Unit 05

INTEGRATION

Integration as accumulator:

The velocity gives the rate of change of displacement of a body, which is the change in displacement per unit time. If we need to find out the total distance covered by a body, we need to add the distance covered at every instance thus distance can be obtained by accumulating velocities.

Integration is the inverse process of differentiation as stated above. The velocity is the rate of change in displacement i.e.

$$v = \frac{dx}{dt} \rightarrow (i)$$

Thus to obtain the expression for the distance x at any time t, we have to perform the inverse process of derivative on (i) i.e.

$$x(t) = \int v dt = \int \frac{dx}{dt} dt$$

This inverse process is called integration. Thus if $f'(x)$ is the derivative of a function f(x), then f(x) is the anti-derivative or integration of $f'(x)$. i.e.

$$\frac{d}{dx}\left(f(x)\right) = f'(x)$$

$$\int f'(x)dx = f(x)$$

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Example:

For $f(x) = x^n + c$ $\quad$ c=constant

$f'(x) = nx^{n-1}$

Therefore $f(x) = \int f'(x)dx = \int nx^{n-1}dx = x^n + c$

The anti-derivative of a function is not unique and may differ by a constant such as;

$$\int 2xdx = x^2 + 1,\ x^2 + 2,\ x^2 + 3 \text{ etc}$$

Thus, if the anti-derivative of a function are $F_1(x)$ and $F_2(x)$ on an interval [a,b]. Then the difference between them is a constant.

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Standard integrals through standard differentiation formulas:

The following are the standard derivative / integration pairs.

	Integrations formula	**Derivatives formula**
1.	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = x^n$
2.	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$	$\frac{d}{dx}(\ln x + C) = \frac{1}{x}$
3.	$\int \cos kx dx = \frac{\sin kx}{k} + C,$	$\frac{d}{dx}\left(\frac{\sin kx}{k} + C\right) = \cos kx$
4.	$\int \sin kx dx = \frac{-\cos kx}{k} + C,$	$\frac{d}{dx}\left(\frac{-\cos kx}{k} + C\right) = \sin kx$
5.	$\int \frac{1}{\cos^2 x} dx = \int \sec^2 x dx = \tan x + C,$	$\frac{d}{dx}(\tan x + C) = \sec^2 x = \frac{1}{\cos^2 x}$
6.	$\int \frac{1}{\sin^2 x} dx = \int \operatorname{cosec}^2 x dx = -\cot x + C,$	$\frac{d}{dx}(-\cot x + C) = \operatorname{cosec}^2 x = \frac{1}{\sin^2 x}$
7.	$\int \tan dx = \int \frac{\sin x}{\cos x} dx = -\ln\lvert\cos x\rvert + C,$	$\frac{d}{dx}(-\ln(\cos x) + C) = \tan x$
8.	$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \ln\lvert\sin x\rvert + C,$	$\frac{d}{dx}(\ln(\sin x) + C) = \cot x$
9.	$\int e^{mx} dx = \frac{e^{mx}}{m} + C,$	$\frac{d}{dx}\left(\frac{e^{mx}}{m} + C\right) = e^{mx}$
10.	$\int a^x dx = \frac{a^x}{\ln a} + C,$	$\frac{d}{dx}\left(\frac{a^x}{\ln a} + C\right) = a^x$
11.	$\int \frac{dx}{1+x^2} = \tan^{-1} x + C,$	$\frac{d}{dx}(\tan^{-1} + C) = \frac{1}{1+x^2}$
12.	$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C,$	$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a} + C\right) = \frac{1}{a^2+x^2}$

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13. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C,$ $\quad \frac{d}{dx}\left(\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C\right) = \frac{1}{a^2-x^2}$

14. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C,$ $\quad \frac{d}{dx}\left(\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C\right) = \frac{1}{x^2-a^2}$

15. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C,$ $\quad \frac{d}{dx}\left(\sin^{-1}x + C\right) = \frac{1}{\sqrt{1-x^2}}$

16. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C,$ $\quad \frac{d}{dx}\left(\sin^{-1}\frac{x}{a} + C\right) = \frac{1}{\sqrt{a^2-x^2}}$

17. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$ $\quad \frac{d}{dx}\left(\ln\left(\sqrt{x^2 \pm a^2}\right) + C\right) = \frac{1}{\sqrt{x^2 \pm a^2}}$

The rules of integrations

1. **Constant rule** $\quad \int k\,dx = kx + C$

2. **Constant multiple rule** $\quad \int kf(x)dx = k\int f(x)dx + C$

3. **Sum rule** $\quad \int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

4. **Power rule** $\quad \int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$

5. $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C$

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EXERCISE 5.1

Q.1 Evaluate the following indefinite integrals and check the result through differentiation.

a) $\int (x^4 + 3x^3 - 7)dx$

Solution: Let the integral of given function be f(x) then

$f(x) = \int (x^4 + 3x^3 - 7)dx$

$f(x) = \int x^4 dx + 3\int x^3 dx - 7\int 1dx$

$f(x) = \frac{x^{4+1}}{4+1} + 3\frac{x^{3+1}}{3+1} - 7x + c$

$\mathbf{f(x) = \frac{x^5}{5} + \frac{3x^4}{4} - 7x + c}$

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Taking derivative of f(x) to check if it is equal to the given function i.e.

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^5}{5}+\frac{3x^4}{4}-7x+c\right)$$

$$=\frac{1}{5}\frac{d}{dx}(x^5)+\frac{3}{4}\frac{d}{dx}(x^4)-7\frac{d}{dx}(x)+\frac{d}{dx}(c)$$

$$=\frac{1}{5}\cdot 5x^4+\frac{3}{4}\cdot 4x^3-7\cdot 1+0$$

$$=x^4+3x^3-7$$

Which is the given function.

b) $\int\frac{1}{x^3}dx$

Solution: Let the integral of given function be f(x) then,

$$f(x)=\int\frac{1}{x^3}dx=\int x^{-3}dx$$

$$f(x)=\frac{x^{-3+1}}{-3+1}+c$$

$$f(x)=-\frac{1}{2}x^{-2}+c$$

Taking derivative of f(x) to check if it is equal to the given function i.e.

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(-\frac{1}{2}x^{-2}+c\right)$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(-\frac{1}{2}x^{-2}\right)+\frac{d}{dx}(c)$$

$$\frac{d}{dx}f(x)=-\frac{1}{2}(-2)x^{-3}+0$$

$$\frac{d}{dx}f(x)=x^{-3}$$

$$\frac{d}{dx}f(x)=\frac{1}{x^3}$$

Which is the given function.

c) $\int\left(2x^4+x^{-2/3}-x^{-5/3}\right)dx$

Solution: Let the integral of given function be f(x) then

$$f(x)=\int\left(2x^4+x^{-2/3}-x^{-5/3}\right)dx$$

$$f(x)=2\int x^4dx+\int x^{-2/3}dx-\int x^{-5/3}dx$$

$$f(x)=2\frac{x^{4+1}}{4+1}+\frac{x^{-2/3+1}}{-2/3+1}-\frac{x^{-5/3+1}}{-5/3+1}+c$$

$$f(x)=\frac{2x^5}{5}+3x^{1/3}+\frac{3}{2}x^{-2/3}+c$$

Taking derivative of f(x) to check if it is gives the given function i.e.

