Unit 08

CONICS-II

The conics section which includes parabola, ellipse and hyperbola are defined as the locus of a point P(x,y) moving in a plane such that its distance from a fixed point F (called focus) and a fixed line l (called directrix) have a fixed ratio called eccentricity i.e.

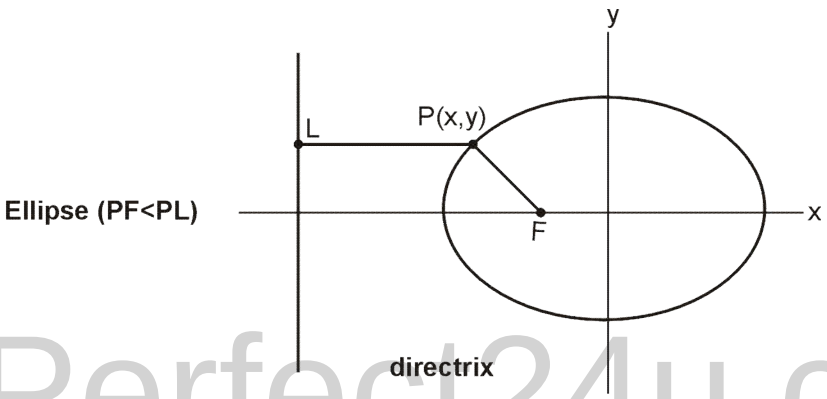
$$e = \frac{\text{distance of P to F}}{\text{distance of P to L}}$$

If $e<1$ / ellipse

If $e=1$ / Parabola

If $e>1$ / hyperbola

As shown in figures below:



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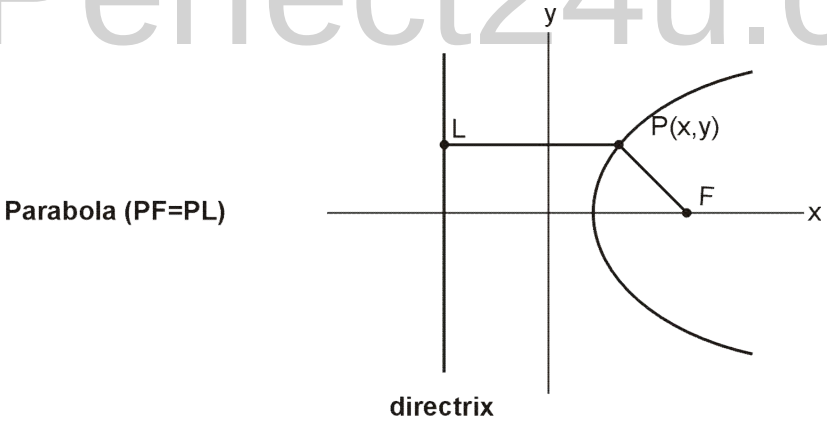

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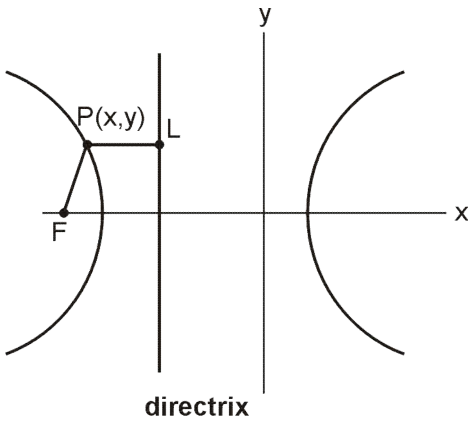
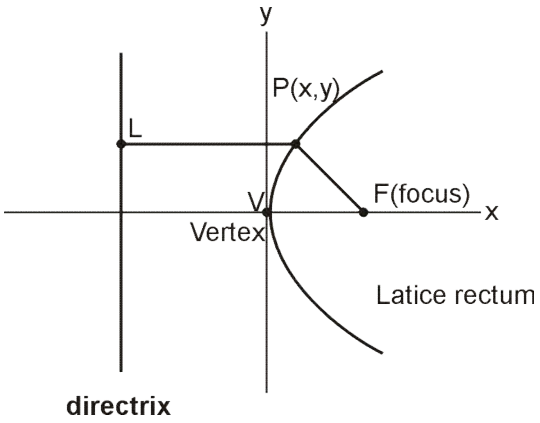
Hyperbola (PF>PL)



Parabola:

A parabola in standard from with vertex at the origin in shown in figure



Equation of the above parabola is;

$y^2=4px$

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A summary of parabola and its characteristics in other standard form are given below

Parabola	Curve	Focus	Directrix	Vertex
$x^2=4py$	Up if $p>0$	$F(0,p)$	$y=-p$	$V(0,0)$
	Down if $p<0$	$F(0,p)$	$y=-p$	$V(0,0)$
$y^2=4py$	Right if $p>0$	$F(p,0)$	$x=-p$	$V(0,0)$
	Left if $p<0$	$F(p,0)$	$x=-p$	$V(0,0)$

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Translation of parabola:

If the standard parabola with equation $y^2=4ax$ is shifted h units and the x-axis and k units along the y-axis, the equation of the parabola becomes, $(y-k)^2=4a(x-h)$, where parabola is

a) shifted right if $h>0$ b) shifted left if $h<0$

c) shifted up if $k>0$ d) shifted down if $k<0$

Parabola	Curve	Focus/Vertex	Directrix
$(x-h)^2=4p(y-k)$	Up if $p>0$ Down if $p<0$	F(h, k+p) V(h,k)	$y=k-p$
$(y-k)^2=4p(x-h)$	Right if $p>0$ Left if $p<0$	F(h+p, k) V(h,k)	$x=h-p$

For the shifted parabola

$(y-k)^2=4p(x-h)$, now

Vertex: v(h,k) focus: F(p+h, k)

Equation of directrix : $x=-p+h$

Similarly for vertical parabola, $x^2=4py$, the shifted equation is $(x-h)^2=4P(y-k)$ with

Vertex: v(h,k), focus: F(h,p+k)

Equation of directrix: $y=-p+k$

Equation of tangent and normal to parabola:

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A line $y=mx+c$ is tangent to parabola $y^2=4px$, if $c=\frac{p}{m}$ and $x^2=4py$ if $c=-pm^2$. The equation of tangent to parabola $y^2=4ax$ at a point $P(x_1,y_1)$ is given by;

$$yy_1=2p(x+x_1)$$

And to parabola $x^2=4py$ is

$$xx_1=2p(y+y_1)$$

The equation of normal to parabola, $y_2=4px$ at a point $p(x_1,y_1)$ is given by; $y-y_1=\frac{y_1}{2p}(x-x_1)$

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EXERCISE 8.1

Q.1 In each case, sketch the parabola represented by the equations, indicate the vertex, the focus the end points of the focal chord (lotus rectum) and the axis of symmetry.

a) $x^2=2y$

Solution: Given that $x^2=2y \quad \rightarrow$ (i)

The standard form of parabola is

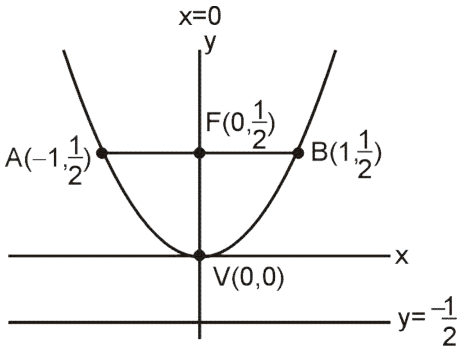
$$(x-h)^2=4p(y-k) \quad \rightarrow \text{(ii)}$$

Comparing (i) and (ii)

$4p=2$

$p=\frac{1}{2}$ (Upward direction)

$h=0, \quad k=0$



Now,

Vertex: $V(h,k)=V(0,0)$

Focus: $F(h, k+p)$

So focus, $f(0,\frac{1}{2})$

End points of Latus rectum:

To find corresponding value of x, put $y=\frac{1}{2}$ in (i)

$$x^2=2\times\frac{1}{2}$$

$x^2=1$

$x=\pm1$

So, ends of latus rectum $A(-1, \frac{1}{2})$, $B(1, \frac{1}{2})$

axis of symmetry $x=0$ (y-axis)

b) $y^2=-3(x+1)$

Solution: Given that $y^2=-3(x+1) \quad \rightarrow$ (i)

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The standard form of parabola is

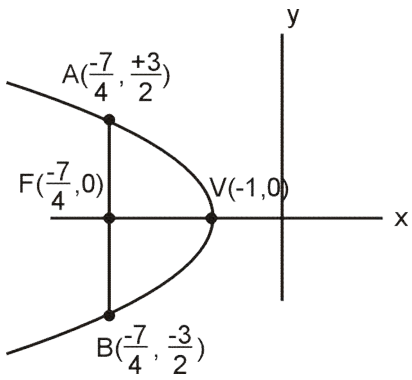
$$(y-k)^2 = 4p(x-h) \quad \rightarrow (ii)$$

Comparing (i) and (ii)

$4p = -3$

$p = \frac{-3}{4}$ (Left direction)

$k=0, h=-1$



Now,

Vertex: $V(h,k) = V(-1,0)$

Focus: $F(p+h, k) = F\left(-\frac{3}{4}-1, 0\right)$

$= F\left(-\frac{7}{4}, 0\right)$

End of Latus rectum:

Putting $x = -\frac{7}{4}$ (from Focus F) in (i)

$y^2 = -3\left(\frac{-3}{4}\right) = \frac{+9}{4}$

$y^2 = \pm\frac{3}{2}$

So, ends of lotus, rectum $A\left(\frac{-7}{4}, \frac{+3}{2}\right)$ and $B\left(\frac{-7}{4}, \frac{-3}{2}\right)$

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Axis of symmetry:

$y-k=0$ put $k=0$

$y=0$

c) $\mathbf{(y-3)^2=x}$

Solution: Given that $(y-3)^2=x \rightarrow$ (i)

The transformed form of parabola is

$$(y-k)^2 = 4p(x-h) \quad \rightarrow (ii)$$

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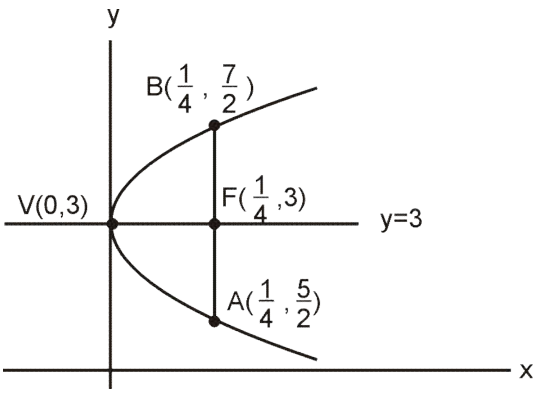
Comparing (i) and (ii)

$4p=1$

$p=\frac{1}{4}$

$k=3,\ h=0$



Now,

Vertex: $V(h,k)=V(0,3)$

Focus: $F(p+h,\ k)=F\left(\frac{1}{4}+0,3\right)$

$=F\left(\frac{1}{4},3\right)$

End of Latus rectum:

Putting $x=\frac{1}{4}$ (from Focus F) in (i)

$(y-3)^2=\frac{1}{4}$

$y-3=\frac{\pm 1}{2}$

$y=3-\frac{1}{2},\ y=3+\frac{1}{2}$

$y=\frac{5}{2},y=\frac{7}{2}$

So, ends points of latus rectum $A\left(\frac{1}{4},\frac{5}{2}\right)$ and $B\left(\frac{1}{4},\frac{7}{2}\right)$

Axis of symmetry:

$y-k=0$

$y-3=0$

$y=3$

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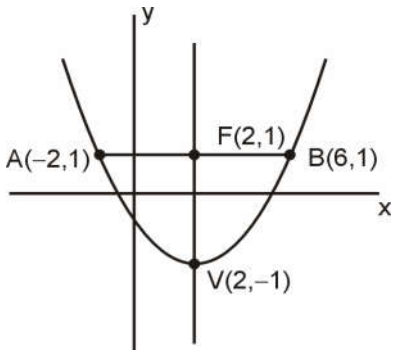
Q.2 In each case, determine the equation of graphed parabola

a) From figure



Length of Latus rectum

$$4P = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Put $x_1 = -2, x_2 = 6, y_1 = 1, y_2 = 1$

$$4P = \sqrt{(6+2)^2 + (1-1)^2}$$

$$4P = \sqrt{8^2 + 0} = \sqrt{8^2} + 0$$

$$4P = 8 \quad \Rightarrow P = 2$$

$$V(h,k) = V(2,-1)$$

h=2, k=–1

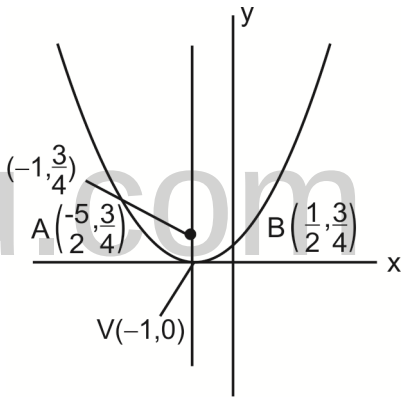
The transformed equation of parabola is

$$(x-h)^2 = 4p(y-k)$$

$$(x-2)^2 = 4(2)(y-(-1))$$

$(x-2)^2 = 8(y+1)$

b) For parabola in the transformed form, in figure



$$V(h,k) = V(-1,0)$$

$$h = -1, \; k = 0$$

$$F = (h, p+k)$$

$$p+k = \frac{3}{4} \quad \Rightarrow \quad p = \frac{3}{4} - k$$

$$p = \frac{3}{4} - 0 = \frac{3}{4}$$

and the transformed equation of parabola, $(x-h)^2 = 4p(y-k)$

$$(x-(-1))^2 = 4\left(\frac{3}{4}\right)(y-0)$$

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$$(x+1)^2 = \cancel{4}\left(\frac{3}{\cancel{4}}\right)(y-0) \qquad \Rightarrow \mathbf{(x+1)^2=3y}$$

Q.3 In each case, write the equation of parabola through the given information

a) Focus at F(0,3), directrix y= –3

Solution: Here F(0,3) so, the parabola is in upward direction

The equation of parabola in upward direction

$(x-h)^2=4p(y-k) \quad \rightarrow$ (i)

Focus, F (h,p+k)=F(0,3)

$h=0,\ p+k=3 \quad \rightarrow$ (ii)

directrix, $y=k-p$, so $p-k=3 \rightarrow$ (iii)

Adding (ii) and (iii)

$p+k=3$

$\underline{p-k=3}$

$2p=6$

$p=3$ putting (ii)

$3+k=3$

k=0

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Putting values of p, h and k in (i)

The equation of parabola becomes

$(x-0)^2=4(3)(y-0)$

$\mathbf{x^2=12y}$

b) Focus at F(4,0), directrix x=–4

Solution: Here focus F(4,0) so, the parabola is directed towards right

The equation of parabola in right direction is

$(y-k)^2=4p(x-h) \quad \rightarrow$ (i)

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Focus, F (p+h,k)=F(4,0)

k=0, p+h=4 → (ii)

Directix, x=–4

As directrix is given by x=h–p

So, h–p=–4 → (iii)

Adding (ii) and (iii)

h+p=4

h–p=–4

2h=0

h=0 put in (ii)

p+0=4 p=4

Thus equation of parabola becomes

$(y-0)^2=4(4)(x-0)$

$y^2=16x$

c) Vertex V(0,0), x-axis is the line of symmetry, passes through (3,6)

Solution: The parabola is the standard form

$y^2=4px$ → (i) putting P(3,6)

$(6)^2=4p(3)$

$36=12p$

$p=3$ putting in (i)

$y^2=4(3)x$

$y^2=12x$

d) Vertex V(0,0), y-axis is the line of symmetry, passes through (–12,–3)

