii)

from (ii)and (iii)

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4} \text{ (Approximately)}$$

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**Q.4 A quarter circle of radius 1 has the equation $y = \sqrt{1-x^2}$ for $0 \le x \le 1$, which means that**

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}, \quad n = 4$$

**Approximate the definite integral by Simpson's rule that equal the right side, when $\pi = 3.1$.**

**Solution:**



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Given that, $\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$ $\rightarrow$ (i)

Here a=0, b=1, 2n=2(4)=8

$\Delta x = \frac{b-a}{2n} = \frac{1-0}{8} = \frac{1}{8}$

The grid points are thus,

$x_0 = a = 0$

$x_1 = a + \Delta x = 0 + \frac{1}{8} = \frac{1}{8}$

$x_2 = a + 2\Delta x = 0 + 2\left(\frac{1}{8}\right) = \frac{1}{4}$

$x_3 = a + 3\Delta x = 0 + 3\left(\frac{1}{8}\right) = \frac{3}{8}$

$x_4 = a + 4\Delta x = 0 + 4\left(\frac{1}{8}\right) = \frac{1}{2}$

$x_5 = a + 5\Delta x = 0 + 5\left(\frac{1}{8}\right) = \frac{5}{8}$

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$x_6 = a + 6\Delta x = 0 + 6\left(\frac{1}{8}\right) = \frac{3}{4}$

$x_7 = a + 7\Delta x = 0 + 7\left(\frac{1}{8}\right) = \frac{7}{8}$

$x_8 = a + 8\Delta x = 0 + 8\left(\frac{1}{8}\right) = 1$

The values of functions at the grid points are then;

$f(x_0) = f(0) = \sqrt{1-0^2} = 1 \Rightarrow f_0 = 1$



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$$f(x_1)=f\left(\frac{1}{8}\right)=\sqrt{1-\left(\frac{1}{8}\right)^2}=\sqrt{\frac{63}{64}}=\frac{\sqrt{63}}{8}\Rightarrow f_1=\frac{\sqrt{63}}{8}$$

$$f(x_2)=f\left(\frac{1}{4}\right)=\sqrt{1-\left(\frac{1}{4}\right)^2}=\sqrt{\frac{15}{16}}=\frac{\sqrt{15}}{4}\Rightarrow f_2=\frac{\sqrt{15}}{4}$$

$$f(x_3)=f\left(\frac{3}{8}\right)=\sqrt{1-\left(\frac{3}{8}\right)^2}=\sqrt{\frac{55}{64}}=\frac{\sqrt{55}}{8}\Rightarrow f_3=\frac{\sqrt{55}}{8}$$

$$f(x_4)=f\left(\frac{1}{2}\right)=\sqrt{1-\left(\frac{1}{2}\right)^2}=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}\Rightarrow f_4=\frac{\sqrt{3}}{2}$$

$$f(x_5)=f\left(\frac{5}{8}\right)=\sqrt{1-\left(\frac{5}{8}\right)^2}=\sqrt{\frac{39}{64}}=\frac{\sqrt{39}}{8}\Rightarrow f_5=\frac{\sqrt{39}}{8}$$

$$f(x_6)=f\left(\frac{3}{4}\right)=\sqrt{1-\left(\frac{3}{4}\right)^2}=\sqrt{\frac{7}{16}}=\frac{\sqrt{7}}{4}\Rightarrow f_6=\frac{\sqrt{7}}{4}$$

$$f(x_7)=f\left(\frac{7}{8}\right)=\sqrt{1-\left(\frac{7}{8}\right)^2}=\sqrt{\frac{15}{64}}=\frac{\sqrt{15}}{8}\Rightarrow f_7=\frac{\sqrt{15}}{8}$$

$$f(x_8)=\sqrt{1-1^2}=0\Rightarrow f_8=0$$

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By Simpson's rule;

$$\text{L.H.S}=\int_0^1\frac{1}{\sqrt{1-x^2}}dx=\frac{\Delta x}{3}\left[f_0+4(f_1+f_3+f_5+f_7)+2(f_2+f_4+f_6)+f_8\right]$$

Putting values

$$=\frac{\left(1/8\right)}{3}\left[1+4\left(\frac{\sqrt{63}}{8}+\frac{\sqrt{55}}{8}+\frac{\sqrt{39}}{8}+\frac{\sqrt{15}}{8}\right)+2\left(\frac{\sqrt{15}}{4}+\frac{\sqrt{3}}{2}+\frac{\sqrt{7}}{4}\right)+0\right]$$

$$=\mathbf{0.7803}\qquad\rightarrow\qquad\text{(i)}$$



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$$\text{R.H.S} = \frac{\pi}{4} = \frac{3.1}{4} = 0.775 \quad \rightarrow \quad \text{(iii)}$$

Thus from (ii)and (iii)

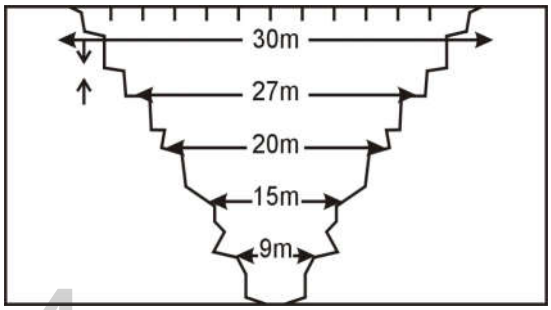
$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4} \textbf{ (Approximately)}$$

Hence approved

**Q.5 The width of an irregularly shaped dam is measured at 5-m intervals, with the results indicated in figure given below. Else Simpson's rule to estimate the one of the face of the dam; Hint:**

**x:$x_0$=0, $x_1$=5, $x_2$=10, $x_3$=15, $x_4$=20**

**f: $f_0$=9, $f_1$=15, $f_2$=20, $f_3$=27, $f_4$=30**




$\Delta x$=5, n=2 (even integrals)

$$S_4 = \frac{\Delta x}{3}\left[f_0 + 4(f_1 + f_3) + 2f_2 + f_4\right]$$

Putting values

$$S_4 = \frac{5}{3}\left[9 + 4(15 + 27) + 2(20) + 30\right]$$

$$\mathbf{S_4 = \frac{1235}{3} = 411.66m^2 = 412m^2}$$

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