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{2x^5}{5}+3x^{1/3}+\frac{3}{2}x^{-2/3}+c\right)$$

$$=\frac{d}{dx}\left(\frac{2}{5}x^5\right)+\frac{d}{dx}\left(3x^{1/3}\right)+\frac{d}{dx}\left(\frac{3}{2}x^{-2/3}\right)\frac{d}{dx}(c)$$

$$=\frac{2}{5}\cdot 5x^4+3\cdot\frac{1}{3}x^{1/3-1}+\frac{3}{2}(-\frac{2}{3})x^{-2/3-1}+0$$

$$=2x^4+x^{-2/3}-x^{-5/3}$$

Which is the given function.

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d) $\int (1+3t)t^3 dt$

Solution: Let the integral of given function be f(t) then

$$f(t) = \int (1+3t)t^3 dt$$

$$f(t) = \int (t^3 + 3t^4) dt$$

$$f(t) = \int t^3 dt + 3\int t^4 dt$$

$$f(t) = \frac{t^{3+1}}{3+1} + \frac{3t^{4+1}}{4+1} + c$$

$$\mathbf{f(t) = \frac{t^4}{4} + \frac{3}{5}t^5 + c}$$

Taking derivative of f(t) to check if it gives the given function i.e.

$$\frac{d}{dx}f(t) = \frac{d}{dt}\left(\frac{t^4}{4} + \frac{3}{5}t^5 + c\right)$$

$$= \frac{d}{dt}\left(\frac{t^4}{4}\right) + \frac{d}{dt}\left(\frac{3}{5}t^5\right) + \frac{d}{dt}(c)$$

$$= \frac{1}{4}\left(4t^3\right) + \frac{3}{5}\left(5t^4\right) + 0$$

$$= t^3 + 3t^4$$

$$\mathbf{= t^3(1+t)}$$

Which is the given function.

e) $\int (t^2-1)^2 dt$

Solution: Let the integral of given function be f(t) then,

$$f(t) = \int (t^2-1)^2 dt$$

Expand by formula

$\because$ $\mathbf{(a-b)^2 = a^2 + b^2 - 2ab}$

$$f(t) = \int (t^4 - 2t^2 + 1) dt$$

$$f(t) = \int t^4 dt - 2\int t^2 dt + \int 1 dt$$

$$f(t) = \frac{t^{4+1}}{4+1} - 2.\frac{t^{2+1}}{2+1} + t + c$$

$$\mathbf{f(t) = \frac{t^5}{5} - \frac{2}{3}t^3 + t + c}$$

Taking derivative of f(t) to check if it gives the given function i.e.

$$\frac{d}{dt}f(t) = \frac{d}{dt}\left(\frac{t^5}{5} - \frac{2}{3}t^3 + t + c\right)$$

$$= \frac{d}{dt}\left(\frac{t^5}{5}\right) - \frac{d}{dt}\left(\frac{2}{3}t^3\right) + \frac{d}{dt}(t) + \frac{d}{dt}(c)$$

$$= \frac{1}{5}\left(5t^4\right) - \frac{2}{3}\left(3t^2\right) + 1 + 0$$

$$= t^4 - 2t^2 + 1$$

$$\mathbf{= (t^2-1)^2}$$

Which is the given function.

f) $\int \frac{x^3+1}{x^3} dx$

Solution: Let the integral of given function be f(x) then

$$f(x) = \int \frac{x^3+1}{x^3} dx$$

$$f(x) = \int \left(\frac{x^3}{x^3} + \frac{1}{x^3}\right) dx$$

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$$f(x)=\int\left(1+x^{-3}\right)dx$$

$$f(x)=\int 1dx+\int x^{-3}dx$$

$$f(x)=x+\frac{x^{-3+1}}{-3+1}+c$$

$$\mathbf{f(x)=x-\frac{x^{-2}}{2}+c}$$

Take derivative of f(x) to check if it is gives the given function i.e.

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(x-\frac{x^{-2}}{2}+c\right)$$

$$=\frac{d}{dx}(x)-\frac{d}{dx}\left(\frac{x^{-2}}{2}\right)+\frac{d}{dx}(c)$$

$$=1-\frac{1}{2}\left(-2x^{-3}\right)+0$$

$$=1+x^{-3}$$

$$=1+\frac{1}{x^3}\quad\text{Taking LCM}$$

$$\mathbf{=\frac{x^3+1}{x^3}}$$

Which is the given function.

g) $\int z^2\sqrt{z}\,dz$

Solution: Let the integral of given function be f(z) then

$$f(z)=\int z^2\sqrt{z}\,dz$$

$$f(z)=\int z^2\cdot z^{1/2}\,dz$$

$$f(z)=\int z^{5/2}\,dz$$

$$f(z)=\frac{z^{5/2+1}}{5/2+1}+c$$

$$f(z)=\frac{z^{7/2}}{7/2}+c$$

$$\mathbf{f(z)=\frac{2}{7}z^{7/2}+c}$$

Take derivative of f(z) to check if it gives the given function i.e.

$$\frac{d}{dz}f(z)=\frac{d}{dz}\left(\frac{2}{7}z^{7/2}+c\right)$$

$$=\frac{d}{dz}\left(\frac{2}{7}z^{7-1/2}\right)+\frac{d}{dz}(c)$$

$$=\frac{2}{7}\left(\frac{7}{2}z^{5/2}\right)+0$$

$$=z^{5/2}$$

We can write

$$=z^2.z^{1/2}$$

$$\mathbf{=z^2\sqrt{z}}$$

Which is the given function.

Q.2 Evaluate the following indefinite integral by method of substitution.

a) $\int(3x+4)^8\,dx$

Solution: $\int(3x+4)^8\,dx$

Integration by substitution

Let t=3x+4

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Diff w.r.t x

$$\frac{dt}{dx} = 3$$

$$\frac{dt}{3} = dx$$

$$\int (3x+4)^8 \, dx$$

Substituting these in given integral becomes.

$$\int t^8 \frac{dt}{3} = \frac{1}{3}\int t^8 dt$$

$$= \frac{1}{3} \cdot \frac{t^{8+1}}{8+1} + c$$

$$= \frac{1}{27} \cdot t^9 + c$$

Putting value of t = (3x+4)

$$= \frac{1}{27}.(3x+4)^9 + c$$

b) $\int 3x^2 (x^3 - 4) dx$

Solution:

Integration by substitution method

Let $t=x^3-4$

Diff w.r.t x

$$\frac{dt}{dx} = 3x^2$$

$$dt = 3x^2 dx$$

$$\int (x^3 - 4) 3x^2 dx$$

Substituting these in given integral becomes.

$$\int t \cdot dt = \frac{t^{1+1}}{1+1} + c$$

$$= \frac{t^2}{2} + c$$

Putting value of t

$$\int 3x^2 (x^3 - 4) dx$$

$$= \frac{(x^3 - 4)^2}{2} + c$$

c) $\int (3x^2 + 7)(x^3 + 7x)^8 \, dx$

$$= \int (x^3 + 7x)^8 (3x^2 + 7x) dx$$

Solution:

Integration by substitution method

Let $t=x^3+7x$

Diff w.r.t x

$$\frac{dt}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(7x)$$

$$\frac{dt}{dx} = 3x^2 + 7$$

$$dt = (3x^2 + 7) dx$$

Substituting t and dt the given integral becomes.

$$\int (x^3 + 7x)^8 (3x^2 + 7) dx$$

$$\int t^8 dt = \frac{t^{8+1}}{8+1} + c$$

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$$= \frac{t^9}{9} + c$$

Putting value of t = $(x^3 + 7x)$

$$\int (x^3+7x)^8 (3x^2+7x)dx$$

$$= \frac{(x^3+7x)^9}{9} + c$$

d) $$\int \frac{(3x^2-7)}{(x^3-7x)^4} dx$$

Solution:

Integration by substitution method

Let $t = x^3 - 7x$

Diff w.r.t x

$$\frac{dt}{dx} = \frac{d}{dx}(x)^3 - \frac{d}{dx}7(x)$$

$$\frac{dt}{dx} = 3x^2 - 7$$

$$\text{and } dt = (3x^2-7)dx$$

Putting t and dt the given integral becomes.

$$\int \frac{(3x^2-7)}{(x^3-7x)^4} dx$$

$$\int \frac{dt}{t^4} = \int t^{-4} dt = \frac{t^{-4+1}}{-4+1} + c = \frac{t^{-3}}{3} + c$$

Putting value of t = $(x^3 - 7x)$

$$\int \frac{(3x^2-7)}{(x^3-7x)^4} dx$$

$$= \frac{-1}{3}(x^3-7x)^{-3} + c$$

e) $$\int \frac{x+3x^2}{\sqrt{x}} dx$$

Solution:

Method-1

$$\int \frac{x+3x^2}{\sqrt{x}} dx$$

Integration by substitution method

Suppose $\sqrt{x} = t$ than $x = t^2$

Diff w.r.t x

$$\frac{d}{dx}(x) = \frac{d}{dt}(t^2)$$

$$1 = 2t.\frac{dt}{dx}$$

$$dx = 2t.dt$$

$$= \int \frac{x+3x^2}{\sqrt{x}} dx$$

$$= \int \left(\frac{t^2+3t^4}{t}\right) 2t.dt$$

$$= 2\int t^2 dt + 6\int t^4 dt$$

$$= \frac{2t^3}{3} + \frac{6t^5}{5} + c \quad \text{put } \sqrt{x} = t$$

$$= \frac{2}{3}(x)^{\frac{3}{2}} + \frac{6}{5}(x)^{\frac{5}{2}} + c$$

Method-2

$$= \int \left(\frac{x}{\sqrt{x}} + \frac{3x^2}{\sqrt{x}}\right) dx$$

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$$= \int \left(\frac{x}{x^{1/2}} + \frac{3x^2}{x^{1/2}} \right) dx$$

$$= \int \left(x^{1-\frac{1}{2}} + 3x^{2-\frac{1}{2}} \right) dx$$

$$= \int \left(x^{1/2} + 3x^{3/2} \right) dx$$

$$= \int x^{1/2} dx + 3\int x^{3/2} dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{3x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$$

$$= \frac{x^{\frac{3}{2}}}{3/2} + \frac{3x^{\frac{5}{2}}}{5/2} + c$$

$$= \frac{2}{3}x^{\frac{3}{2}} + \frac{6}{5}x^{\frac{5}{2}} + c$$

f) $\int \frac{x+1}{(x^2+2x+2)^2} dx$

Solution:

Let $t = x^2 + 2x + 2 \rightarrow (i)$

$$\frac{dt}{dx} = 2x + 2 = 2(x+1)$$

$$\frac{dt}{2} = (x+1)dx \rightarrow (ii)$$

Substituting (i) and (ii) in the given integral becomes.

$$\int \frac{1}{t^2} \cdot \frac{dt}{2} = \frac{1}{2}\int t^{-2} dt = \frac{1}{2} \cdot \frac{t^{-2+1}}{-2+1} + c$$

$$= -\frac{1}{2}t^{-1} + c = -\frac{1}{2t} + c$$

Substituting value of t

$$= \frac{x+1}{(x^2+2x+2)^2}$$

$$= \frac{-1}{2(x^2+2x+2)} + c$$

Q.3 Evaluate the following integral by method of substitution.

a) $\int 6e^{6t} dt = \int e^{6t} 6dt$

Solution: Given integral is

$\int 6e^{6t} dt$

Integration by substitution method

Let $u = 6t \rightarrow (i)$

Diff w.r.t t

$$\frac{du}{dt} = 6$$

$du = 6dt \rightarrow (ii)$

Substituting (i) and (ii) in the given integral becomes

$$\int 6e^{6t} dt = \int e^u du = e^u + c$$

Substituting the value of u from (i)

$$\int e^{6t} 6dt = e^{6t} + c$$

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b) $\int xe^{(5x^2+1)}dx$

Solution: Given integral is

$$\int xe^{(5x^2+1)}dx = \int e^{(5x^2+1)}xdx$$

Integration by substitution method

Let $u=5x^2+1 \rightarrow$ (i)

Diff w.r.t t

$$\frac{du}{dx} = 10x \Rightarrow \frac{du}{10} = xdx \rightarrow (ii)$$

Substituting (i) and (ii) in the given integral becomes

$$= \int e^u \frac{du}{10} = \frac{1}{10}\int e^u du = \frac{1}{10}e^u + c$$

Substituting the value of u from (i)

$$\int e^{(5x^2+1)}xdx = \frac{1}{10}e^{(5x^2+1)} + c$$

c) $\int (x^2-2)e^{(x^3-6x+4)}dx$

Solution: Given integral is

$$\int (x^2-2)e^{(x^3-6x+4)}dx$$

Integration by substitution method

Let $t=x^3-6x+4 \rightarrow$ (i)

Diff w.r.t x

$$\frac{dt}{dx} = 3x^2-6 = 3(x^2-2)$$

$$\frac{dt}{3} = (x^2-2)dx \rightarrow (ii)$$

Substituting (i) and (ii) the given integral becomes

$$\int e^t \frac{dt}{3} = \frac{1}{3}\int e^t dt = \frac{1}{3}e^t + c$$

Substituting the value of t from (i)

$$\int e^{(x^3-6x+4)}(x^2-2)dx = \frac{1}{3}e^{(x^3-6x+4)} + c$$

d) $\int 8^{(7-3x^2)}(-6x)dx$

Solution: Given integral is

$$\int 8^{(7-3x^2)}(-6x)dx$$

Let $t=7-3x^2 \rightarrow$ (i)

Diff w.r.t x

$$\frac{dt}{dx} = -6x$$

$$dt = (-6x)dx \rightarrow (ii)$$

Substituting (i) and (ii) is given integral

$$= \int 8^t dt = \frac{8^t}{\ln 8} + c$$

Substituting back the value of t from (i)

$$\int e^{(7-3x^2)}(-6x)dx = \frac{8^{(7-3x^2)}}{\ln 8} + c$$

Q.4 Find the equation of the particular curve that has a slope $4x^3+6x^2$ at a point P(1,0).

Solution: Let the required curve is $y=f(x)$

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Then, slope $= \frac{dy}{dx} = 4x^3 + 6x^2$

$$dy = \left(4x^3 + 6x^2\right)dx$$

Taking integral on both sides

$$\int dy = \int (4x^3 + 6x^2)dx$$

$$y = \int 4x^3 dx + \int 6x^2 dx$$

$$y = 4\int x^3 dx + 6\int x^2 dx$$

$$y = 4\frac{x^4}{4} + 6\frac{x^3}{3} + c$$

$$\mathbf{y = x^4 + 2x^3 + c} \rightarrow (i)$$

Since point P(1,0) lies on the curve, substitute it in equation of the curve

$$0=(1)^4+2(1)^3+c$$

$$0=3+c$$

$$\mathbf{c=-3}$$

Putting in eq. of curve (i)

$$\mathbf{y = x^4 + 2x^3 - 3}$$

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Q.5 A certain curve has a slope of $x(2x^2-1)^2$ that passes through the point P(3,3). What is the equation of the specific curve?