Solution: Since vertex is V(0,0) and is symmetric about y-axis, so the equation is

$x^2=4py$ → (i) putting p(–12,–3)

$(-12)^2=4p(-3)$

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$144=12p$

$p=-12$ putting in (i) we get equation of parabola as

$x^2=4(-12)y$

$\mathbf{x^2=-48y}$

e) Line of symmetry is vertical, passes through P(–3,4), vertex V(5, 1)

Solution: As the line of symmetry is vertical the equation of parabola will be

$(x-h)^2=4p(y-k) \quad \rightarrow \quad$ (i)

Since vertex $V(h,k)=V(5,1)$, $h=5$, $k=1$

Putting in (i)

$(x-5)^2=4p(y-1) \quad \rightarrow \quad$ (ii)

Now putting P(–3,4) in (ii)

$(-3-5)^2=4p(4-1)$

$64=12p$

$p=\frac{64}{12}$

$\mathbf{p=\frac{16}{3}}$

Putting in (i)

$(x-5)^2=4\left(\frac{16}{3}\right)(y-1)$

$(x-5)^2=\frac{64}{3}(y-1)$

f) Line of symmetry is horizontal, passes through P(7,9), vertex V(3,–7)

Solution: As the line of symmetry is horizontal the equation of parabola will be;

$(y-k)^2=4p(x-h) \quad \rightarrow \quad$ (i)

Here vertex $V(h,k)=V(3,-7)$

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h=3, k=–7, putting in (i)

$(y+7)^2=4p(x-3)$ → (ii)

Putting p(7,9) in (ii)

$(9+7)^2=4p(7-3)$

$256=16p$

P=16 putting in (i)

$(y+7)^2=4(16)(x-3)$

$(y+7)^2=64(x-3)$

Q.4 Find the equation of the set of all points with distances from (4,3) that equals their distance from (–2, 1)

Solution: Let P(x,y) be the set of all points which is at equal distance from A (4,3) and B(–2,1) then, |PA|=|PB|

$$\sqrt{(x-4)^2+(y-3)^2}=\sqrt{(x+2)^2+(y-1)^2}$$

Taking square on both sides

$$(x-4)^2+(y-3)^2=(x+2)^2+(y-1)^2$$

$x^2-8x+16+y^2-6y+9=x^2+4x+4+y^2-2y+1$

$-12x-4y+20=0$, dividing by –4

$3x+y-5=0$

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Q.5 Find an equation of a parabola, whose focal chord has length 6, if it is known that the parabola has focus (4, –2) and its direction is parallel to the y-axis.

Solution: As the directrix is parallel to y-axis, the equation of parabola is

$(y-k)^2=4p(x-h)$ → (i)

Given, that length of focal chord = 4p =6

$$p=\frac{6}{4}=\frac{3}{2}$$

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Also, focus $F(p+h,k)=(4,-2)$

Thus, $k=-2$,

$p+h=4$, putting $p=\frac{3}{2}$

$\frac{3}{2}+h=4 \quad \Rightarrow h=4-\frac{3}{2}=\frac{5}{2}$ Putting h,k and p in (i)

$(y-(-2))^2=4\left(\frac{3}{2}\right)\left(x-\frac{5}{2}\right)$

$\mathbf{(y+2)^2=6\left(x-\frac{5}{2}\right)}$

Q.6 In each case, find the points of intersection in between the line and the parabola.

a. $y^2=9x$, $x-y+2=0$

Solution: Given line, $l=x-y+2=0 \rightarrow$ (i)

and parabola, $y^2=9x \quad \rightarrow$ (ii)

From line in (i), $y=x+2 \rightarrow$(iii) Putting in (ii)

$(x+2)^2=9x \quad \Rightarrow \quad x^2+4x+4=9x$

$x^2-5x+4=0$

$x^2-4x-x+4=0$

$(x-4)(x-1)=0$

$x=1, x=4$

For $x=1$, from (iii), $y=x+2=1+2=3$

So, point $P_1(1,3)$

For $x=4$, from (iii), $y=x+2=4+2=6$

So, point $P_2(4,6)$

Thus, the points of intersection are

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$P_1(1,3)$ and $P_2(4,6)$

b. **$y^2+3x=-8$, $l=x-y+2=0$**

Solution: Given that line, $l=x-y+2=0 \rightarrow$ (i)

Parabola is $y^2+3x=-8 \quad \rightarrow$ (ii)

From line in (i), $y=x+2 \quad \rightarrow$ (iii)

Putting in (ii)

$(x+2)^2+3x=-8$

$x^2+4x+4+3x+8=0$

$x^2+7x+12=0$

$x^2+3x+4x+12=0$

$(x+3)(x+4)=0$

$x+3=0 \quad \Rightarrow \quad x=-3$

$x+4=0 \quad \Rightarrow \quad x=-4$

For $x=-3$, from (iii), $y=x+2=-3+2=-1$

So, point $P_1(-3,-1)$

For $x=-4$, from (iii), $y=x+2=4+2=-2$

So, point $P_2(-4,-2)$

Thus, the points of intersection are

$P_1(-3,-1)$ and $P_2(-4,-2)$

c. **$x^2=2y$, $x-y-2=0$**

Solution: Given that line, $l=x-y-2=0 \rightarrow$ (i)

and parabola, $x^2=2y \quad \rightarrow$(ii)

from (i), $y=x-2 \quad \rightarrow$ (iii)

putting in (ii)

$x^2=2(x-2)$

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$x^2=2x-4$

$x^2-2x+4=0$

$a=1, b=-2, c=4$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(4)}}{2(1)}$$

$$x=\frac{2\pm\sqrt{-12}}{2}=\frac{2\pm2\sqrt{-3}}{2}=\frac{2\left(1\pm\sqrt{3}i\right)}{2}$$

$\mathbf{x=1\pm\sqrt{3}i}$

Which is imaginary so the line does not intersect the parabola.

d. $\mathbf{y^2=-2(x+1),\ x+y-2=0}$

Solution: Given that line, $l_1=x+y-2=0\rightarrow$ (i)

and parabola, $y^2=-2(x+1)\rightarrow$(ii)

from (i), $y=-x+2 \quad \rightarrow$ (iii)

putting in (ii)

$(-x+2)^2=-2(x+1)$

$\Rightarrow \quad x^2-4x+4+2x+2=0$

$\Rightarrow \quad x^2-2x+6=0$

$a=1, b=-2, c=6$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(6)}}{2(1)}$$

$$x=\frac{2\pm\sqrt{-20}}{2}=\frac{2\left(1\pm\sqrt{-5}\right)}{2}$$

$\mathbf{x=1\pm\sqrt{5}i}$

Which is imaginary so the line does not intersect the parabola.

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Q.7 For what value of c,

a. The line x–y+c=0 will touch the parabola $y^2=9x$?

Solution: Given the line, $l=x-y+c=0 \rightarrow$ (i)

and parabola, $y^2=9x \rightarrow$ (ii)

From (i), $y=x+c$

Putting in (ii)

$(x+c)^2=9x$

$x^2+c^2+2cx-9x=0$

$x^2+(2c-9)x+c^2=0$

Which is quadratic equation in x. For the line to touch the parabola, $b^2-4ac=0 \rightarrow$(iv)

$a=1, b=2c-9, c=c^2$ putting in (iv)

$(2c-9)^2-4(1)(c^2)=0$

$4c^2-36c+81-4c^2=0$

$36c=81$

$c=\frac{81}{36} \Rightarrow \mathbf{c=\frac{9}{4}}$

b. The line x–y+c=0, will touch the parabola $y^2=8x$?

Solution: Given the line, $l=x-y+c=0 \rightarrow$ (i)

and parabola, $y^2=8x \rightarrow$ (ii)

From (i), $y=x+c \rightarrow$ (iii)

Putting in (ii)

$(x+c)^2=8x$

$x^2+2cx+c^2=8x$

$x^2+(2c-8)x+c^2=0$

Which is quadratic equation in x. For the line to touch the parabola, $b^2-4ac=0 \rightarrow$(iv)

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$a=1, b=2c-8, c=c^2$ putting in (iv)

$(2c-8)^2-4(1)(c^2)=0$

$4c^2-32c+64-4c^2=0$

$32c=64$

c=2

c. The line x+y+c=0, will touch the parabola $x^2=-8y$?

Solution: Given the line, $l=x+y+c=0 \rightarrow$ (i)

and parabola, $x^2=-8y \quad \rightarrow$ (ii)

From (i), $y=-x-c \rightarrow$ (iii)

Putting in (ii)

$x^2=-8(-x-c)$

$x^2=8x+8c$

$x^2-8x-8c=0$

Which is quadratic in x. For the line to touch the parabola

$b^2-4ac=0 \rightarrow$(iv)

$a=1, b=-8, c=-8c$

Putting in (iv)

$(-8)^2-4(1)(-8c)=0$

$64+32c=0$

$32c=-64$

$c=\frac{-64}{32}=-2$

c = –2

d. The line x–y+c=0, will touch the parabola $x^2=\frac{2}{3}y$?

Solution: Given the line, $l=x-y+c=0 \rightarrow$ (i)

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and parabola, $x^2=\frac{2}{3}y \quad \rightarrow$ (ii)

From (i), $y=x+c \rightarrow$ (iii)

Putting in (ii)

$$x^2=\frac{2}{3}(x+c)$$

$$3x^2=2x+2c$$

$$3x^2-2x-2c=0$$

Which is quadratic in x, for the line to touch the parabola

$$b^2-4ac=0 \rightarrow \text{(iv)}$$

$$a=3,\ b=-2,\ c=-2c$$

Putting in (i)

$$(-2)^2-4(3)(-2c)=0$$

$$4+24c=0$$

$$24c=-4$$

$$c=\frac{-4}{24}$$

$$\mathbf{c=\frac{-1}{6}}$$

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Q.8 In each case, find the tangent equation and normal equation

a. At a point p(1,2) to parabola, $y^2=4x$

Solution: The equation of tangent to parabola

$y^2=4px$ at a point $p(x_1,y_1)$ is given by

$$\mathbf{yy_1=2p(x+x_1)} \rightarrow \text{(i)}$$

Here $x_1=1,\ y_1=2,\ 4p=4 \quad \Rightarrow \quad p=1$

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Putting these value in (i), equation of tangent becomes

$y(2)=2(1)(x+1)$

$2y=2x+2$ divide by 2

$\mathbf{y=x+1}$

$x-y+1=0$

The equation of normal is

$$\mathbf{y-y_1=\frac{-y_1}{2p}(x-x_1)} \quad \rightarrow (ii)$$

Putting x_1, y_1 and p in (ii), the equation of normal becomes

$$y-2=\frac{-2}{2(1)}(x-1)$$

$$y-2=-1(x-1)$$

$$y-2=-x+1$$

$$\mathbf{x+y-3=0}$$

b. At a point p(3,6) to parabola, $y^2=12x$

Solution: The equation of tangent to parabola

$y^2=4px$ at a point $p(x_1,y_1)$ is given by

$$\mathbf{yy_1=2p(x+x_1)} \rightarrow (i)$$

Here $x_1=3$, $y_1=6$, $4p=12 \Rightarrow p=3$

Putting these value in (i), equation of tangent becomes

$y(6)=2(3)(x+3)$

$6y=6x+18$ divide by 6

$y=x+3$

$\mathbf{x-y+3=0}$

The equation of normal is given by

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$$y-y_1=\frac{y_1}{2p}(x-x_1) \quad \rightarrow (ii)$$

Putting x_1, y_1, p the equation of normal becomes

$$y-6=\frac{-6}{2(3)}(x-3)$$

$$y-6=-1(x-3)$$

$$y-6=-x+3$$

$$\mathbf{x+y-9=0}$$

c. At a point $p\left(\frac{-3}{4},3\right)$ to parabola, $y^2=-12x$

Solution: The equation of tangent to parabola

$y^2=4px$ at a point $p(x_1,y_1)$ is given by

$$\mathbf{yy_1=2p(x+x_1)} \rightarrow (i)$$

Here $x_1=\frac{-3}{4}$, $y_1=3$, $4p=-12 \quad \Rightarrow \quad p=-3$

Putting these value in (i), equation of tangent becomes

$$y(3)=2(-3)(x+\left(\frac{-3}{4}\right)) \quad \text{divide by 3}$$

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$$y=-2\left(x-\frac{3}{4}\right) \text{ multiplying by 2}$$

$$2y=-4x+3$$

$$\mathbf{4x+2y-3=0}$$

The equation of normal is given by

$$y-y_1=\frac{-y_1}{2p}(x-x_1)$$

$$y-3=-\frac{3}{2(-3)}\left(x-\left(-3/4\right)\right)$$

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$$y-3=\frac{1}{2}\left(x+\frac{3}{4}\right)$$

$$y-3=\frac{1}{2}x+\frac{3}{8} \text{ multiplying by 8}$$

$$8y-24=4x+3$$

$$\mathbf{4x-8y+27=0}$$

d. At a point $p\left(\frac{1}{2},\frac{-1}{3}\right)$ to parabola, $x^2=\frac{-3y}{4}$

Solution: The equation of tangent to parabola $x^2=4py$

at a point $p(x_1,y_1)$ is given by

$$\mathbf{xx_1=2p(y+y_1)} \rightarrow \text{(i)}$$

Here $x_1=\frac{1}{2}$, $y_1=\frac{-1}{3}$, $4p=\frac{-3}{4}$ $\Rightarrow$ $p=\frac{-3}{16}$

Putting these value in (i), equation of tangent becomes

$$x\left(\frac{1}{2}\right)=2\left(\frac{-3}{16}\right)\left(y+\left(\frac{-1}{3}\right)\right)$$

$$\frac{x}{2}=\frac{-3}{8}\left(y-\frac{1}{3}\right)$$

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$$\frac{x}{2}=\frac{-3}{8}y+\frac{1}{8} \text{ Multiplying by 8}$$

$$4x=-3y+1$$

$$\mathbf{4x+3y-1=0}$$

The equation of normal is given by

$$y-y_1=\frac{-y_1}{2p}(x-x_1)$$

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$$y-\left(\frac{-1}{3}\right)=\frac{-\left(\frac{-1}{3}\right)}{2\left(\frac{-3}{16}\right)}\left(x-\frac{1}{2}\right)$$

$$y+\frac{1}{3}=\frac{-8}{9}x+\frac{4}{9} \qquad \text{multiplying by 9}$$

$$9y+3=-8x+4$$

$$\mathbf{8x+9y-1=0}$$

Q.9 Find the tangent equation

a. To the parabola $y^2=8x$ which is perpendicular to the $2y-x+1=0$. Find also the point of contact.

Solution: The equation of tangent to parabola

$y^2=4px$ in slope form is given by

$$\mathbf{y=mx+\frac{p}{m}} \rightarrow \text{(i)}$$

Here $4p=8 \Rightarrow p=2$

Slope of given line, $m_1=\frac{-a}{b}=\frac{-(-1)}{2}=\frac{1}{2}$

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Slope of required tangent, perpendicular to given line is

$$m=\frac{-1}{m_1}=\frac{-1}{\left(1/2\right)}=-2$$

Putting $m=-2$, and $p=2$ in (i), the equation of tangent becomes

$$y=(-2)x+\frac{2}{(-2)}$$

$$y=-2x-1$$

$$\mathbf{2x+y+1=0} \rightarrow \text{(ii)}$$

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For point of contact, put (ii) $(y=-2x-1)$ in equation of parabola;

$$(-2x-1)^2=8x$$

$$4x^2+4x+1=8x \quad \Rightarrow \quad 4x^2-4x+1=0$$

$$(2x-1)^2=0 \quad \Rightarrow \quad 2x-1=0$$

$x=\frac{1}{2}$ putting in (ii)

$$2\left(\frac{1}{2}\right)+y+1=0$$

$$y+1+1=0$$

$$\mathbf{y = -2}$$

The point of contact is $P(\frac{1}{2},-2)$

b. To the parabola $y^2=5x$ which is parallel to the line $x+4y+1=0$. Find also the point of contact.