Solution: Let the curve is y=f(x)

Then, slope of curve is given by $\frac{dy}{dx}$,

$$\frac{dy}{dx} = x(2x^2-1)^2$$

$$dy = x(2x^2-1)^2 dx$$

Taking integral on both sides

$$\int dy = \int x(2x^2-1)^2 dx \rightarrow (A)$$

By substitution method

Let $t=(2x^2-1) \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx} = \frac{d}{dx}(2x^2-1)$$

$$\frac{dt}{dx} = 4x$$

$$\frac{dt}{4} = xdx \rightarrow (ii)$$

Substituting (i) and (ii) in (A)

$$\int dy = \int (2x^2-1)^2 xdx \int t^2 \frac{dt}{4}$$

$$y = \frac{1}{4}\cdot\frac{t^3}{3} + c$$

$$y = \frac{t^3}{12} + c$$

Substituting the value of t from (i)

$$\mathbf{y = \frac{\left(2x^2-1\right)^3}{12} + c} \rightarrow (B)$$

Since the curve passes through P(3,3), it will satisfy the equation of curve.

Put x = 3, y = 3

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$$3 = \frac{\left(2(3)^2 - 1\right)^3}{12} + c$$

$$3 = \frac{(17)^3}{12} + c$$

$$c = -406.42$$

And $\mathbf{y = \frac{\left(2x^2 - 1\right)^3}{12} - 406.42}$

Q.6 A certain curve has a slope of $x\sqrt{2x^2-1}$ that passes through the point P(3,3). What is the equation of the specific curve?

Solution: Let the curve is y=f(x), then the slope of curve is given by $\frac{dy}{dx}$ so,

$$\frac{dy}{dx} = x\sqrt{2x^2 - 1}$$

$$dy = x\sqrt{2x^2 - 1}\,dx$$

$$dy = \left(2x^2 - 1\right)^{1/2} x dx$$

Taking integral on both sides

$$\int dy = \int (2x^2 - 1)^{1/2} x dx \rightarrow (A)$$

Let $t = 2x^2 - 1 \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx} = 4x$$

$$\frac{dt}{4} = x dx \rightarrow (ii)$$

Substituting (i) and (ii) in (A)

$$\int dy = \int t^{1/2} \frac{dt}{4} = \frac{1}{4}\int t^{1/2}\,dt$$

$$y = \frac{1}{4} \cdot \frac{t^{3/2}}{3/2} + c$$

$$y = \frac{1}{6} t^{3/2} + c$$

Substituting the value of t from (i)

$$\mathbf{y = \frac{1}{6}\left(2x^2 - 1\right)^{3/2} + c \rightarrow (B)}$$

Since the curve passes through P(3,3), it will satisfy the equation i.e.

$$3 = \frac{1}{6}\left(2(3)^2 - 1\right)^{3/2} + c$$

$$3 = \frac{1}{6}(17)^{3/2}$$

$$3 = 11.89 + c$$

$$\mathbf{c = -8.89}$$

Putting in equation of curve i.e. (B)

$$\mathbf{y = \frac{1}{6}\left(2x^2 - 1\right)^{3/2} - 8.89}$$

Q.7 For an average person, the rate of change of weight w (in pounds) with respect to

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height h (in inches) is approximated by

$$\frac{dW}{dh} = 0.0015h^2$$

a) Find W(h), if the weight is W=108 points in response of height h=60 inches.

Solution: $\frac{dW}{dh} = 0.0015h^2$

$dW = 0.0015h^2 dh$

taking integration on both sides

$$\int dW = \int 0.0015h^2 dh$$

$$W = 0.0015\frac{h^3}{3} + K$$

$$W = 0.0005h^3 + K \rightarrow (1)$$

To find K, put W=108 and h=60

$108 = 0.0005(60)^3 + K$

$108 = 108 + K$

$K = 0$ putting in (1)

$W = 0.0005h^3$

b) Find weight of a person who is 5 feet 10 inches tall.

Solution: From part a, the weight W is given by.

$W = 0.0005h^3 \rightarrow (i)$

Here h=5feet, 10 inches

$h = 5 \times 12 + 10$ inches

$h = 70$ inches

putting in (i)

$W(70) = 0.0005\ (70)^3$

W(70)=171.5 pounds

Q.8 The rate of growth of the population N(t) of a newly incorporated city t years after incorporation is estimated to be?

$$\frac{dN}{dt} = 400 + 600\sqrt{t},\ 0 \le t \le 9$$

If the population was 5,000 at the time of incorporation, find the population 9 years later.

Solution: Here given that

$$\frac{dN}{dt} = 400 + 600\sqrt{t} \rightarrow (i)$$

$N(0) = 5{,}000$, $N(9) = ?$

From (i) $\frac{dN}{dt} = 400 + 600\sqrt{t}$

$$\frac{dN}{dt} = 400 + 600t^{1/2}$$

$$dN = \left(400 + 600t^{1/2}\right)dt$$

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taking integral on both sides

$$\int dN = \int \left(400 + 600t^{1/2}\right) dt$$

$$N(t) = 400t + 600 \cdot \frac{t^{3/2}}{3/2} + c$$

$$\mathbf{N(t) = 400t + 400\ t^{3/2} + c} \rightarrow (ii)$$

To find c put N(t)=5000 at t=0

$$5{,}000 = 400(0) + 400(0)^{3/2} + c$$

c=5,000

Thus eq.(i) becomes

$$N(t) = 400t + 400t^{3/2} + 5{,}000$$

Now, $N(9) = 400(9) + 400(9)^{3/2} + 5{,}000$

N(9)=19400

EXERCISE 5.2

Q.1 Evaluate the following indefinite integrals by method of substitution.

a) $\int \sin^4 x \cos x dx$

Solution: Given integral is

$$\int \sin^4 x \cos x dx \rightarrow (A)$$

By substitution method

Let $t = \sin x \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx} = \frac{d}{dx}(\sin x)$$

$$\frac{dt}{dx} = \cos x$$

$$dt = \cos dx \rightarrow (ii)$$

Substituting (i) and (ii) then (A) becomes

$$\int \sin^4 x \cos x dx$$

$$\int t^4 dt = \frac{t^5}{5} + c$$

Putting the value of t from (i)

$$= \frac{\sin^5 x}{5} + c$$

b) $\int \frac{\cos x . \ln(\sin x)}{\sin x} dx$

Solution: Given integral is

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$$\int \frac{\cos x \cdot \ln(\sin x)}{\sin x} dx \rightarrow (A)$$

By substitution method

Let $t = \ln(\sin x) \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx} = \frac{d}{dx}(\ln \sin x)$$

$$\frac{dt}{dx} = \frac{1}{\sin x} \cdot \cos x$$

$$dt = \frac{\cos x}{\sin x} dx \rightarrow (ii)$$

Substituting (i) and (ii) in (A) becomes

$$\int \frac{\cos x \cdot \ln(\sin x)}{\sin x} dx$$

$$\int t \cdot dt = \frac{t^2}{2} + c$$

Putting the value of t from (i)

$$= \frac{(\ln(\sin x))^2}{2} + c$$

c) $\int e^x \sin e^x \, dx$

Solution: Given integral is

$$\int e^x \sin e^x \, dx \rightarrow (A)$$

By substitution method

Let $t = e^x \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx} = \frac{d}{dx} e^x$$

$$\frac{dt}{dx} = e^x$$

$$dt = e^x dx \rightarrow (ii)$$

Substituting (i) and (ii) in (A) becomes

$$\int \sin e^x \cdot e^x dx$$

$$\int \sin t \cdot dt = -\cos t + c$$

Putting the value of t from (i)

$$= -\cos e^x + c$$

d) $\int (t+3) \cos(t+3)^2 dt$

Solution: Given integral is

$$\int (t+3) \cos(t+3)^2 dt$$

$$= \int \cos(t+3)^2 \ (t+3) dt \rightarrow (A)$$

By substitution method

Let $u = (t+3)^2 \rightarrow (i)$

Diff w.r.t t

$$\frac{du}{dt} = 2(t+3)\frac{d}{dt}(t+3)$$

$$\frac{du}{dt} = 2(t+3)$$

$$\frac{du}{2} = (t+3)dt \rightarrow (ii)$$

Substituting (i) and (ii) in (A) becomes

$$\int \cos(t+3)^2 \ (t+3) dt$$

$$\int \cos u \frac{du}{2} = \frac{1}{2}\int \cos u \, du$$

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$$= \frac{1}{2}\sin u + c$$

Putting the value of u from (i)

$$= \frac{1}{2}\sin(t+3)^2 + c$$

Q.2 Evaluate the following indefinite integrals by method of substitution.

a) $\int \tan 2x \sec 2x\,dx = \int \sec 2x \tan 2x\,dx$

Solution: The given integral is

$$\int \sec 2x \tan 2x\,dx \rightarrow (A)$$

By substitution method

Let $t=2x \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx} = 2$$

$$\frac{dt}{2} = dx \rightarrow (ii)$$

Substituting (i) and (ii) then (A) becomes

$$\int \tan 2x \sec 2x\,dx$$

$$\int \sec t \tan t \frac{dt}{2}$$

$$= \frac{1}{2}\int \sec t \tan t\,dt$$

$$= \frac{1}{2}\sec t + c$$

Substituting the value of t from (i)

$$= \frac{1}{2}\sec 2x + c$$

b) $\int 4\sec^2 4x\,dx$

Solution: The given integral is

$$\int 4\sec^2 4x\,dx \rightarrow (A)$$

By substitution method

Let $t=4x \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx} = 4$$

$$dt = 4dx \rightarrow (ii)$$

Substituting (i) and (ii) in (A) becomes

$$\int \sec^2 4x.4dx$$

$$\int \sec^2 t\,dt = \tan t + c$$

Substituting the value of t from (i)

$$= \tan 4x + c$$

c) $\int \tan x \sec^2 x\,dx$

Solution: The given integral is

$$\int \tan x \sec^2 x\,dx \rightarrow (A)$$

By substitution method

Let $t=\tan x \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx} = \sec^2 x$$

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$$dt = \sec^2 x\,dx \rightarrow (ii)$$

Substituting (i) and (ii), in (A) becomes

$$\int t \cdot dt = \frac{t^2}{2} + c$$

Substituting the value of t from (i)

$$= \frac{\tan^2 x}{2} + c$$

d) $\int (\tan 3x + \sec 3x)\,dx$

Solution: The given integral is

$$\int (\tan 3x + \sec 3x)\,dx \rightarrow (A)$$

By substitution method

Let $t = 3x \rightarrow (i)$

$$\frac{dt}{dx} = 3$$

$$\frac{dt}{3} = dx \rightarrow (ii)$$

Substituting (i) and (ii) in (A) becomes

$$\int (\tan t + \sec t)\frac{dt}{3} = \frac{1}{3}\int \tan dt + \frac{1}{3}\int \sec t dt \rightarrow (B)$$

We will solve the two integrate separately.

$$\frac{1}{3}\int \tan dt = \frac{1}{3}\int \frac{\sin t}{\cos t} dt \rightarrow (C)$$

Let $u = \cos t \rightarrow (iii)$

$$\frac{du}{dt} = -\sin t$$

$$-du = +\sin t\,dt \rightarrow (iv)$$

Substitute (iii) and (iv) in (C)

$$\frac{1}{3}\int \frac{-du}{u} = \frac{-1}{3}\int \frac{1du}{u} = \frac{-1}{3}|\ln u| + c_1$$

Substituting the value of u from (iii)

$$\frac{1}{3}\int \tan t dt = \frac{-1}{3}\ln(\cos t) + c_1 \rightarrow (D)$$

$$\frac{1}{3}\int \sec t dt$$

Multiplying and dividing by Sect+tant

$$= \frac{1}{3}\int \sec t \frac{\sec t + \tan t}{\sec t + \tan t} dt \rightarrow (E)$$

$$= \frac{1}{3}\int \frac{\sec^2 t + \sec t \tan t}{\sec t + \tan t} dt$$

By substitution method

Let $r = \sec t + \tan t \rightarrow (v)$

Diff w.r.t t

$$\frac{dr}{dt} = \sec t \tan t + \sec^2 t$$

$$dr = (\sec t \tan t + \sec 2t)\,dt \rightarrow (vi)$$

Substituting (v) and (vi) in (E) becomes

$$= \frac{1}{3}\int \frac{dr}{r} = \frac{1}{3}\ln(r) + c_2$$

Substituting the value of r from (v)

$$\frac{1}{3}\int \sec t dt = \frac{1}{3}\ln|\sec t + \tan t| + c_2 \rightarrow (F)$$

Putting (D) and (F) in (B)

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$$\frac{1}{3}\int(\sec t+\tan t)dt$$

$$=\frac{1}{3}\ln|\sec t+\tan t|\frac{-1}{3}\ln|\cos t|+c_1+c_2$$

$$=\frac{1}{3}\ln\left|\frac{\sec t+\tan t}{\cos t}\right|+c$$

Putting the value of t from (i)

$$\int(\tan 3x+\sec 3x)dx=\frac{1}{3}\ln\left|\frac{\sec 3x+\tan 3x}{\cos 3x}\right|+c$$

e) $\int\frac{\cos^2 x}{\operatorname{cosec} x}dx$

$$\int\cos^2 x\frac{1}{\operatorname{cosec} x}dx=$$

$$\int\cos^2 x\sin dx\rightarrow(A)$$

Solution:

By substitution method

Let $t=\cos x\rightarrow(i)$

Diff w.r.t x

$$\frac{dt}{dx}=-\sin x$$

$$-dt=\sin x dx\rightarrow(ii)$$

Substituting (i) and (ii) in (A) becomes

$$\int t^2(-dt)=-\int t^2dt=\frac{-t^3}{3}+c$$

Substituting the value of t

$$\int\frac{\cos^2 x}{\operatorname{cosec} x}dx=-\frac{\cos^3 x}{3}+c$$

f) $\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx\rightarrow(A)$

Solution:

By substitution method

Let $t=\sqrt{x}\rightarrow(i)$

$$\frac{dt}{dx}=\frac{1}{2\sqrt{x}}$$

$$2dt=\frac{1}{\sqrt{x}}dx\rightarrow(ii)$$

Substituting (i) and (ii) then (A) becomes

$$\int\cot t(2dt)=2\int\cot t\,dt=2\int\frac{\cos t}{\sin t}dt\rightarrow(B)$$

Again substitution method

Let $u=\sin t\rightarrow(iii)$

Diff w.r.t x

$$\frac{du}{dt}=\cos t$$

$$du=\cos t dt\rightarrow(iv)$$

Substitute (iii) and (iv) in (B)

$$2\int\frac{du}{u}=2\ln|u|+c$$

Putting the value of u from (iv)

$$=2\ln|\sin t|+c$$

Substituting the value of t from (i)

$$\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx=2\ln|\sin\sqrt{x}|+c$$

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g) $\int \frac{\sin x - \cos x}{\sin x + \cos x} dx \rightarrow (A)$