Solution: The equation of tangent to parabola,

$y^2=4px$ in slope form is given by

$$\mathbf{y=mx+\frac{p}{m}} \rightarrow (i)$$

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Here $4p=5 \Rightarrow p=\frac{5}{4}$

and slope of given line, $x+4y+1=0$ is

$$m_1=\frac{-a}{b}=\frac{-1}{4}$$

The slope of required tangent is same as it is parallel to given line, so

$$\mathbf{m=\frac{-1}{4}}$$

Putting these values in (i) gives

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$$y=\frac{-1}{4}x+\frac{\left(5/4\right)}{\left(-1/4\right)}$$

$$\mathbf{y=\frac{-1}{4}x-5} \quad \rightarrow (ii)$$

For point of contact put (ii), is equation of parabola

$$\left(\frac{-1}{4}x-5\right)^2=5x$$

$$\frac{x^2}{16}+25+\frac{5}{2}x=5x$$

Multiplying by 16

$$x^2+400+40x=80x$$

$$x^2-40x+400=0$$

$$(x-20)^2=0$$

$$x=20$$

Putting in (ii)

$$y=-\frac{1}{4}(20)-5=-5-5=-10$$

The point of contact is thus P(20,–10)

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Q.10 Find the tangent equation

a. To the parabola $y^2=x$, which makes angle 45° with the x-axis.

Solution: The equation of tangent to parabola,

$y^2=4px$ in slope form is given by:

$$\mathbf{y=mx+\frac{p}{m}} \rightarrow (i)$$

Here $4p=1 \Rightarrow p=\frac{1}{4}$ and $m=\tan 45^\circ=1$

Putting these values in (i), the equation of tangent becomes,

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$$y = 1 \cdot x + \frac{\left(\frac{1}{4}\right)}{1}$$

$$y = x + \frac{1}{4}$$

Multiplying by 4

$4y=4x+1$

$4x-4y+1=0$

b. To parabola $y^2=x$, which makes angle 60° with the x-axis.

Solution: The equation of tangent to parabola,

$y^2=4px$ in slope form is given by:

$$\mathbf{y=mx+\frac{p}{m}} \rightarrow (i)$$

Here $4p=1 \Rightarrow p=\frac{1}{4}$ and $m=\tan 60^o=\sqrt{3}$

Putting these values in (i), the equation of tangent becomes,

$$y = \sqrt{3}x + \frac{\left(\frac{1}{4}\right)}{\sqrt{3}}$$

$$\mathbf{y = \sqrt{3}x + \frac{1}{4\sqrt{3}}}$$

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c. To parabola $y^2=x$, which makes an angle 135° with the x-axis.

Solution: The equation of tangent to parabola,

$y^2=4px$ in slope form is given by:

$$\mathbf{y=mx+\frac{p}{m}} \rightarrow (i)$$

Here $4p=1 \Rightarrow p=\frac{1}{4}$ and $m=\tan 135^o=-1$

Putting these values in (i), the equation of tangent becomes,

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$$y = -1x + \frac{\left(\frac{1}{4}\right)}{(-1)}$$

$$y = -x - \frac{1}{4}$$

Multiplying by 4

$$4y = -4x - 1$$

$$\mathbf{4x + 4y + 1 = 0}$$

Q.11 Find the tangent equation

a. To the parabola $x^2=y$, which makes an angle of 45° with the x-axis.

Solution: The equation of tangent to parabola,

$x^2=4py$ in slope form is given by:

$$y=mx-pm^2 \rightarrow (i)$$

Here $m=\tan 45^o=1$

$$4p=1 \Rightarrow \quad \mathbf{p=\frac{1}{4}}$$

Putting these values in (i), the equation of tangent becomes,

$$y = 1 \cdot x - \frac{1}{4}(1)^2$$

$$\mathbf{y = x - \frac{1}{4}}$$

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b. To the parabola $x^2=y$, which makes an angle of 60° with the x-axis.

Solution: The equation of tangent to parabola,

$x^2=4py$ in slope form is given by:

$$y=mx+pm^2 \rightarrow (i)$$

Here $4p=1 \Rightarrow \quad p=\frac{1}{4}$ and $m=\tan 60^o=\sqrt{3}$

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Putting these values in (i), the equation of tangent becomes,

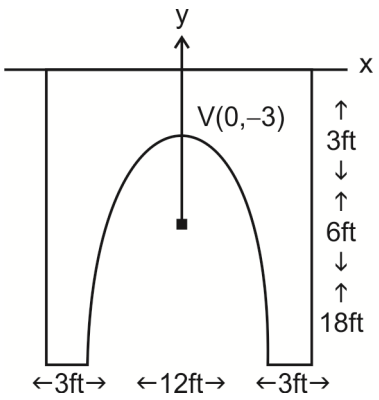
$$y = \sqrt{3}x - \frac{1}{4}(\sqrt{3})^2$$

$$y = \sqrt{3x} - \frac{3}{4}$$

$$\mathbf{4y = 4\sqrt{3}\,x - 3}$$

Q.12 Find the equation of the parabolic portion of the arc way, if parabolic arch way has the dimensions shown in the figure below.

Solution: Here the Venter is v(0,–3)



The length of focal chord, 4p=12

p=3

The parabola opens down wards, so the equation of parabola will be

$$(x-h)^2 = 4p(y-k) \rightarrow (i)$$

With p=3, h=0, k=–3

$$(x-0)^2 = -4(3)(y-(-3))$$

$$\mathbf{x^2 = -12(y+3)}$$

Book answer is wrong

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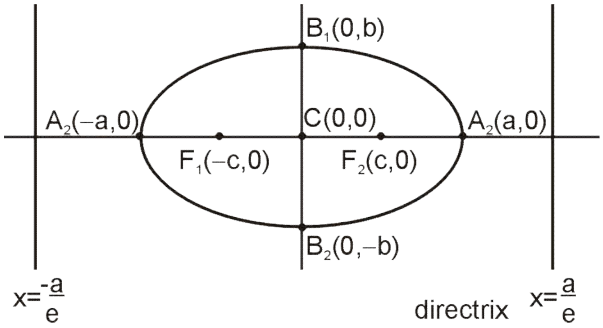
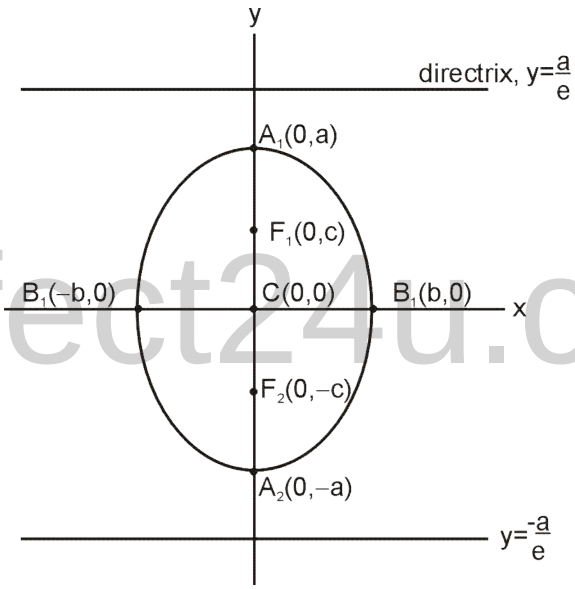
Ellipse:

An ellipse in standard form with major axis at x-axis is shown in figure;



The equation above ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $c^2=a^2-b^2$

Similarly in vertical form the ellipse is



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And the equation of ellipse in vertical form is, $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

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Orientation	Foci	Vertices	Semiminor axis
Horizontal	$F_1(-c,0)$, $F_2(c,0)$	$(-a,0)$, $(a,0)$	$(0,b)$, $(0,-b)$
Vertical	$F_1(0,-c)$, $F_2(0,c)$	$(0,-a)$, $(0,a)$	$(b,0)$, $(-b,0)$

Translation of ellipse:

The equation of ellipse in translated form is;

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Where h is translation along x-axis and k is translation along y-axis, for the translated ellipse.

Centre: $C(h,k)$, **Foci:** $F_1(-c+h,k)$, $(c+h,k)$

Vertices: $V_1(-a+h,k)$, $V_2(a+h,k)$

Ends of minor axis: $B_1(h,b+k)$, $B_2(h,-b+k)$

Equation of tangent and normal:

A line $y=mx+c$ is tangent to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c=\pm\sqrt{a^2m^2+b^2}$

The equation of tangent to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point $p(x_1,y_1)$ is given by:

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \text{ and}$$

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The equation of normal to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point $p(x_1,y_1)$ is given by:

$$\frac{y-y_1}{\left(\frac{y_1}{b^2}\right)}=\frac{x-x_1}{\left(\frac{x_1}{a^2}\right)}$$

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EXERCISE 8.2

Q.1 In each case, sketch the ellipse represented by the equation. Indicate the centre, foci, endpoints of the major axis and end points of the minor axis;

a) $\frac{x^2}{9}+\frac{y^2}{4}=1$

Solution: Comparing with standard equation of ellipse, i.e.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{(Horizontal axis)}$$

$a^2=9 \Rightarrow a=\pm 3$

$b^2=4 \Rightarrow b=\pm 2$

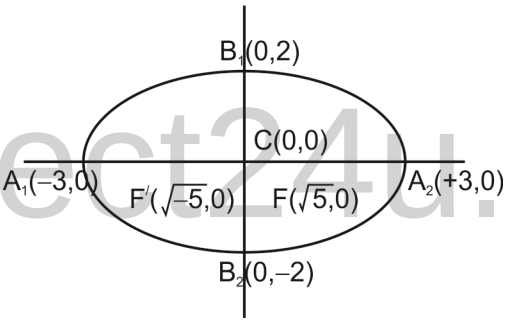
$c^2=a^2-b^2=9-4=5 \Rightarrow \mathbf{c=\pm\sqrt{5}}$

Centre: The centre is C(0,0) as the parabola is centered at origin

Foci: $F(\pm c,0) = F(\pm\sqrt{5},0)$

End points of major axis: $A(\pm a,0) = A(\pm 3,0)$

End points of minor axis: $B(0,\pm b) = B(0,\pm 2)$



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b) $\frac{x^2}{16}+\frac{y^2}{25}=1$

Solution: Comparing with standard equation of ellipse, i.e.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{(Vertical axis)}$$

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$b^2=16 \quad \Rightarrow \quad b=\pm 4$

$a^2=25 \quad \Rightarrow \quad a=\pm 5$

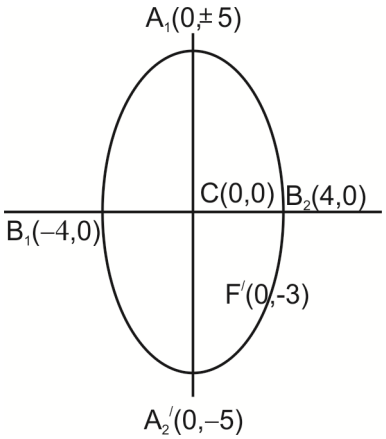
$c^2=a^2-b^2=25-16=9 \quad \Rightarrow \quad c=\pm 3$

Centre: The centre is C(0,0) as the parabola is centered at origin (no-shifting)

Foci: The foci arc F(0, ±c) = F(0, ±3)

End points of major axis: A(0, ±a) = A(0, ±5)

End points of minor axis: B(±b,0) = B(±4,0)



c) $\dfrac{(x+1)^2}{16}+\dfrac{(y-2)^2}{9}=1$

Solution: Comparing with general equation of ellipse: i.e.

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$a^2=16 \quad \Rightarrow \quad a=\pm 4$

$b^2=9 \quad \Rightarrow \quad b=\pm 3$

$c^2=a^2-b^2=16-9=7 \Rightarrow \quad c=\pm\sqrt{7}$, $h=-1$, $k=2$

Centre: The centre is C(h,k) = C(–1,2)

Foci: foci are F_1 (c+h, 0+k), F_2(–c+h, 0+k) i.e.,

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$F_1\left(\sqrt{7}-1,0+2\right)=F_1\left(\sqrt{7}-1,2\right)$

$F_2\left(-\sqrt{7}-1,0+2\right)=F_2\left(-\sqrt{7}-1,2\right)$

Ends of major axis: A(a+h,0+k), A′(–a+h, 0+k)

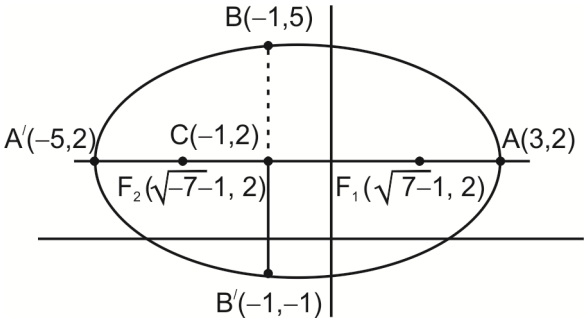
A(4+(–1), 0+2), A′(–4+(–1), 0+2)

A(3,2) A′(–5,2)

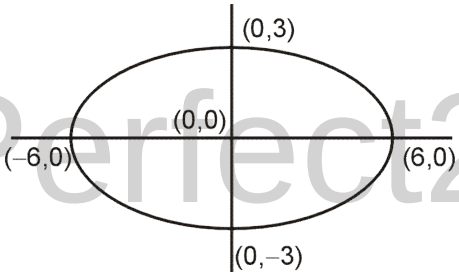
Ends of minor axis: B(0+h, b+k) , B′(0+h, –b+k)

B(0+(–1), 3+2) , B′(0+(–1), –3+2)

B(–1,5) B′(–1,–1)



Q.2 In each case, determine the equation of graphed ellipse



a)

Solution: The ellipse is centered at origin C(0,0) with end of major axis A(a,0)=A(6,0) and end of minor axis B(0,b)=B(0,3)

Thus, a=6, b=3

So, the equation of ellipse in horizontal direction

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \rightarrow$ (i) Putting values of a and b in (i)

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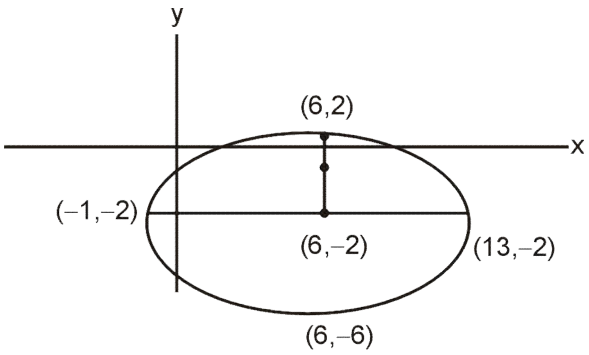
$$\frac{x^2}{(6)^2}+\frac{y^2}{(3)^2}=1$$

$$\frac{x^2}{36}+\frac{y^2}{9}=1$$



b)

Solution: From the given figure

C(h,k)=C(6,–2)

h=6, k=–2

End of major axis: A (h+a, k+0)=A(13,–2)

h+a=13 Putting h=6

6+a=13 ⇒ a=7

End of minor axis: B(0+h, b+k)=B(6,2)

b+k=2 putting k=–2

b–2=2

b=4

Thus, the equation of ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$ (Horizontal axis)