Solution:

By substitution method

Let $t = \sin x + \cos x \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx} = \cos x - \sin x = -(\sin x - \cos x)$$
$$-dt = (\sin x - \cos x)dx \rightarrow (ii)$$

Substituting (i) and (ii) then (A) becomes

$$\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$$
$$\int \frac{-dt}{t} = -\int \frac{1}{t} dt = -\ln t + c$$

Substituting the value of t from (i)

$$\int \frac{\sin x - \cos x}{\sin x + \cos x} dx = -\ln|\sin x + \cos x| + c$$

h) $\int \frac{\sin x}{3 + 2\cos x} dx \rightarrow (A)$

Solution:

By substitution method

Let $t = 3 + 2\cos x \rightarrow (i)$

$$\frac{dt}{dx} = \frac{d}{dx}(3 + 2\cos x)$$
$$\frac{dt}{dx} = -2\sin x\, dx$$
$$\frac{dt}{-2} = \sin x\, dx \rightarrow (ii)$$

Substituting (i) and (ii) in (A)

$$\int \frac{dt}{-2}\frac{1}{t} = \frac{-1}{2}\int \frac{1}{t} dt = \frac{-1}{2}\ln t + c$$

Substituting the value of t from (i)

$$\int \frac{\sin x}{3 + 2\cos x} dx = \frac{-1}{2}\ln|3 + 2\cos x| + c$$

Q.3 Use suitable substitutions and tables to evaluate the following indefinite integrals.

a) $\int \frac{dx}{x^2 + 16} \rightarrow (A)$

Solution: Since this is the form of $\frac{1}{a^2 + x^2}$

Let $x = 4\tan\theta \rightarrow (i)$

Diff w.r.t θ

$$\frac{dx}{d\theta} = 4\frac{d}{d\theta}(\tan\theta)$$

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$\frac{dx}{d\theta} = 4\sec^2\theta$

$dx = 4\sec^2\theta d\theta \rightarrow (ii)$

Substitute (i) and (ii) then (A) becomes

$$\int \frac{dx}{x^2+16} = \int \frac{4\sec^2\theta d\theta}{16\tan^2\theta+16} = \int \frac{4\sec^2\theta d\theta}{16(\tan^2\theta+1)} = \int \frac{4\sec^2\theta d\theta}{16\sec^2\theta}$$

$$= \frac{1}{4}\int 1 d\theta = \frac{1}{4}\theta + c \rightarrow (B)$$

From (1) x=4 tanθ

$\frac{x}{4} = \tan\theta$

$\theta = \tan^{-1}\frac{x}{4}$

Substituting in (B)

$\frac{\theta}{4} + c \rightarrow (B)$

$= \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$

OR

$$\int \frac{dx}{x^2+16} = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$$

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b) $\int \frac{\sin x}{\cos^2 x + 1} dx \rightarrow (A)$

Solution:

Integration by substitution method

Let t=cosx→(i)

Diff w.r.t x

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$$\frac{dt}{dx} = \frac{d}{dx}(\cos x)$$

$$\frac{dt}{dx} = -\sin x$$

$$-dt = \sin x\, dx \rightarrow (ii)$$

Substituting (i) and (ii) then (A) becomes

$$\int \frac{\sin x}{\cos^2 x + 1} dx = \int \frac{-dt}{t^2 + 1} = -\int \frac{1}{t^2 + 1} dt = -\tan^{-1}(t) + c$$

Substituting the value of t from (i)

$$\boldsymbol{\int \frac{\sin x}{\cos^2 x + 1} dx = -\tan^{-1}(\cos x) + c}$$

c) $\boldsymbol{\int \frac{dx}{\sqrt{5 - 2x^2}} \rightarrow (A)}$

$$\int \frac{dx}{\sqrt{5 - 2x^2}} = \int \frac{dx}{\sqrt{5\left(1 - \frac{2}{5}x^2\right)}}$$

Now we can write

$$\int \frac{1}{\sqrt{5}\sqrt{\left(1 - \frac{\sqrt{2}}{\sqrt{5}}x\right)^2}} dx \ldots\ldots\ldots\ldots (1)$$

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Integration by substitution method

$$\frac{\sqrt{2}}{\sqrt{5}}x = \sin\theta \ldots\ldots\ldots\ldots (2)$$

Diff w.r.t θ

$$\frac{\sqrt{2}}{\sqrt{5}}\frac{dx}{d\theta} = \cos\theta \qquad \Rightarrow \frac{\sqrt{2}}{\sqrt{5}} dx = \cos\theta d\theta$$

$$dx = \frac{\sqrt{5}}{\sqrt{2}} \cos\theta d\theta \ldots\ldots\ldots\ldots (3)$$

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Substituting (2) and (3) and (1)

$$\int \frac{dx}{\sqrt{5}\sqrt{\left(1-\frac{\sqrt{2}}{\sqrt{5}}x\right)^2}} = \frac{\sqrt{5}}{\sqrt{2}}\int \frac{1.\cos\theta d\theta}{\sqrt{5}\sqrt{1-\sin^2\theta}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{\cancel{\cos\theta} d\theta}{\cancel{\cos\theta}} = \frac{1}{\sqrt{2}}\int 1 d\theta = \frac{1}{\sqrt{2}}\theta + c$$

$$\frac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}}\theta + c = \frac{\sqrt{2}}{2}\theta + c \ldots\ldots\ldots\ldots (4)$$

Now from equation (2)

$$\frac{\sqrt{2}}{\sqrt{5}}x = \sin\theta$$

$$\theta = \sin^{-1}\frac{\sqrt{2}}{\sqrt{5}}x$$

$$\theta = \sin^{-1}\frac{\sqrt{10}}{5}x$$

$$\int \frac{\mathbf{dx}}{\sqrt{\mathbf{5-2x^2}}} = \frac{\sqrt{\mathbf{2}}}{\mathbf{2}}\mathbf{\sin^{-1}}\frac{\sqrt{\mathbf{10}}\mathbf{x}}{\mathbf{5}} + \mathbf{c}$$

d) $\int \frac{\mathbf{dx}}{\sqrt{\mathbf{e^{2x}-4}}}$

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Integration by substitution method

Let $t=e^x \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx} = \frac{d}{dx}e^x$$

$$\frac{dt}{dx} = e^x$$

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$$\frac{dt}{e^x} = dx \Rightarrow \frac{dt}{t} = dx \rightarrow (ii)$$

Substituting (i) and (ii) in (A)

$$\int \frac{dx}{\sqrt{e^{2x}-4}} = \int \frac{dt}{t\sqrt{t^2-4}} = \int \frac{dt}{t\sqrt{t^2-(2)^2}}$$

As $\int \frac{1}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\sec -1\left(\frac{x}{a}\right)+c$

So, $\int \frac{dt}{t\sqrt{t^2-(2)^2}} = \frac{1}{2}\sec^{-1}\left(\frac{t}{2}\right)+c$

Substituting the value of t from (i) $t = e^x$

$$\int \frac{dx}{t\sqrt{e^{2x}-4}} dx = \frac{1}{2}\sec^{-1}\left(\frac{e^x}{2}\right)+c$$

e) $\int \frac{2x+5}{x^2+4x+5} dx$

$$= \int \frac{2x+4+1}{x^2+4x+1+4} dx$$

$$= \int \frac{2(x+2)+1}{(x+2)^2+1} dx = \int \frac{2(x+2)}{(x+2)^2+1} dx + \int \frac{1}{(x+2)^2+1} dx \rightarrow (A)$$

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Integration by substitution method

Let t=x+2→(i)

Diff w.r.t x

$$\frac{dt}{dx} = 1$$

dt=dx→ (ii) Substituting (i) and (ii) in (A)

$$\int \frac{2t}{t^2+1} dt + \int \frac{1}{t^2+1} dt \rightarrow (B)$$

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Evaluating the two integrals separately

$$\int \frac{2t}{t^2+1} dt \rightarrow (B)$$

Using substitution method

Let $u=t^2+1 \rightarrow$ (iii)

Diff w.r.t x

$$\frac{du}{dt} = 2t \Rightarrow dx = 2tdt \rightarrow (iv)$$

Substituting (iii) and (iv) in (B)

$$\int \frac{2t}{t^2+1} dt = \int \frac{du}{u} = \ln|u| + c_1$$ Putting the value of u from (iii)

$$\int \frac{\mathbf{2t}}{\mathbf{t^2+1}} \mathbf{dt} = \mathbf{\ln|t^2+1| + c_1} \ldots\ldots\ldots\ldots (1)$$

$$\int \frac{1}{t^2+1} dt = \tan^{-1}(t) + c_2$$ putting in (B)

$$\int \frac{2t}{t^2+1} dt + \int \frac{1}{t^2+1} dt = \ln(t^2+1) + c_1 + \tan^{-1}(t) + c_2$$

$$\int \frac{2t}{t^2+1} dt + \int \frac{1}{t^2+1} dt = \ln(t^2+1) + \tan^{-1}(t) + c_1 + c_2$$

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$$\int \frac{2t}{t^2+1} dt + \int \frac{1}{t^2+1} dt = \ln(t^2+1) + \tan^{-1}(t) + c$$

Substituting the value of t form (i)

$$\int \frac{2x+5}{x^2+4x+5} dx = \ln|(x+2)^2+1| + \tan^{-1}(x+2) + c$$

$$\int \frac{\mathbf{2x+5}}{\mathbf{x^2+4x+5}} \mathbf{dx} = \mathbf{\ln|x^2+4x+5| + \tan^{-1}(x+2) + c}$$

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f) $\int \frac{2+x}{\sqrt{4-2x-x^2}}dx$

$$\int \frac{(2+x)dx}{\sqrt{4-2x-x^2}} = \int \frac{(x+1+1)dx}{\sqrt{4-(2x+x^2)}} = \int \frac{(x+1+1)dx}{\sqrt{4+1-(x^2+2x+1)}}$$

$$= \int \frac{(x+1+1)dx}{\sqrt{5-(x+1)^2}} = \int \frac{(x+1)dx}{\sqrt{5-(x+1)^2}} + \int \frac{1}{\sqrt{5-(x+1)^2}}dx \rightarrow (i)$$

Let $t=x+1$

$\frac{dt}{dx}=1 \qquad \Rightarrow dt=dx$ putting in (i)

$$= \int \frac{t\,dt}{\sqrt{5-t^2}} + \int \frac{dt}{\sqrt{5-t^2}} \rightarrow (ii)$$

For $\int \frac{t\,dt}{\sqrt{5-t^2}} \rightarrow (iii)$

Let $u=5-t^2$

$\frac{du}{dt}=-2t \qquad \Rightarrow \frac{du}{-2}=tdt$, Substituting in (iii)

$$\int \frac{du}{-2\sqrt{u}} = \frac{-1}{2}\int u^{\frac{-1}{2}}du = \frac{-1}{2}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}} = -u^{\frac{1}{2}}+c_1$$ Putting value of u

$$\int \frac{tdt}{\sqrt{5-t^2}} = -(5-t^2)^{\frac{1}{2}}+c_1 \rightarrow (iv)$$

For $\int \frac{dt}{\sqrt{5-t^2}} = \int \frac{dt}{\sqrt{(\sqrt{5})^2-t^2}} = \sin^{-1}\left(\frac{t}{\sqrt{5}}\right)+c_2 \rightarrow (v)$

Putting $t=x+1$ in (iv) and (v)

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$$\int \frac{(x+1)dx}{\sqrt{5-(x+1)^2}} = -\left(5-(x+1)^2\right)^{\frac{1}{2}} + c_1 \text{ and } \int \frac{dx}{\sqrt{5-(x+1)^2}} = \sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c_2$$

Putting in (i)

$$\int \frac{(2+x)dx}{\sqrt{4-2x-x^2}} = -\left(5-(x+1)^2\right)^{\frac{1}{2}} + c_1 + \sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c_2$$

$$\int \frac{(2+x)dx}{\sqrt{4-2x-x^2}} = \sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) - \left(5-(x+1)^2\right)^{\frac{1}{2}} + c, \qquad c_1+c_2=c$$

g) $\int \frac{dx}{x\sqrt{7x^2-5}}$

$$= \frac{1}{\sqrt{7}} \int \frac{dx}{x\sqrt{x^2 - 5/7}}$$

$$= \frac{1}{\sqrt{7}} \int \frac{dx}{x\sqrt{x^2 - \left(\sqrt{5/7}\right)^2}}$$

$$= \frac{1}{\sqrt{7}} \frac{1}{\sqrt{5/7}} \operatorname{cosec}^{-1}\left(\frac{x}{\sqrt{5/7}}\right) + c$$

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$$= \frac{1}{\sqrt{5}} \operatorname{cosec}^{-1}\left(\sqrt{\frac{7}{5}}x\right) + c$$

$$= \frac{\sqrt{5}}{5} \operatorname{cosec}^{-1}\left(\frac{\sqrt{35}}{5}x\right) + c$$

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Integration by part:

For the product to two functions, f(x) and g(x);

$$\frac{d}{dx}\big(f(x)\cdot g(x)\big)=f(x)\frac{d}{dx}\big(g(x)\big)+g(x)\frac{d}{dx}\big(f(x)\big)$$

$$\frac{d}{dx}\big(f(x)\cdot g(x)\big)=f(x)g'(x)+g(x)f'(x)$$

$$f(x)g'(x)=\frac{d}{dx}\big(f(x)g(x)\big)-g(x)\cdot f'(x)$$

Taking integral on both sides

$$\int f(x)g'(x)dx=\int\frac{d}{dx}\big(f(x)g(x)\big)dx-\int g(x)\cdot f'(x)dx$$

$$\int f(x)g'(x)dx=f(x)g(x)-\int g(x)f'(x)dx \quad \rightarrow (i)$$

Let U=f(x) and V=g'(x) then

$$\frac{dU}{dx}=f'(x) \text{ and } \int Vdx=g(x)$$

Substituting in (i) we get

$$\int UVdx=U\int Vdx-\int\left(\frac{d}{dx}(U)\int Vdx\right)dx \quad \rightarrow (ii)$$

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The integral form in equation (ii) is called integration by part formula.