Putting values the equation of ellipse becomes

$$\frac{(x-6)^2}{(7)^2}+\frac{(y-(-2))^2}{(4)^2}=1$$

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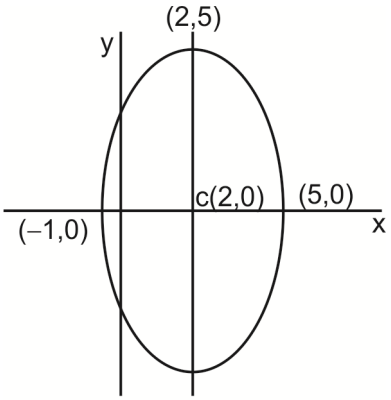
$$\frac{(x-6)^2}{49}+\frac{(y-(-2))^2}{16}=1 \quad \Rightarrow \quad \frac{(x-6)^2}{49}+\frac{(y+2)^2}{16}=1$$



c)

Solution: From the given figure

C(h,k)=C(2,0)

h=2, k=0

End of minor axis: B (h+b, k+0)=B(5,0)

h+b=5 Putting h=2

2+b=5 $\Rightarrow$ b=3

End of major axis: A(0+h, a+k)=A(2,5)

a+k=5 putting k=0

a+0=5

a=5

Thus, the equation of ellipse is

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$ (Vertical axis) Putting values

$$\frac{(x-2)^2}{(3)^2}+\frac{(y-0)^2}{(5)^2}=1$$

$$\frac{(x-2)^2}{9}+\frac{y^2}{25}=1$$

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Q.3 In each case, write the equation of ellipse through the given information

a) center is at c(–3,2), a=2, b=1, major axis is horizontal

Solution: Given centered C(–3,2)=C(h,k)

h=–3, k=2

a=2, b=1, Horizontal

The equation of ellipse will be

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \qquad \text{(Horizontal axis)}$$

Putting values

$$\frac{(x-(-3))^2}{(2)^2}+\frac{(y-2)^2}{(1)^2}=1$$

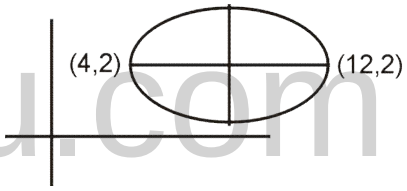
$$\frac{(x+3)^2}{4}+\frac{(y-2)^2}{1}=1$$

b) Vertices at A(4,2) and A′(12,2), b=2

Solution: As the centre is the midpoint of vertices A(4,2) and A′(12,2)

Then,

$$C(h,k)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{4+12}{2},\frac{2+2}{2}\right)$$



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$C(h,k)=(8,2)$, h=8, k=2

The length of major axis=2a=|AA′|

$$2a=\sqrt{(12-4)^2+(2-2)^2}$$

$$2a=\sqrt{64}$$

$$2a=8 \quad \Rightarrow \quad a=4$$

The equation of ellipse is given by $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

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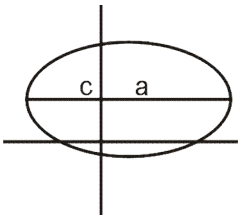
$$\frac{(x-8)^2}{(4)^2}+\frac{(y-2)^2}{(2)^2}=1$$

$$\frac{(x-8)^2}{16}+\frac{(y-2)^2}{4}=1$$

c) A focus is at $F_1(-2,3)$ a vertices is at A(6,3) length of minor axis is 6.

Solution: length of minor axis, 2b=6, b=3



$F_1(-2,3)=F_1(-c+h, 0+k)$

h–c=–2 and k=3

c–h=2 → (i)

Vertex A(6,3)=A(a+h, 0+k)

a+h=6 → (ii)

Adding (i) and (ii)

a+h=6

c–h=2

a+c=8 → (iii)

But $c=\sqrt{a^2-b^2}$, put b=3 and substitute in (iii)

$$a+\sqrt{a^2-3^2}=8$$

$$a+\sqrt{a^2-9}=8$$

$$\sqrt{a^2-9}=8-a$$

Taking square on both sides

$$\left(\sqrt{a^2-9}\right)^2=(8-a)^2$$

$$a^2-9=64-16a+a^2$$

16a=73

$a=\frac{73}{16}$ Putting in (ii)

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$$\frac{73}{16}+h=6 \quad \Rightarrow \quad h=6-\frac{73}{16}=\frac{96-73}{16}=\frac{23}{16}$$

Now the equation of ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{\left(x-\frac{23}{16}\right)^2}{\left(\frac{73}{16}\right)^2}+\frac{(y-3)^2}{(3)^2}=1$$

Note: The book answer is according to F(–3,3)

d) Vertices are at A (0,8) and A′(0,2), c= $\sqrt{5}$

Solution: As the vertices are along y-axis, the ellipse lies at y-axis

Now distance between vertices $2a=|AA'|$

$$2a=\sqrt{(0-0)^2+(8-2)^2}$$

$$2a=\sqrt{36}$$

$$2a=6 \quad \Rightarrow \quad a=3$$

as $b^2=a^2-c^2=(3)^2-\left(\sqrt{5}\right)^2=9-5=4$

$b=2$

The centre of ellipse is the midpoint of vertices A and A′. i.e.

$$C(h,k)=\left(\frac{0+0}{2},\frac{8+2}{2}\right)$$

$$C(h,k)=(0,5)$$

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The equation of ellipse is thus

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

$$\frac{(x-0)^2}{(2)^2}+\frac{(y-5)^2}{(3)^2}=1$$

$$\frac{x^2}{4}+\frac{(y-5)^2}{9}=1$$

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Q.4 The shape of an ellipse depends on the eccentricity of the ellipse $e=\frac{c}{a}$. Determine?

a) The eccentricity of ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$

Solution: Comparing the given ellipse with standard equation of ellipse i.e.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$a^2=25 \quad \Rightarrow \quad a=5$

$b^2=16 \quad \Rightarrow \quad b=4$

$c^2=a^2-b^2=25-16=9 \Rightarrow \quad c=3$

Now eccentricity, $e=\frac{c}{a}=\frac{3}{5}$

$$\mathbf{e=\frac{3}{5}}$$

b) The equation of the ellipse with vertices are A(–5,0) and A/(5,0)and the eccentricity $e=\frac{3}{5}$

Solution: The length of major axis is the distance between the vertices

Length of major axis$=2a=|AA'|=\sqrt{(5-(-5))^2+(0-0)^2}$

$2a=\sqrt{100}$

$2a=10 \quad \Rightarrow \quad a=5$

$c=ae=5\left(\frac{3}{5}\right)=3$

$b^2=a^2-c^2=(5)^2-(3)^2=25-9=16$

$b=4$

Thus the equation of ellipse is

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$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (Horizontal axis)

$\frac{x^2}{(5)^2}+\frac{y^2}{(4)^2}=1$

$\mathbf{\frac{x^2}{25}+\frac{y^2}{16}=1}$

c) The eccentricity of the ellipse if the length of the semi-major axis is a=4 and the length of semi minor axis is b=2

Solution: Given a=4, b=2

$c^2=a^2-b^2=(4)^2-(2)^2=16-4=12$

$c=\sqrt{12}=2\sqrt{3}$

Now eccentricity, $e=\frac{c}{a}=\frac{2\sqrt{3}}{4}$

$\mathbf{e=\frac{\sqrt{3}}{2}}$

Q.5 For what value of c

a) The line x–y+c=0 will touch the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

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Solution: Given the line, l=x–y+c=0 → (i)

and ellipse: $\frac{x^2}{4}+\frac{y^2}{1}=1$ → (ii)

from (i) y=x+c

Putting in (ii)

$\frac{x^2}{4}+\frac{(x+c)^2}{1}=1$

Multiplying by 4

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$x^2+4(x^2+c^2+2cx)=4$

$x^2+4x^2+8cx+4c^2-4=0$

$5x^2+8cx+4(c^2-1)=0$

Which is quadratic equation is x. For the line to touch the ellipse,

$b^2-4ac=0 \quad \rightarrow$ (ii)

$a=5,\ b=8c \quad c=4(c^2-1)$

Putting in (iii)

$(8c)^2-4(5)(4(c^2-1))=0$

$64c^2-80c^2+80=0$

$-16c^2=-80$

$c^2=5$

$\mathbf{c=\pm\sqrt{5}}$

Method 2: From the line, $y=x+c$

Comparing with $y=mx+c$

$m=1,\ c=c$

From the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

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Comparing with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$a^2=4,\ b^2=1$

If the line touches the ellipse then

$c=\pm\sqrt{a^2m^2+b^2}$

Putting values

$c=\pm\sqrt{(4)^2(1)^2+1}$

$\mathbf{c=\pm\sqrt{5}}$

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b) The line 2x–y+c=0 will touch the ellipse $\frac{x^2}{3}+\frac{y^2}{4}=1$

Solution: Given the line, l=2x–y+c=0 → (i)

and ellipse: $\frac{x^2}{3}+\frac{y^2}{4}=1$ → (ii)

from (i) y=2x+c

Comparing with, y=mx+c

m=2, c=c

Comparing (ii) with equation of ellipse

$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

$a^2=4, b^2=3$

If the line y=mx+c, to touch the ellipse

$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, then $c=\sqrt{a^2+b^2m^2}$

Putting values

$c=\sqrt{4+(2)^2(3)}=\sqrt{16}$

c=±4

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c) The line x+y+c=0, will touch the ellipse $\frac{x^2}{25}+\frac{y^2}{11}=1$

Solution: Given the line, l=x+y+c=0 → (i)

and ellipse: $\frac{x^2}{25}+\frac{y^2}{11}=1$ → (ii)

from (i) y=–x–c

Comparing with, y=mx+c

m=–1, c=–c

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Comparing (ii) with standard equation of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$a^2=25,\ b^2=11$

Now, If the line $y=mx+c$, touch the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\text{, then } c=\sqrt{a^2m^2+b^2}$$

Putting values

$$-c=\sqrt{(25)(-1)^2+11}$$

$$-c=\sqrt{36}$$

$$-c=\pm 6$$

$$\mathbf{c=\pm 6}$$

Q.6 In each case, find the tangent equation and normal equation

a) At a point P($\frac{2\sqrt{5}}{3}$,2) to ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

Solution: The equation of tangent to ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

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at a point $p(x_1,y_1)$ is given by;

$$\frac{x.x_1}{b^2}+\frac{y.y_1}{a^2}=1 \qquad \rightarrow \text{(i)}$$

and the equation of normal is given by

$$\frac{x-x_1}{\frac{x_1}{b^2}}=\frac{y-y_1}{\frac{y_1}{a^2}} \qquad \rightarrow \text{(ii)}$$

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Here $a^2=9$, $b^2=4$, $x_1=\frac{2\sqrt{5}}{3}$, $y_1=2$

So, the equation of tangent is, from (i)

$$\frac{x\left(\frac{2\sqrt{5}}{3}\right)}{4}+\frac{y.2}{9}=1$$

$$\frac{2\sqrt{5}x}{12_6}+\frac{2y}{9}=1$$

Multiplying by 18

$$3\sqrt{5}\,x+4y=18$$

$$3\sqrt{5}\,x+4y-18=0 \qquad \text{Equation of tangent}$$

and equation of normal is, from (ii)

$$\frac{x-\frac{2\sqrt{5}}{3}}{\frac{2\sqrt{5}}{4}}=\frac{y-2}{\frac{y2}{9}}$$

$$\frac{4x-\frac{8\sqrt{5}}{3}}{\frac{2\sqrt{5}}{3}}=\frac{9y-18}{2}$$

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$$4x-9y-\frac{8\sqrt{5}}{3}+18=0 \qquad \text{Equation of normal}$$

b) At a point P(0,2) to ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$

Solution: The equation of tangent at a point $P(x_1,y_1)$ to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by;

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$$\frac{x.x_1}{a^2}+\frac{y.y_1}{b^2}=1 \qquad \rightarrow (i)$$

And the equation of normal is given by:

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}} \qquad \rightarrow (ii)$$

Here $x_1=0$, $y_1=2$, $a^2=9$, $b^2=4$

So, the equation of tangent, from (i) is

$$\frac{x(0)}{9}+\frac{y(2)}{4}=0$$

$$0+\frac{y}{2}=0 \qquad \Rightarrow \qquad y=0$$

$y = 0$ Equation of tangent

and equation of normal according to (ii) is

$$\frac{x-0}{\left(\frac{0}{9}\right)}=\frac{y-2}{\left(\frac{2}{4}\right)}$$

$$\frac{x}{0}=\frac{y-2}{\frac{1}{2}}$$

$x = 0$ Equation of normal

c) At a point $P(\sqrt{3},\frac{1}{2})$ to the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

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Solution: The equation of tangent at a point $P(x_1,y_1)$ to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by;

$$\frac{x.x_1}{a^2}+\frac{y.y_1}{b^2}=1 \rightarrow (i)$$

and equation of normal is given by:

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$$

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Here $x_1=\sqrt{3}$, $y_1=\frac{1}{2}$, $a^2=4$, $b^2=1$

So, the equation of tangent according to (i) is

$$\frac{x.\sqrt{3}}{4}+\frac{y.\left(1/2\right)}{1}=1$$

$$\frac{x\sqrt{3}}{4}+\frac{y}{2}=1 \qquad \text{Multiplying by 4}$$

$$x\sqrt{3}+2y=4$$

$$\sqrt{3}x+2y-4=0$$

and equation of normal according to (ii) is

$$\frac{x-\sqrt{3}}{\frac{\sqrt{3}}{4}}=\frac{y-1/2}{\frac{1/2}{1}}$$

$$\frac{1}{2}\left(x-\sqrt{3}\right)=\frac{\sqrt{3}}{4}\left(y-\frac{1}{2}\right)$$

$$2(x)-2\sqrt{3}=\sqrt{3}y-\frac{\sqrt{3}}{2}$$

$$2x-2\sqrt{3}-\sqrt{3}y+\frac{\sqrt{3}}{2}=0$$

$$\mathbf{2x-\sqrt{3}y-2\sqrt{3}+\frac{\sqrt{3}}{2}=0}$$

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Q.7 Find the tangent equation

a) At a point (1,2) to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ which is perpendicular to the line $9x+8y-36=0$

Solution: The equation of tangent to ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ with slope m is given by $y=mx\pm\sqrt{a^2+b^2m^2}$

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Here slope of the given line, $m_1 = \frac{-a}{b} = \frac{-9}{8}$

Slope of required tangent line, perpendicular to given line in,

$$m = \frac{-1}{m_1} = \frac{8}{9}$$

Using point slope form

$$y - y_1 = m(m - x_1)$$

$$x_1 = 1,\ y_1 = 2, m = \frac{8}{9}$$

$$y - 2 = = \frac{8}{9}(x - 1) \qquad \Rightarrow 9y - 18 = 8x - 8$$

$8x - 9y + 10 = 0$

b) At a point (3,6) to the ellipse $\frac{x^2}{7} + \frac{y^2}{4} = 1$ which is parallel to the line $6x + 21y - 14 = 0$

Solution: Equation of given line

$$6x + 21y - 14 = 0$$

Slope of given line $m = \frac{-6}{21}$

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$m_1 = m_2$ for parallel line with slopes m_1 and m_2

Hence required equation of tangent

$$y - 6 = \frac{-6}{21}(x - 3)$$

$6x + 21y - 144 = 0$ Book answer is wrong

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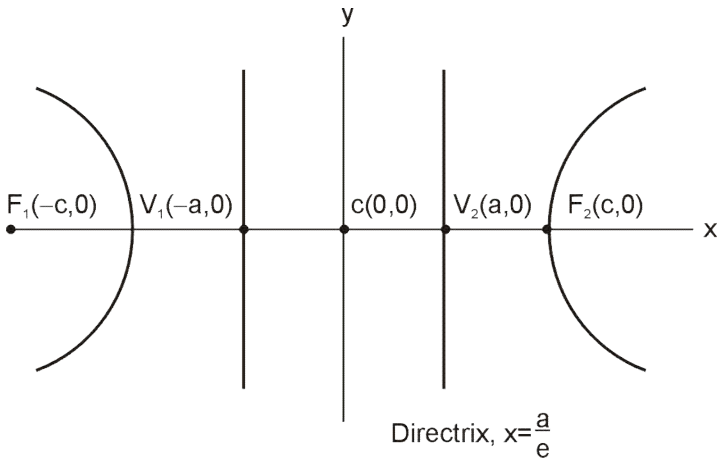
Hyperbola:

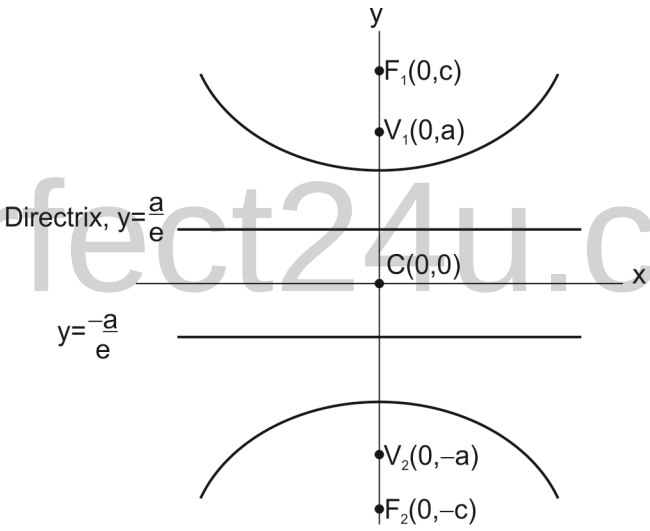
A hyperbola in standard form with transverse axis (Major axis) at x-axis is shown in figure;



The equation above hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ c^2=a^2+b^2$$

Similarly in their vertical form the hyperbola is



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And the equation of hyperbola in vertical form is, $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

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Orientation	Foci	Vertices
Horizontal	$F_1(c,0)$ $F_2(-c,0)$	$V_1(a,0)$ $V_2(a,0)$
Vertical	$F_1(0,c)$ $F_2(0,-c)$	$V_1(0,a)$ $V_2(0,-a)$

Translation of hyperbola:

The equation of hyperbola in translated form is;

$$\frac{(x-h)^2}{a^2}-\frac{y^2}{b^2}=1$$

Where h is translation along x-axis and k is translation along y-axis. For the translated hyperbola

Centre: C(h,k), Foci: $F_1(-c+h, k)$, $F_2(c+h,k)$

Vertices: $V_1(-a+h,k)$, $V_2(a+h, k)$

Equation of tangent and normal:

A line y=mx+c is tangent to hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ if } c=\sqrt{a^2m^2-b^2}$$

And equation of tangent is $y=mx\pm\sqrt{a^2m^2-b^2}$

The equation of tangent to hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is given by

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \text{ and}$$

The equation of normal to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point $p(x_1,y_1)$ is given by:

$$\frac{y-y_1}{\left(\frac{y_1}{b^2}\right)}=\frac{(x-x_1)}{\left(\frac{x_1}{a^2}\right)}=1$$

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EXERCISE 8.3

Q.1 In each case, sketch the hyperbola represent by the equation. Indicate the centre, vertices, foci and equations of the asymptotes.

a) $\frac{x^2}{4}-\frac{y^2}{9}=1$

Solution: Comparing with standard equation of hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$a^2=4,\ b^2=9,\ c^2=a^2+b^2=4+9=13 \Rightarrow a=2,\ b=3, c=\sqrt{13}$

Centre: The hyperbola is centered at origin, so C(0,0)

Vertices: As vertices are given by $V_1(-a,0)$ $V_2(a,0)$

Here $a^2=4 \Rightarrow a=2$

So, $V_1(-2,0)$ $V_2(2,0)$

Foci: The foci are given by, $F_1(-c,0)$, $F_2(c,0)$

Here $c^2=13 \Rightarrow c=\sqrt{13}$

So, $F_1\left(-\sqrt{13},0\right)$ $F_2\left(\sqrt{13},0\right)$

Equation of asymptotes: Equations of asymptotes are given by,

$$y=\pm\frac{b}{a}x$$

Here $a=2,\ b=3$

So, $y=\pm\frac{3}{2}x$

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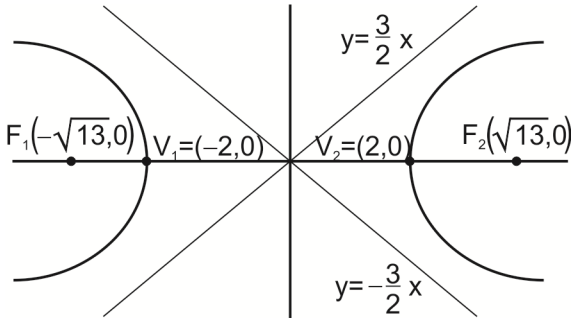

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b) $\frac{y^2}{25}-\frac{x^2}{4}=1$

Solution: Comparing with standard equation of hyperbola:

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1 \text{ (Vertical axis)}$$

$a^2=25 \quad \Rightarrow \quad a=\pm 5$

$b^2=4 \quad \Rightarrow \quad b=\pm 2$

$c^2=a^2+b^2=25+4=29 \quad \Rightarrow \quad c=\pm\sqrt{29}$

Now,

Centre: The hyperbola is centered at origin, so C(0,0)

Vertices: As vertices are given by $V_1(0,-a)$ $V_2(0,a)$

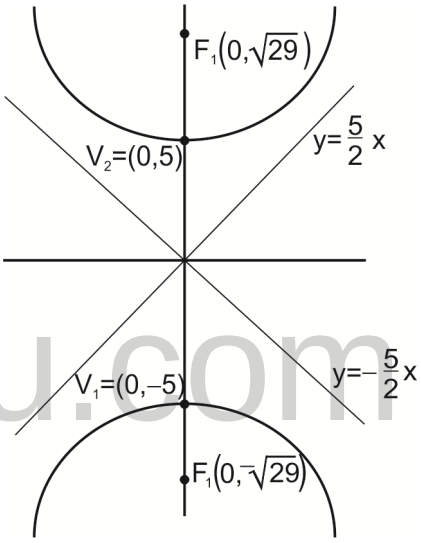
Here $a=5$

So, $V_1(0,-5)$ $V_2(0,5)$

Foci: The foci are given by, $F_1(0,-c)$, $F_2(0,c)$

Here $c=\sqrt{29}$

So, $F_1\left(0,-\sqrt{29}\right)$ $F_2\left(0,\sqrt{29}\right)$



Equation of asymptotes: The equations of asymptotes are given by,

$$y=\pm\frac{a}{b}x$$

Here $a=5$, $b=2$

So, $y=\pm\frac{5}{2}x$ are equations of asymptotes

c) $\frac{(x-2)^2}{9}-\frac{(y-3)^2}{16}=1$

Solution: Comparing with standard equation of hyperbola,

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

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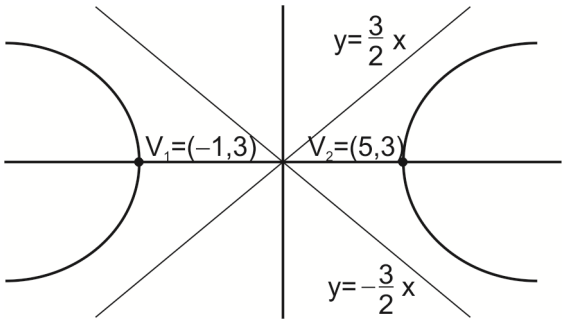
$h=2,\ k=3,\ a^2=9 \Rightarrow a=3$

$b^2=16 \Rightarrow b=4$

$c^2=a^2+b^2=19+16=25 \Rightarrow c=\pm5$



Now,

Centre: As h=2, and k=3, so centre C(h,k) = C(2,3)

Vertices: vertices are given by $V_1(-a,+h,k)$ $V_2(a+h,k)$

Here a=3, h=2, k=3

So, $V_1(-3+2,3) = V_1(-1,3)$, $V_2(3+2,3) = V_2(5,3)$,

Foci: The foci is given by, $F_1(-c+h,k)$, $F_2(c+h,k)$

Putting c, h and k

So, $F_1(-5+2,3)=F_1(-3,3)$, $F_2(5+2,3)=F_2(7,3)$

Equation of asymptotes: The equations of asymptotes are given by,

$$y-k=\pm\frac{b}{a}(x-h)$$

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Putting a, b, h and k

$$y-3=\pm\frac{4}{3}(x-2)$$

(i) $y-3=+\frac{4}{3}(x-2) \Rightarrow 3y-9=4x-8 \Rightarrow 4x-3y+1=0$

(ii) $y-3=-\frac{4}{3}(x-2) \Rightarrow 3y-9=-4x+8 \Rightarrow 4x+3y-17=0$

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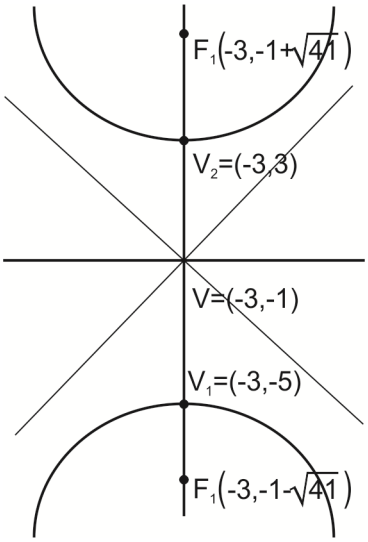
d) $\dfrac{(y+1)^2}{16}-\dfrac{(x+3)^2}{25}=1$



Solution: Comparing with, $\dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1$

$k=-1,\ h=-3,\ a^2=16 \quad \Rightarrow a=4$

$b^2=25 \quad \Rightarrow b=5$

$c^2=a^2+b^2=16+25=41 \quad \Rightarrow c=\sqrt{41}$

Now,

Centre: centre is given by C(h,k), so C(–3,–1)

Vertices: */ vertices are given by $V_1(h,k-a)$ $V_2(h,k+a)$*

Putting values of a, h and k

$V_1(-3,-1-4) = V_1(-3,-5), V_2(-3,-1+4) = V_2(-3,3)$

Foci: The foci is given by, $F_1(h,k-c)$, $F_2(h,k+c)$

Putting values of h, k and c

So, $F_1(-3,-1-\sqrt{41}), F_2(-3,-1+\sqrt{41})$

Equation of asymptotes: The equations of asymptotes are given by,

$$y-k=\pm\frac{a}{b}(x-h)$$

$$y-(-1)=\pm\frac{4}{5}(x-(-3))$$

$$y+1=\pm\frac{4}{5}(x+3)$$

(i) $y+1=\dfrac{4}{5}(x+3) \quad \Rightarrow 5y+5=4x+12 \quad \Rightarrow 4x-5y+7=0$

(ii) $y+1=-\dfrac{4}{5}(x+3) \quad \Rightarrow 5y+5=-4x-12 \quad \Rightarrow 4x+5y+17=0$

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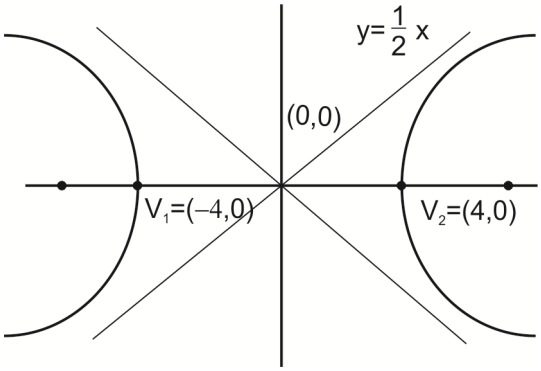
Q.2 In each case, determine the equation of the graphed hyperbola?

a)



Solution: From the given figure the hyperbola has its transverse axis along the x-axis

Centre: $C(h,k) = C(0,0)$

$h=0,\ k=0$

Vertex: $V(a,0) = V(4,0), \quad a=4$

Equation of asymptote, $y=\frac{b}{a}x=\frac{1}{2}x$

$\frac{b}{a}=\frac{1}{2} \quad \Rightarrow \quad b=\frac{1}{2}a$ putting $a=4$

$b=\frac{1}{2}(4) \quad \Rightarrow b=2$

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Putting these values in standard equation of hyperbola,

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-0)^2}{(4)^2}-\frac{(y-0)^2}{(2)^2}=1$$

$$\mathbf{\frac{x^2}{16}-\frac{y^2}{4}=1}$$

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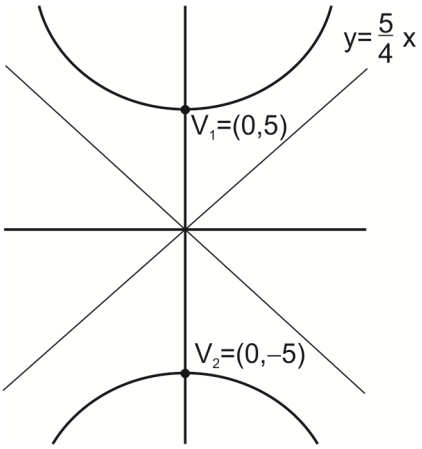

b)

Solution: From the given figure the hyperbola has its transverse axis along the y-axis

Centre: C(h,k)= C(0,0)

h=0, k=0

Vertex: V (0,a) = V(0,5), a=5

Equation of asymptote, $y=\frac{a}{b}x=\frac{5}{4}x$

$\frac{a}{b}=\frac{5}{4} \quad \Rightarrow \quad \frac{b}{a}=\frac{4}{5}$

$b=\frac{4}{5}a$ putting a=5

$b=\frac{4}{5}(5) \quad \Rightarrow \quad b=4$

Putting these values in standard equation of hyperbola,

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \quad \Rightarrow \quad \frac{(y-0)^2}{(5)^2}-\frac{(x-0)^2}{(4)^2}=1$$

$$\mathbf{\frac{y^2}{25}-\frac{x^2}{16}=1}$$

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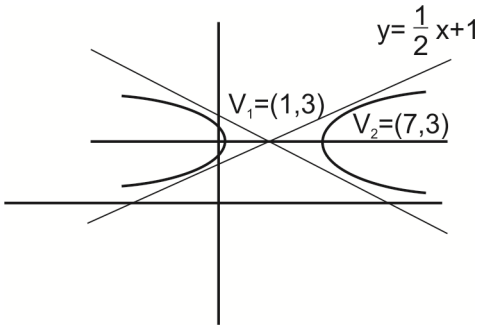

c)

Solution: From the given figure the hyperbola has its transverse axis along y=3, parallel to x-axis

Centre: C(h,k) is the midpoint of vertices, V_1 (1,3) and V_2(7,3)

$$C(h,k)=\left(\frac{1+7}{2},\frac{3+3}{2}\right)=(4,3)$$

h=4, k=3

Vertex: $V_1(-a+h, k) = V_1(1,3)$

$-a+h=1$ putting h=4

$-a+4=1$ a=3

Equation of asymptote, $y-k=\frac{b}{a}(x-h)$

$$y-3=\frac{b}{3}(x-4)$$

$$y-3=\frac{b}{3}x-\frac{4b}{3}$$

$$y=\frac{b}{3}x+3-\frac{4b}{3}$$ comparing with

$$y=\frac{1}{2}x+1$$

$$3-\frac{4b}{3}=1 \quad \Rightarrow \quad \frac{4b}{3}=2 \quad \Rightarrow \quad b=\frac{6}{4}$$

$$b=\frac{3}{2}$$

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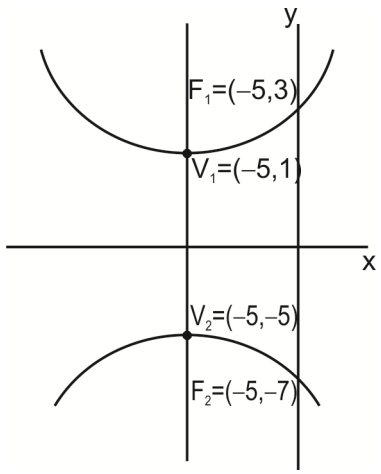
Putting these values in standard equation of hyperbola

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-4)^2}{(3)^2}-\frac{(y-3)^2}{\left(\frac{3}{2}\right)^2}=1$$

$$\frac{(x-4)^2}{9}-\frac{4(y-3)^2}{9}=1$$



d)

Solution: From the given figure the hyperbola has its transverse axis along x=–5, parallel to y-axis

Centre: C(h,k) is the midpoint of vertices, V_1 (–5,1) and V_2(–5,–5)

$$C(h,k)=\left(\frac{-5+(-5)}{2},\frac{1+(-5)}{2}\right)=c(-5,-2)$$

h=–5, k=–2

Vertex: $V_1(h, a+k) = V_1(-5,1)$

a+k=1 putting k=–2

a–2=1 ⇒ a=3

Focus: $F_1(h, c+k) = F(-5,3)$

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$c+k=3$ putting $k=-2$

$c-2=3 \Rightarrow c=5$

$b^2=c^2-a^2=25-9=16 \Rightarrow b=4$

Putting these values in standard equation of hyperbola,

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

$$\frac{(y-(-2))^2}{(3)^2}-\frac{(x-(-5))^2}{(4)^2}=1$$

$$\frac{(y+2)^2}{9}-\frac{(x+5)^2}{16}=1$$

Q.3 In each case, write the equation of hyperbola through the given information?

a) Foci are at $F_1(0,3)$, $F_2(0,-3)$, one vertex is at $V(0,-2)$

Solution: As foci are symmetric about the origin the hyperbola has its centre at origin, and transverse axis along y-axis.

Centre: $C(h,k) = C(0,0)$

$h=0, k=0$

Focus: $F_1(h, c+k) = F(0,3)$

$c+k=3$ putting $k=0$

$c=3$

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Vertex: $V(h, -a+k) = V(0,-2)$

$-a+k=-2$ putting $k=0$

$-a+0=-2 \Rightarrow a=2$

$b^2=c^2-a^2=(3)^2-(2)^2=5$

Putting these values is standard equation of hyperbola,

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

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$$\frac{(y-0)^2}{(2)^2}-\frac{(x-0)^2}{5}=1$$

$$\frac{y^2}{4}-\frac{x^2}{5}=1$$

b) Foci at $F_1(\sqrt{2},0)$, $F_2(-\sqrt{2},0)$, one vertex is at V(1,0)

Solution: As foci are symmetric about the origin and lies at x-axis, the hyperbola has its centre at origin and transverse axis at x-axis.

Centre: $C(h,k) = C(0,0)$

$h=0,\ k=0$

Focus: $F_1(h+c,\ k) = F(\sqrt{2},0)$

$h+c=\sqrt{2}$ putting $h=0$

$c+0=\sqrt{2}$

Vertex: $V(h+a,\ k) = V(1,0)$

$h+a=1$ putting $h=0$

$a+0=1 \Rightarrow a=1$

$b^2=c^2-a^2=\left(\sqrt{2}\right)^2-(1)^2=1$

$b=1$

Putting these values is standard equation of hyperbola,

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-0)^2}{(1)^2}-\frac{(y-0)^2}{(1)^2}=1$$

$$\frac{x^2}{1}-\frac{y^2}{1}=1$$

$x^2-y^2=1$

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c) Vertices at $V_1(5,0)$ and $V_2(-5,0)$, one focus is v(–7,0)

Solution: As vertices are symmetric about the origin and lies on x-axis, so the hyperbola has its centre at origin one having transverse axis as x-axis thus.

Centre: $C(h,k) = C(0,0)$

$h=0,\ k=0$

Vertex: $V(a+h, k) = V(5,0)$

$a+h=5$ putting $h=0$

$a+0=5 \Rightarrow a=5$

Focus: $F_2(c+h, k) = F(-7,0)$

$-c+h=-7$ putting $h=0$

$-c+0=-7 \Rightarrow c=7$

Now $b^2=c^2-a^2=(7)^2-(5)^2=24$

$b=\sqrt{24}=2\sqrt{6}$

Putting these values in standard equation of hyperbola,

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-0)^2}{(5)^2}-\frac{(y-0)^2}{(2\sqrt{6})^2}=1$$

$$\frac{x^2}{25}-\frac{y^2}{24}=1$$

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d) Vertices at $V_1(3,0)$ and $V_2(-3,0)$, and asymptotes are the lines y=3x and y=–3x

Solution: As vertices are symmetric about the origin and lies at x-axis, so the hyperbola has its centre at origin and axis as its transverse axis.

Centre: $C(h,k) = C(0,0)$

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$h=0, k=0$

Vertex: $V_1 (a+h, k) = V(3,0)$

$a+h=3$ putting $h=0$

$a=3$

Equation of asymptotes, $y=\pm\frac{b}{a}x$, comparing with given $y=\pm 3x$

$\frac{b}{a}=3$ putting $a=3$

$\frac{b}{3}=3 \Rightarrow \quad b=9$

Putting these values is standard equation of hyperbola,

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-0)^2}{(3)^2}-\frac{(y-0)^2}{(9)^2}=1$$

$$\mathbf{\frac{x^2}{9}-\frac{y^2}{81}=1}$$

e) Transverse axis is the x-axis, asymptotes are line y=3x and y=–3x

Solution: As the equation of asymptotes with transverse axis at x-axis is given by, $y=\pm\frac{b}{a}x$

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Comparing with given asymptotes, $y=\pm 3x$

$\frac{b}{a}=\frac{3}{1}$

$b=3, a=1$

Putting these values is standard equation of hyperbola,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

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$$\frac{x^2}{(1)^2}-\frac{y^2}{(3)^2}=1$$

$$\frac{x^2}{1}-\frac{y^2}{9}=1$$

$$\mathbf{9x^2-y^2=9}$$

f) Transverse axis is the y-axis, asymptotes are line y=3x and y=–3x

Solution: As the equation of asymptotes with transverse axis at y-axis is given by, $y=\pm\frac{a}{b}x$

Comparing with given asymptotes, $y=\pm 3x$

$$\frac{a}{b}=\frac{3}{1} \qquad \Rightarrow \qquad a=3,\ b=1$$

Putting these values is standard equation of hyperbola,

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

$$\frac{y^2}{(3)^2}-\frac{x^2}{(1)^2}=1$$

$$\mathbf{\frac{y^2}{9}-\frac{x^2}{1}=1}$$

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g) Foci at $F_1(5,0)$ and $F_2(-5,0)$, eccentricity $e=\frac{5}{3}$

Solution: As the Foci are symmetric about origin and lie at x-axis, the hyperbola has its centre at origin and transverse axis at x-axis.

$C(h, k) = C(0, 0)$

$h = 0, k = 0$

Focus, $F_1\ (c+h, k) = f(5,0)$

$c+h=5$ putting $h=0$

$c=5$

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Also eccentricity $e=\frac{c}{a}=\frac{5}{3}$ putting c=5

$\frac{5}{a}=\frac{5}{3}$ $\Rightarrow \frac{a}{5}=\frac{3}{5}$ $\Rightarrow a = 3$

Now, $b^2=c^2-a^2=(5)^2-(3)^2=16$ $\Rightarrow$ $b=4$

Putting these values in standard equation of hyperbola,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\frac{x^2}{(3)^2}-\frac{y^2}{(4)^2}=1$$

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

×ing both sides by 144

$$\frac{144x^2}{9}-\frac{144y^2}{16}=144$$

$16x^2 - 9y^2 = 144$

h) Vertices at $V_1(9,0)$ and $V_2(-9,0)$, whose asymptotes are perpendicular to each others

Solution: As the vertices are symmetric about origin and lie at x-axis, the centre of hyperbola is origin and it transverse axis as x-axis.

$C(h, k) = C(0, 0)$

$h = 0, k = 0$

Vertex, $V(a+h, k) = V(a,0)$

$a+h=9$ putting $h=0$

$a=9$

The equations of asymptotes are, $y=\pm\frac{b}{a}x$

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The slopes of asymptotes are $m_1=\frac{b}{a}$ and $m_2=-\frac{b}{a}$

As asymptotes are perpendicular to each other,

$$\left(\frac{b}{a}\right)\left(-\frac{b}{a}\right)=-1$$

$$-\frac{b^2}{a^2}=-1 \quad \Rightarrow \quad b^2=a^2$$

$b=a=9$

Putting these values in standard equation of hyperbola,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\frac{x^2}{(9)^2}-\frac{y^2}{(9)^2}=1$$

$$\mathbf{\frac{x^2}{81}-\frac{y^2}{81}=1}$$

Q.4 Determine the path of a point that moves so that the difference of its distances from

a) The points $P_1(-5,0)$ and $P_2(5,0)$ is 8

Solution: Let the given point A(–5, 0) and B(5,0) and P(x,y) to the point whose path is to be determined

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According to given condition

$|PA| - |PA| = 8$

$|PA| = 8 + |PA|$

$$\sqrt{(x+5)^2+(y-0)^2}=8+\sqrt{(x-5)^2+(y-0)^2}$$

Squaring on both sides

$$x^2 + 10x + 25 + y^2 = 64 + ((x-5)^2+y^2)+2\left(8\sqrt{(x-5)^2+y^2}\right)$$

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$$x^2 + 10x + 25 + y^2 = 64 + x^2 - 10x + 25 + y^2 + 216\sqrt{(x-5)^2 + y^2}$$

$$20x - 64 = +16\sqrt{x^2 + 25 - 10x + y^2}$$

$$5x - 16 = 4\sqrt{x^2 + 25 - 10x + y^2}$$

Squaring on both sides

$$(5x - 16)^2 = (4)^2\left(\sqrt{x^2 + 25 - 10x + y^2}\right)^2$$

$$25x^2 + 256 - 160x + y^2 = 16(x^2 + 25 - 10x + y^2)$$

$$25x^2 + y^2 - 160x + 256 = 16x^2 + 400 - 160x + 16y^2$$

$$9x^2 - 16y^2 = 144$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

b) The points $F_1(0,-13)$ and $F_2(0,13)$ is 10

Solution: Let the given point A(0, –13) and B(0, 13) and P(x,y) to the point whose path is to be determined

According to given condition

$$|PA| - |PB| = 10$$

$$|PA| = 10 + |PB|$$

$$\sqrt{(x-0)^2 + (y+13)^2} = 10 + \sqrt{(x-0)^2 + (y-13)^2}$$

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Squaring on both sides

$$\left((x-0)^2 + (y+13)^2\right) = (10)^2 + \left(\sqrt{x^2 + (y-13)^2}\right)^2 + 2(10)\left(\sqrt{x^2 + (y-13)^2}\right)$$

$$x^2 + (y^2 + 26y + 169) = 100 + (x^2 + (y - 13)^2 - 20\left(\sqrt{x^2 + y^2 + 169 - 26y}\right)$$

$$x^2 + y^2 + 169 + 26y = 100 + x^2 + y^2 + 169 - 26y + 20\sqrt{x^2 + y^2 + 169 - 26y}$$

$$26y + 26y - 100 = +20\sqrt{x^2 + y^2 + 169 - 26y}$$

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$52y - 100 = +20\sqrt{x^2 + y^2 + 169 - 26y}$

$13y - 25 = +20\sqrt{x^2 + y^2 + 169 - 26y}$

Again squaring on both sides

$(13y - 25)^2 = (20)^2\left(\sqrt{x^2 + y^2 + 169 - 26y}\right)^2$

$169y^2 + 625 - 650y = 25(x^2 + y^2 + 169 - 26y)$

$169y^2 + 625 - 650y = 25x^2 + 25y^2 + 4225 - 650y$

$144y^2 - 25x^2 = 3600$

$\frac{144y^2}{3600} - \frac{25x^2}{3600} = 1$

$\mathbf{\frac{y^2}{25} - \frac{x^2}{144} = 1}$

Q.5 Write the equation of hyperbola

a) With vertices at $V_1(2, -2)$, $V_2(-4,-2)$ and that passes through the point with coordinates (5,1)

Solution: The centre of the hyperbola is the midpoint of vertices

$C(h, k) = \left(\frac{2+(-4)}{2}, \frac{-2+(-2)}{2}\right) = (-1,-2)$

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The length of transverse axis, $2a = |V_1V_2|$

$2a = \sqrt{(2-(-4))^2 + (2-(-2))^2} = \sqrt{36+0} = 6$

$a = 3$

Let the equation of hyperbola is

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Putting h, k and a

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$$\frac{(x-(-1))^2}{(3)^2}-\frac{(y-(-2))^2}{b^2}=1$$

$$\frac{(x+1)^2}{9}-\frac{(y+2)^2}{b^2}=1 \qquad \rightarrow \text{(i) Putting P (5,1)}$$

$$\frac{(5+1)^2}{9}-\frac{(1+2)^2}{b^2}=1 \qquad \Rightarrow \frac{36}{9}-\frac{9}{b^2}=1 \qquad \Rightarrow 4-\frac{9}{b^2}=1$$

$$\frac{9}{b^2}=4-1 \quad \Rightarrow \quad \frac{9}{b^2}=3$$

$$\frac{b^2}{9}=\frac{1}{3} \quad \Rightarrow \quad b^2=3$$

Putting in (i)

$$\frac{(x+1)^2}{9}-\frac{(y+2)^2}{3}=1$$

b) With vertices at $V_1(-3, 1)$, $V_2(-3, 3)$ and that passes through the point with coordinates (0,4)

Solution: The centre of the hyperbola is the midpoint of vertices

$$C(h, k)=\left(\frac{-3+(-3)}{2},\frac{1+3}{2}\right)=(-3,2)$$

$h=-3, k=2$

The length of transverse axis, $2a=|V_1V_2|$

$$2a=\sqrt{(-3-(-3))^2+(3-1)^2}=\sqrt{0+4}=2$$

$a=1$

Let the equation of hyperbola is given by

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \qquad \text{(Vertical)}$$

Putting h, k and a

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$$\frac{(y-2)^2}{(1)^2}-\frac{(x+3)^2}{b^2}=1 \qquad \rightarrow \text{(i) Putting P (0,4)}$$

$$\frac{(4-2)^2}{1}-\frac{(0+3)^2}{b^2}=1$$

$$4-\frac{9}{b^2}=1 \quad \Rightarrow \quad \frac{9}{b^2}=4-1$$

$$\frac{b^2}{9}=\frac{1}{3} \quad \Rightarrow \quad b^2=\frac{9}{3}=3$$

Putting in (i)

$$\frac{(y-2)^2}{1}-\frac{(x+3)^2}{3}=1$$

Q.6 In each case, sketch the rectangular hyperbola and identity the vertices, the foci and the asymptotes

a) $(x+1)^2-(y-2)^2=1$

Solution: Comparing with standard equation of hyperbola,

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$h=-1, k=2, a^2=1 \Rightarrow a=1$

$b^2=1 \Rightarrow b=1$

$c^2=a^2+b^2=1+1=2 \Rightarrow c=\sqrt{2}$

Vertices: Vertices are $V_1(-a+h, k), V_2(a+h, k)$

$V_1(-1+(-1), 2) = V(-2,2)$

$V_2(1+(-1)), 2) = V(0,2)$

Foci: The foci are $F_1(-c+h, k), F_2(c+h, k)$

$F_1(-\sqrt{2}-1, 2)$, $F_2(\sqrt{2}-1, 2)$

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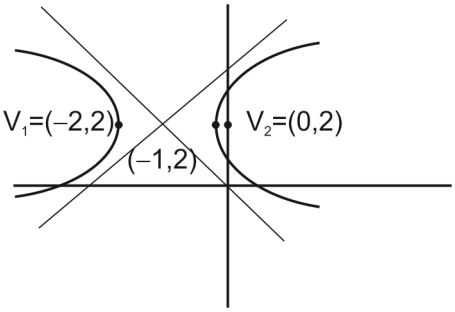
The equations of asymptotes

$$y-k=\pm\frac{b}{a}(x-h)$$

$$y-2=\pm\frac{1}{1}(x-(-1)) \Rightarrow y-2=\pm1(x+1)$$

(i) $y-2=x+1 \Rightarrow x-y+3=0$

(ii) $y-2=-x-1 \Rightarrow x+y-1=0$



b) $\frac{(x-3)^2}{4}-\frac{(y+1)^2}{4}=1$

Solution: Comparing with standard equation of hyperbola,

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$h=3, k=-1, \ a^2=4 \Rightarrow a=2$

$b^2=4 \Rightarrow b=2$

$c^2=a^2+b^2=4+4=8 \Rightarrow c=\sqrt{8}=2\sqrt{2}$

Vertices: Vertices are $V_1(-a+h, k), V_2(a+h, k)$

$V_1(-2+3, -1) = V_1(1,-1)$

$V_2(2+3, -1) = V_2(5,-1)$

Foci: Foci are $F_1(-c+h, k), F_2(c+h, k)$

$F_1(-\sqrt{2}+3, -1), F_2(\sqrt{2}+3, -1)$

The equations of asymptotes

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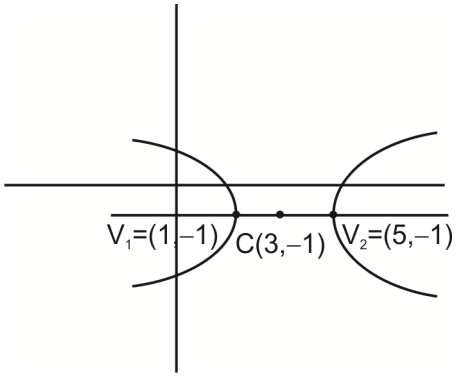
$$y-k=\pm\frac{b}{a}(x-h)$$

$$y-(-1)=\pm\frac{2}{2}(x-3) \Rightarrow y+1=\pm1(x-3)$$

(i) $y+1=x-3 \Rightarrow x-y-4=0$

(ii) $y+1=-x+3 \Rightarrow x+y-2=0$



Q.7 In each case, find the point of intersection in between the line and hyperbola;

a) xy =4, y=x–3

Solution: Given hyperbola, xy=4 →(i)

and line l: y=x–3 →(ii)

Putting (ii) in (i)

$x(x-3)=4$

$x^2-3x-4=0$

$x^2-4x+x-4=0$

$(x-4)(x+1)=0$

$x=4, x=-1$

When x=4, putting in (ii), y=x–3=4–3=1

When x=–1, putting in (ii), y=–1–3=–4

So, the points of intersection are

$P_1(4,1), (-1,-4)$

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b) $x^2 - y^2 = 1 \rightarrow$ **(i)**

$y = \sqrt{3}(x-1) \rightarrow$ **(ii)**

Solution: Putting (ii) in (i)

$$x^2 - \left(\sqrt{3}(x-1)\right)^2 = 1$$

$$x^2 - 3(x^2 - 2x + 1) = 1$$

$$x^2 - 3x^2 + 6x - 3 = 1$$

$$2x^2 - 6x + 4 = 0$$

Divide by 2

$$x^2 - 3x + 2 = 0$$

$$x^2 - 2x - x + 2 = 0$$

$$(x-2)(x-1) = 0$$

$x = 1$ and $x = 2$

When $x = 1$, putting in (ii), $y = \sqrt{3}(1-1) = 0$

When $x = 2$, putting in (ii), $y = \sqrt{3}(2-1) = \sqrt{3}$

So, the points of intersection are

$P_1(1,0)$, $P_2(2, \sqrt{3})$

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c) $\frac{y^2}{1} - \frac{x^2}{4} = 1 \rightarrow$ **(i)**

$x + 4y - 4 = 0 \rightarrow$ (ii)

Solution: From (ii), $x = -4y + 4 = 4(1-y)$

Putting in (i)

$$\frac{y^2}{1} - \frac{\left(4(1-y)\right)^2}{4} = 1$$

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$$y^2 - \frac{16(1-2y+y^2)}{4} = 1 \quad \Rightarrow \quad y^2-4(1-2y+y^2)=1$$

$$y^2 - 4y^2 - 4 + 8y = 1$$

$$3y^2 - 8y + 5 = 0$$

$$3y^2 - 3y - 5y + 5 = 0$$

$$(3y-5)(y-1) = 0$$

$$y = \frac{5}{3}, y = 1$$

When $y=1$, putting in (ii), $x+4(1)-4=0 \quad \Rightarrow x=0$

When $y=\frac{5}{3}$, putting in (ii), $x+4\left(\frac{5}{3}\right)-4=0 \quad \Rightarrow x=\frac{-8}{3}$

The points of intersection is

$\mathbf{P_1(0,1), P_2\left(\frac{-8}{3}, \frac{5}{3}\right)}$

d) $\frac{y^2}{9} - \frac{x^2}{4} = 1 \rightarrow$ **(i)**

y=3(x+1) →(ii)

Solution: Putting (ii) in (i)

$$\frac{[3(x+1)]^2}{9} - \frac{x^2}{4} = 1$$

$$\frac{9(x^2+2x+1)}{9} - \frac{x^2}{4} = 1 \quad \text{Multiplying by 4}$$

$4x^2+8x+4-x^2=4$

$3x^2+8x=0 \quad \Rightarrow \quad x(3x+8)=0$

$x=0$ and $x=\frac{-8}{3}$

When $x=0$, putting in (ii), $y=3\ (0+1)=3$

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When $x=\frac{-8}{3}$, putting in (ii), $y=3\left(\frac{-8}{3}+1\right)=-5$

So, the points of intersection are

$\mathbf{P_1(0,3),\ P_2\left(\frac{-8}{3},-5\right)}$

Q.8 For what value of c;

a) The line y=x+c will touch the hyperbola $\frac{x^2}{16}-\frac{y^2}{4}=1$?

Solution: Given line l: $y=x+c \quad \rightarrow$(i)

and hyperbola, $\frac{x^2}{16}-\frac{y^2}{4}=1 \rightarrow$(ii)

Putting (i) in (ii)

$\frac{x^2}{16}-\frac{(x+c)^2}{4}=1$ Multiplying by 16

$x^2-4(x^2+2xc+c^2)=16$

$x^2-4x^2-8cx-(4c^2+16)=0 \Rightarrow 3x^2+8cx+(4c^2+16)=0 \rightarrow$(iii)

Which is quadratic in x, for the line to touch the hyperbola, the roots of (iii) should be equal i.e. $b^2-4ac=0 \rightarrow$ (iv)

Here $a=3,\ b=8c,\ c=4c^2+16$

Putting in (iv)

$(8c)^2-4(3)(4c^2+16)=0$

$64c^2-48c^2-192=0$

$16c^2=192 \Rightarrow c^2=12$

$c=\pm\sqrt{12} = \pm 2\sqrt{3}$

b) The line y=–x+c will touch the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$

Solution: Given line l: $y=-x+c \quad \rightarrow$(i)

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and hyperbola, $\frac{x^2}{16}-\frac{y^2}{9}=1 \rightarrow$(ii)

Putting (i) in (ii)

$\frac{x^2}{16}-\frac{(-x+c)^2}{9}=1$ Multiplying by 144

$9x^2-16(x^2-2cx+c^2)=144$

$9x^2-16x^2+32cx-16c^2=144$

$7x^2-32cx+16c^2+144=0$

$7x^2-32cx+16(c^2+9)=0 \quad \rightarrow \quad$ (iii)

Which is quadratic in x, for the line to touch the hyperbola, the roots of (iii) should be equal i.e. $b^2-4ac=0 \rightarrow$ (iv)

Here $a=7, b=-32c, c=16(c^2+9)$

$(-32c)^2-4(7)\ 16(c^2+9)=0$

$1024c^2-64\times7(c^2+9)=0$ divide by 64

$16c^2-7c^2-63=0$

$9c^2=63$

$c^2=7 \Rightarrow \quad \mathbf{c=\pm\sqrt{7}}$

c) The line y=x+c, will touch the hyperbola $\frac{x^2}{4}-\frac{y^2}{1}=1$

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Solution: Given line l: $y=x+c \quad \rightarrow$(i)

and hyperbola, $\frac{x^2}{4}-\frac{y^2}{1}=1 \rightarrow$(ii)

Putting (i) in (ii)

$\frac{x^2}{4}-(x+c)^2=1$ Multiplying by 4

$x^2-4(x^2+2cx+c^2)=4$

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$x^2 - 4x^2 - 8cx - 4c^2 = 4$

$3x^2 + 8cx + 4c^2 + 4 = 0$

$3x^2 + 8cx + 4(c^2 + 1) = 0 \quad \rightarrow \quad$ (iii)

Which is quadratic in x, for the line to touch the hyperbola, the roots of (iii) should be same, i.e. $b^2-4ac=0 \rightarrow$ (iv)

Here $a=3$, $b=8c$, $c=4(c^2+1)$

Putting in (iv)

$(8c)^2-4(3)\ (4(c^2+1))=0$

$64c^2-48(c^2+1)=0$ divide by 16

$4c^2-3(c^2+1)=0$

$4c^2-3c^2-3=0$

$c^2=3$

$c=\pm\sqrt{3}$

Q.9 In each case, find the tangent equation and normal equation.

a) At a point $\left(-\sqrt{13}, \frac{9}{2}\right)$ to hyperbola, $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Solution: The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

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At point P (x_1, y_1) is given by,

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \rightarrow \quad$ (i)

Given parabola is $\frac{x^2}{4} - \frac{y^2}{9} = 1$

$a^2=4$, $b^2=9$

and $x_1=-\sqrt{13}$, $y_1=\frac{9}{2}$ Putting these values in (i)

The equation of tangent is

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$$\frac{x\left(-\sqrt{13}\right)}{4}-\frac{y\left(9/2\right)}{9}=1 \qquad \text{Multiplying by 4}$$

$$-\sqrt{13}x-2y=4 \quad \Rightarrow \quad \sqrt{13}x+2y+4=0$$

And equation of normal is given by

$$\frac{\dfrac{x-x_1}{x_1}}{a^2}=-\frac{\dfrac{(y-y_1)}{y_1}}{b^2} \qquad \text{Putting values}$$

$$\frac{x-\left(-\sqrt{13}\right)}{\left(\dfrac{-\sqrt{13}}{4}\right)}=-\frac{\left(y-9/2\right)}{\left(\dfrac{9/2}{9}\right)}$$

$$\frac{4\left(x+\sqrt{13}\right)}{\left(-\sqrt{13}\right)}=-\frac{-\left(y-9/2\right)}{1/2}$$

$$4\left(1/2\right)\left(x+\sqrt{13}\right)=\sqrt{13}\left(y-9/2\right)$$

$$2x+2\sqrt{13}=\sqrt{13}y-9\frac{\sqrt{13}}{2} \qquad \text{Multiplying by 2}$$

$$4x+4\sqrt{13}=2\sqrt{13}y-9\sqrt{13}$$

$$\mathbf{4x-2\sqrt{13}y+13\sqrt{13}=0}$$

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b) At a point $P_1\left(\frac{16}{3},5\right)$ to hyperbola, $\frac{y^2}{9}-\frac{x^2}{16}=1$

Solution: The equation of tangent at (x_1, y_1) to hyperbola

$\frac{y^2}{a}-\frac{x^2}{b^2}=1$ is

$$\frac{y.y_1}{a^2}-\frac{x.x_1}{b^2}=1$$

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Where $(x_1, y_1) = \left(\frac{16}{5}, 5\right)$

$$\frac{y(5)}{9} - \frac{x(16/3)}{16} = 1$$

$$\frac{5y}{9} - \frac{16x}{16 \times 3} = 1$$

$$\frac{5y}{9} - \frac{x}{3} = 1$$

Multiplying both sides by 9

$5y - 3x = 9$

$5y = 3x + 9$

$$y = \frac{3x}{5} + \frac{9}{5}$$

Equation of normal

$$\frac{(y - y_1)}{\frac{y_1}{a^2}} = -\frac{x - x_1}{\frac{x_1}{b^2}}$$ Putting values

$$\frac{(y-5)}{(5/9)} = \frac{x - (16/5)}{\frac{(16/5)}{(16)}}$$

$$\frac{9(y-5)}{5} = \frac{16(x - 16/5)}{\left(16/5\right)}$$

$$y = \frac{5x}{3} + \frac{125}{9}$$

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Translation and rotation of Axes:

For the translation of axis to point h,k the original axes x and y and the transformed axes X and Y are related as;

$$x=X+h,\ y=Y+k$$

For the rotation of axis by an angle θ, the original axes x and y and the transformed axis X and Y are related as;

$$x=X\cos\theta-Y\sin\theta$$

$$y=X\sin\theta+Y\cos\theta$$

EXERCISE 8.4

Q.1 Translate to parallel axis through the

a) Point (0,2,) the equation $2x-y+2=0 \rightarrow$ (i)

Solution: Here h=0, and k=2

For new axis, $x=X+h=X+0$

and $y=Y+k=Y+2$

Putting in (i)

$2(X+0)-(Y+2)+2=0$

$2X-Y-2+2=0$

$2X-Y=0$

b) Point (–1,2), the conic $x^2+y^2+2x-4y+1=0 \rightarrow$ (i)

Solution: Here h=–1, and k=2

For new axis, $x=X+h=X-1$

and $y=Y+k=Y+2$

Putting in (i)

$(X-1)^2+(Y+2)^2+2(X-1)-4(Y+2)+1=0$

$X^2-2X+1+Y^2+4Y+4+2X-2-4Y-8+1=0$

$X^2+Y^2-4=0$

$X^2+Y^2=4$

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c) Point (3,–4), the conic $x^2+2y^2-6x+16y+39=0 \rightarrow$ (i)

Solution: Here h=3, and k=–4

For new axis, $x=X+h=X+3$

and $y=Y+k=Y-4$

Putting in (i)

$(X+3)^2+2(Y-4)^2-6(X+3)+16(Y-4)+39=0$

$X^2+6X+9+2Y^2-16Y+32-6X-18+16Y-64+39=0$

$X^2+2Y^2-16=0$

$\mathbf{X^2+2Y^2=16}$

d) Point (–2,2), the conic $x^2+y^2-3xy+10x+10y+21=0 \rightarrow$ (i)

Solution: Here h=–2, and k=2

For new axis, $x=X+h=X-2$

and $y=Y+k=Y+2$

Putting in (i)

$(X-2)^2+(Y+2)^2-3(X-2)\ (Y+2)+10(X-2)-10(Y+2)+21=0$

$X^2-4X+4+Y^2+4Y+4-3XY-6X+6Y+12+10X-20-10Y-20+21=0$

$\mathbf{X^2+Y^2-3XY+1=0}$

Q.2 Transform to axes inclined at an angle

a) Point 45° to the original axes of the conic $x^2-y^2=a^2$

Solution: Here conic is $x^2-y^2=a^2 \rightarrow$ (i)

And angle of rotation is $\theta=45^\circ$

The equations of rotation are

$x=X\cos\theta-Y\sin\theta \rightarrow$ (ii)

and $y=X\sin\theta+Y\cos\theta \rightarrow$ (iii) putting $\theta=45^\circ$ in (ii) and (iii)

$$x=X\cos 45^\circ - Y\sin 45^\circ = X\cdot\frac{1}{\sqrt{2}}-Y\frac{1}{\sqrt{2}}=\frac{X}{\sqrt{2}}-\frac{Y}{\sqrt{2}} \text{ and}$$

$$y=X\sin 45^\circ + Y\cos 45^\circ = X\cdot\frac{1}{\sqrt{2}}+Y\frac{1}{\sqrt{2}}=\frac{X}{\sqrt{2}}+\frac{Y}{\sqrt{2}}$$

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Putting x and y in (i)

$$\left(\frac{X}{\sqrt{2}}-\frac{Y}{\sqrt{2}}\right)^2-\left(\frac{X}{\sqrt{2}}+\frac{Y}{\sqrt{2}}\right)^2=a^2$$

$$\left(\frac{X^2}{2}+\frac{Y^2}{2}-2\frac{X}{\sqrt{2}}\cdot\frac{Y}{\sqrt{2}}\right)-\left(\frac{X^2}{2}+\frac{Y^2}{2}+2\frac{X}{\sqrt{2}}\cdot\frac{Y}{\sqrt{2}}\right)=a^2$$

$$\frac{X^2}{2}+\frac{Y^2}{2}-\frac{2XY}{2}-\frac{X^2}{2}-\frac{Y^2}{2}-2\frac{XY}{2}=a^2$$

$-XY-XY=a^2$

$-2XY=a^2 \quad \Rightarrow \quad \mathbf{2XY+a^2=0}$

b) 90° to the original axes of the conic $y^2=4px$

Solution: Here conic is, $y^2=4px \rightarrow$ (i) and $\theta=90^o$

The equations of rotation are,

$x=X\cos\theta - Y\sin\theta \rightarrow$ (ii) and

$y=X\sin\theta + Y\cos\theta \rightarrow$ (iii)

Putting $\theta=90^o$ in (ii) and (iii)

$x= X\cos90^o-Y\sin 90^o=X(0)-Y(1)=-Y$

$y= X\sin90^o-Y\cos 90^o=X(1)+Y(0)=X$

Putting x and y in (i)

$(X)^2=4p(-Y)$

$\mathbf{X^2=-4pY}$

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c) 45° to the original axes of the conic $x^2+y^2+4xy=-1$

Solution: Here conic is, $x^2+y^2+4xy-1=0 \rightarrow$ (i)

and angle of rotation $\theta=45^o$

The equations of rotation are given by,

$x=X\cos\theta - Y\sin\theta \rightarrow$ (ii) and

$y=X\sin\theta + Y\cos\theta \rightarrow$ (iii)

Putting $\theta=45^o$ in (ii) and (iii)

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$x = X\cos 45^\circ - Y\sin 45^\circ = X\cdot\frac{1}{\sqrt{2}} - Y\cdot\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(X-Y)$

$y = X\sin 45^\circ + Y\cos 45^\circ = X\cdot\frac{1}{\sqrt{2}} + Y\cdot\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(X+Y)$

Putting x and y in (i)

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 + \left(\frac{X+Y}{\sqrt{2}}\right)^2 + 4\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) - 1 = 0$$

$$\frac{1}{2}(X^2+Y^2-2XY) + \frac{1}{2}(X^2+Y^2+2XY) + \frac{4}{2}(X^2-Y^2) - 1 = 0$$

Multiplying by 2

$X^2+Y^2-2XY+X^2+Y^2+2XY+4X^2-4Y^2-2=0$

$6X^2-2Y^2-2=0$ divide by 2

$3X^2-Y^2-1=0$

d) 45° to the original axes of the conic $x^2-y^2-2\sqrt{2}x-10\sqrt{2}y+2=0$

Solution: Here conic is, $x^2-y^2-2\sqrt{2}x-10\sqrt{2}y+2=0 \rightarrow$ (i)

and angle of rotation $\theta = 45^\circ$

The equations of rotation are;

$x = X\cos\theta - Y\sin\theta \rightarrow$ (ii) and

$y = X\sin\theta + Y\cos\theta \rightarrow$ (iii)

Putting $\theta = 45^\circ$ in (ii) and (iii)

$x = X\cos 45^\circ - Y\sin 45^\circ = X\cdot\frac{1}{\sqrt{2}} - Y\cdot\frac{1}{\sqrt{2}} = \frac{X-Y}{\sqrt{2}}$

$y = X\sin 45^\circ + Y\cos 45^\circ = X\cdot\frac{1}{\sqrt{2}} + Y\cdot\frac{1}{\sqrt{2}} = \frac{X+Y}{\sqrt{2}}$

Putting x and y in (i)

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 - \left(\frac{X+Y}{\sqrt{2}}\right)^2 - 2\sqrt{2}\left(\frac{X-Y}{\sqrt{2}}\right) - 10\sqrt{2}\left(\frac{X+Y}{\sqrt{2}}\right) + 2 = 0$$

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$$\frac{X^2+Y^2-2XY}{2}-\frac{X^2+Y^2+2XY}{2}-2X+2Y-10X-10Y+2=0$$

$$\frac{X^2}{2}+\frac{Y^2}{2}-XY-\frac{X^2}{2}-\frac{Y^2}{2}-XY-12X-8Y+2=0$$

$-2XY-12X-8Y+2=0$ Divide by -2

$$\mathbf{XY+6X+4Y-1=0}$$

Q.3 Transform to new axes inclined at an angle

a) $\tan^{-1}(1/2)$ to the original axes of the conic

$14x^2 + 11y^2 - 36x + 48y - 4xy + 41 = 0$ through $(1,-2)$

Solution: Here the transformation involves both translation and rotation of axes, thus

$x = h + X\cos\theta - Y\sin\theta \rightarrow$ (i)

$y = k + X\sin\theta + Y\cos\theta \rightarrow$ (ii)

Here $h = 1$, $k = -2$ and $\theta = \tan^{-1}(-\frac{1}{2})$

$$\tan\theta = -\frac{1}{2}$$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \pm\frac{\sqrt{5}}{2}$$

$$\cos\theta = \pm\frac{1}{\sec\theta} = \pm\frac{2}{\sqrt{5}} = -\frac{2}{\sqrt{5}}$$

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$$\text{and } \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

Putting these values in (i) and (ii)

$$x = 1-\frac{2}{\sqrt{5}}X-\frac{1}{\sqrt{5}}Y$$

and $$y = -2+\frac{1}{\sqrt{5}}X-\frac{2}{\sqrt{5}}Y$$

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Putting in the equation of conic

$$14\left(1-\frac{2}{\sqrt{5}}X-\frac{1}{\sqrt{5}}Y\right)^2+11\left(-2+\frac{1}{\sqrt{5}}X-\frac{2}{\sqrt{5}}Y\right)^2-36\left(1-\frac{2}{\sqrt{5}}X-\frac{1}{\sqrt{5}}Y\right)$$
$$+48\left(-2+\frac{1}{\sqrt{5}}X-\frac{2}{\sqrt{5}}Y\right)-4\left(1-\frac{2}{\sqrt{5}}X-\frac{1}{\sqrt{5}}Y\right)\left(-2+\frac{1}{\sqrt{5}}X-\frac{2}{\sqrt{5}}Y\right)+41=0$$

$$14\left(1+\frac{4X^2}{5}+\frac{Y^2}{5}-\frac{4X}{\sqrt{5}}-\frac{2Y}{\sqrt{5}}+\frac{4XY}{5}\right)+11\left(4+\frac{X^2}{5}+\frac{4Y^2}{5}-\frac{4X}{\sqrt{5}}-\frac{8Y}{\sqrt{5}}-\frac{4XY}{5}\right)$$
$$-36\left(1-\frac{2X}{\sqrt{5}}-\frac{Y}{\sqrt{5}}\right)+48\left(-2+\frac{X}{\sqrt{5}}-\frac{2Y}{\sqrt{5}}\right)-4\left(-2+\frac{X}{\sqrt{5}}-\frac{2Y}{\sqrt{5}}+\frac{4X}{\sqrt{5}}-\frac{2X^2}{5}\right.$$
$$\left.+\frac{4XY}{5}+\frac{2Y}{\sqrt{5}}-\frac{XY}{5}+\frac{2Y^2}{5}\right)=0$$

$$\frac{X^2}{5}(56+11+8)+\frac{Y^2}{5}(14+44-8)+\frac{X}{5}(56-44-12)+\frac{X}{\sqrt{5}}(-56-44+72+$$
$$48-4-16)+\frac{Y}{\sqrt{5}}(-28+88+36-98+8-8)+(14+44-36-96+8+41)=0$$

$$\frac{75X^2}{5}+\frac{50Y^2}{5}+\frac{XX}{5}[0]+\frac{X}{\sqrt{5}}[0]+\frac{Y}{\sqrt{5}}[0]-25=0$$

$15X^2 + 10Y^2 - 25 = 0$

$3X^2 + 2Y^2 = 5$

b) $\tan^{-1}(-4/3)$ to the original axes of the conic

$11x^2 + 4y^2 - 20x - 40y + 24xy - 5 = 0$ through $(2,-1)$

Solution: Here the transformation involves both translation and rotation of axes, thus

$x = h + X\cos\theta - Y\sin\theta \rightarrow$ (i)

$y = k + X\sin\theta + Y\cos\theta \rightarrow$ (ii)

Here $h = 2$, $k = -1$ and $\theta = \tan^{-1}(-4/3)$

$$\tan\theta = \frac{-4}{3}$$

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$$\text{Sec}\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{-4}{3}\right)^2} = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$$

$$\text{Cos}\theta = \frac{1}{\text{Sec}\,\theta} = \pm\frac{3}{5} = -\frac{3}{5}$$

$$\text{and } \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{-4}{5}$$

Putting these values in (i) and (ii)

$$x = 2-\frac{3}{5}X-\frac{4}{5}Y$$

and $$y = -1+\frac{4}{5}X-\frac{3}{5}Y$$

Putting in the equation of conic

$$11\left(2-\frac{3}{5}X-\frac{4}{5}Y\right)^2+4\left(-1+\frac{4}{5}X-\frac{3}{5}Y\right)^2-20\left(2-\frac{3}{5}X-\frac{4}{5}Y\right)$$

$$+40\left(-1+\frac{4}{5}X-\frac{3}{5}Y\right)+24\left(2-\frac{3}{5}X-\frac{4}{5}Y\right)\left(-1+\frac{4}{5}X-\frac{3}{5}Y\right)-5=0$$

$$11\left(4+\frac{9X^2}{25}+\frac{16Y^2}{25}-\frac{12X}{5}-\frac{16Y}{5}+\frac{24XY}{25}\right)+4\left(1+\frac{16X^2}{25}+\frac{9Y^2}{25}-\frac{8X}{5}\right.$$

$$\left.+\frac{6Y}{5}-\frac{24XY}{25}\right)-20\left(2-\frac{3X}{5}-\frac{4Y}{5}\right)-40\left(-1+\frac{4X}{5}-\frac{3Y}{5}\right)+24=0$$

$$\left[-2+\frac{8X}{5}-\frac{6Y}{5}+\frac{3X}{5}-\frac{12X^2}{25}+\frac{9XY}{25}+\frac{4Y}{5}-\frac{16XY}{25}+\frac{12Y^2}{25}\right]-5=0$$

$$\frac{X^2}{25}(99-64-288)+\frac{Y^2}{25}(176+36+288)+\frac{XY}{25}(264-96-168)+\frac{X}{5}(-132$$

$$-32+60-160+264)+\frac{Y}{5}(-176+24+80+120-48)-5-\frac{125X^2}{25}+\frac{500Y^2}{25}$$

$$+\frac{XX}{25}[0]+\frac{X}{5}[0]+\frac{Y}{5}[0]=0$$

$$-5X^2 + 20Y^2 - 5 = 0$$

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$-5X^2 + 20Y^2 = 5$

Dividing both sides by 5

$$\frac{-5X^2}{5} + \frac{20Y^2}{5} = \frac{5}{5}$$

$$\mathbf{-X^2 + 4Y^2 = 1}$$

c) **$\tan^{-1}(3/4)$ to the original axes of the conic**

$3x^2 + 10y^2 + 6x + 52y - 24xy = 0$ through (3,1)

Solution: Here the transformation involves both translation and rotation of axes, thus

$x = h + X\cos\theta - Y\sin\theta \rightarrow$ (i)

$y = k + X\sin\theta + Y\cos\theta \rightarrow$ (ii)

Here $h = 3$, $k = 1$ and $\theta = \tan^{-1}(3/4)$

$$\tan\theta = \frac{3}{4}$$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{3}{4}\right)^2} = \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

$$\cos\theta = \frac{1}{\sec\theta} = \frac{4}{5}$$

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$$\text{and } \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

Putting these values in (i) and (ii)

$$x = 3 + \frac{4}{5}X - \frac{3}{5}Y$$

and $$y = 1 + \frac{3}{5}X + \frac{4}{5}Y$$

Putting in the equation of conic

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$$3\left(3+\frac{4}{5}X-\frac{3}{5}Y\right)^2+10\left(1+\frac{3}{5}X+\frac{4}{5}Y\right)^2+6\left(3+\frac{4}{5}X-\frac{3}{5}Y\right)$$
$$+52\left(1+\frac{3}{5}X+\frac{4}{5}Y\right)-24\left(3+\frac{4}{5}X-\frac{3}{5}Y\right)\left(1+\frac{3}{5}X+\frac{4}{5}Y\right)=0$$

$$3\left(9+\frac{16}{25}X^2+\frac{9}{25}Y^2+\frac{24}{5}X-\frac{24}{25}XY-\frac{18}{5}Y\right)+10\left(1+\frac{9}{25}X^2+\frac{16}{25}Y^2\right.$$
$$\left.+\frac{6}{5}X+\frac{24}{25}XY+\frac{8}{5}Y\right)+18+\frac{24}{5}X-\frac{18}{5}Y+52+\frac{156}{5}X+\frac{208}{5}Y-24$$
$$\left(3+\frac{9}{5}X+\frac{12}{5}Y+\frac{4}{5}X+\frac{12}{25}X^2+\frac{16}{25}XY-\frac{3}{5}Y-\frac{9}{25}XY-\frac{12}{25}Y^2\right)=0$$

$$\frac{X^2}{25}[48+90-288]=\frac{Y^2}{25}[27+160+288]+\frac{XY}{25}[-72+240-168]$$
$$+\frac{X}{5}[72+60+24+156-312]+\frac{Y}{5}[-54+80-18+208-216]$$
$$+[27+10+18+52-72]=0$$

$$\frac{-150X^2}{25}+\frac{475Y^2}{25}+\frac{XX}{25}[0]+\frac{X}{5}[0]+\frac{Y}{5}[0]+25=0$$

$$\frac{-150X^2}{25}+\frac{475Y^2}{25}+25=0$$

$$-6X^2+19Y^2+25=0$$

$6X^2-19Y^2=25$

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Q.4 At what angle the axes are rotated about the origin so that the transformed equation of the conic

a) $11x^2+4y^2-20x-40y+24xy-5=0$ does not contain the term involving XY?

Solution: For the conic of the form

$ax^2+2hxy+by^2=0 \rightarrow$ (i) the xy term is removed by rotation of axes by

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$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right) \rightarrow (ii)$$

Comparing (i) with the given conic

$a = 11, b = 4, 2h = 24 \quad \Rightarrow h = 12$

Putting these values in (ii)

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2(12)}{11-4}\right)$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{24}{7}\right)$$

$\theta = 36.87$

b) $5x^2 + 7y^2 + 2\sqrt{3}xy - 16 = 0$ does not contain the term involving XY?

Solution: For the conic of the form

$ax^2 + 2hxy + by^2 = 0 \rightarrow$ (i) the xy term is removed by rotation of axes by

$$\tan 2\theta = \frac{2h}{a-b}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right) \rightarrow (ii)$$

Comparing (i) with the given conic

$a = 5, b = 7, 2h = 2\sqrt{3} \quad \Rightarrow h = \sqrt{3}$

Putting these values in (ii)

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2(\sqrt{3})}{5-7}\right)$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right)$$

$$\theta = \frac{1}{2}\tan^{-1}\left(-\sqrt{3}\right)$$

$$\theta = \frac{1}{2}\left(-60^\circ\right) \quad \Rightarrow \quad \theta = -30^\circ$$

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