Integration by partial fractions:

The expression of the form $f(x)=\frac{P(x)}{D(x)}$ is called a rational function. If order of P(x) is smaller than order of D(x), then it called a proper rational function, otherwise it is called an improper rational function. To integrate a

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rational function it is usually useful to decompose the rational into simpler rational functions called the partial fractions. The partial fractions of rational function.

$$\mathbf{f(x) = \frac{2x}{x^2 - 4}} \text{ are;}$$

$$\frac{1}{x-2} \text{ and } \frac{1}{x+2} \text{ such that}$$

$$\mathbf{\frac{2x}{x^2 - 4} = \frac{1}{x-2} + \frac{1}{x+2}}$$

The partial fraction of a proper rational function can be obtained by fractionizing the denominator and taking partial fraction for each factor in denominator.

The rules for partial fraction decomposition are;

1. **Convert the improper fraction to a proper fraction by division, such as;**

$$\frac{x^4 + 2x^3 - 4x^2 + x - 3}{x^2 - x - 2} = \left(x^2 + 3x + 1\right) + \frac{8x - 1}{x^2 - x - 2}$$

2. **Factorize the denominator of proper rational function of**

$$\frac{8x-1}{x^2 - x - 2} = \frac{8x-1}{(x-2)(x+1)}$$

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3. **Represent the function as a sum of partial fraction with numerator polynomial one degree less than the denominator.**

$$\frac{8x-1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

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EXERCISE 5.3

Q.1 Evaluate the indefinite integral after decomposing the following rational functions into partial fractions.

a) $\int \frac{1}{x(x-3)}dx$

Solution: The give rational function is $\frac{1}{x(x+3)}$ it is a proper rational fraction, So

$$\frac{1}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3} \rightarrow (i)$$ *Multiplying both sides by x(x–3)*

$1 = A(x-3) + B(x) \rightarrow$ (ii)

To find A, put x=0 in (ii)

$1 = A(0-3) + B(0)$

$1 = -3A$

$A = \frac{-1}{3}$ *Now, to find B, put x–3=0, i.e. x=3 in (ii)*

$1 = A(3-3) + B(3)$

$1 = 3B$

$B = \frac{1}{3}$ *Put values of A and B in eq. (i)*

$$\frac{1}{x(x-3)} = \frac{-1/3}{x} + \frac{1/3}{x-3}$$ *Taking integral on both sides*

$$\int \frac{1}{x(x-3)} = \int \left(\frac{-1}{3x} + \frac{1}{3(x-3)} \right) dx = \int \frac{-1}{3} x dx + \frac{1}{3} \int \frac{1}{x-3} dx$$

$$= \frac{-1}{3} \ln x + \frac{1}{3} \ln |x-3| + c = \frac{1}{3} [\ln |x-3| - \ln x]$$

$$= \frac{1}{3} \ln \left| \frac{x-3}{x} \right| + c$$

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b) $\int \frac{3x^2+2x-1}{x(x+1)}dx$

Solution: The given rational function is $\frac{3x^2+2x-1}{x(x+1)}$

It is not proper rational fraction, so we make is proper RF.

$$\frac{3x^2+2x-1}{x(x+1)} = \frac{3x^2+3x-x-1}{x(x+1)}$$

$$\frac{3x^2+2x-1}{x(x+1)} = \frac{3x(x+1)-1(x+1)}{x(x+1)} = \frac{(3x-1)(x+1)}{x(x+1)} = \frac{3x-1}{x} = 3-\frac{1}{x}$$

Integrating both sides

$$\int \frac{3x^2+2x-1}{x(x+1)}dx = \int\left(3-\frac{1}{x}\right)dx = \int 3dx - \int \frac{1}{x}dx$$

$$= 3x - \ln x + c$$

c) $\int \frac{4x^3+4x^2+x-1}{x^2(x+1)^2}dx$

Solution: The given fraction is $\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}$ since this is a proper fraction, so we put

$$\frac{4x^3+4x^2+x-1}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2} \rightarrow (i)$$

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Multiplying both sides by $x^2(x+1)^2$.

$$4x^3+4x^2+x-1 = Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2 \rightarrow (ii)$$

Put x=0 in (ii)

$$4(0)^3+4(0)^2+0-1 = A(0)(0+1)^2 + B(0+1)^2 + C(0)^2(0+1) + D(0)^2$$

$$-1 = B$$

B=–1

Putting x=–1 in (ii)

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$4(-1)^3+4(-1)^2+(-1)-1=A(-1)(-1+1)^2+B(-1+1)^2+C(-1)^2(-1+1)+D(-1)^2$

$-2=D$

D=–2

To find C and D, we have to expand (ii)

$4x^3+4x^2+x-1=Ax(x^2+2x+1)+B(x^2+2x+1)+Cx^2(x+1)+Dx^2$

$4x^3+4x^2+x-1=Ax^3+2Ax^2+Ax+Bx^2+2Bx+B+Cx^3+Cx^2+Dx^2$

$4x^3+4x^2+x-1=(A+C)x^3+(2A+B+C+D)x^2+(A+2B)x+B$

Equating coefficients on both sides

A+2B=1 but B=–1

A+2(–1)=1

A=3

Similarly, A+C=4 but A=3

3+C=4

C=1

Putting values of A,B,C and D in (i)

$\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}=\frac{3}{x}+\frac{-1}{x^2}+\frac{1}{x+1}+\frac{-2}{(x+1)^2}$ Taking integral on both sides

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$\int\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}dx=\int\left(\frac{3}{x}-x^{-2}+\frac{1}{x+1}-2(x+1)^{-2}\right)dx$

$=\int\frac{3}{x}dx-\int x^{-2}dx+\int\frac{dx}{x+1}-\int 2(x+1)^{-1}dx$

$=3\ln x-\frac{x^{-1}}{-1}+\ln(x+1)-2\frac{(x+1)^{-1}}{-1}+c$

$$\mathbf{=3\ln x+\ln(x+1)+\frac{1}{x}+\frac{2}{x+1}+c}$$

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d) $\int \frac{1}{x^3-1}dx$

Given fraction is $\frac{1}{x^3-1}$ which is in proper form

$$\frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}=\frac{A}{(x-1)}+\frac{Bx+C}{(x^2+x+1)} \rightarrow (i)$$

Multiplying by $(x-1)\ (x^2+x+1)$

$1=A(x^2+x+1)+(Bx+C)(x-1) \rightarrow$ (ii)

Put $x=1$ in (ii) to find A,

$1=A(1^2+1+1)+(B(1)+C)\ (1-1)$

$1=3A$

$$A=\frac{1}{3}$$

To find B and C, we will expand (ii)

$1=Ax^2+Ax+A+Bx^2-Bx+Cx-C$

$1=(A+B)x^2+(A-B+C)x+(A-C)$

Equating coefficients on both sides of x^2

$A+B=0$ Put $A=\frac{1}{3}$

$\frac{1}{3}+B=0$

$$B=-\frac{1}{3}$$

Equating coefficients of x

and $A-B+C=0$ put $A=\frac{1}{3}$ and $B=-\frac{1}{3}$

$\frac{1}{3}+\frac{1}{3}+C=0$

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$$C = \frac{-2}{3}$$

Putting values of A,B and C in (i)

$$\frac{1}{x^3-1} = \frac{1/3}{x-1} + \frac{-1/3\,x - 2/3}{x^2+x+1}$$

Taking integral on both sides

$$\int \frac{1}{x^3-1}dx = \frac{1}{3}\int\left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1}\right)dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{6}\int \frac{2x+4}{x^2+x+1}dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{6}\int \frac{2x+1+3}{x^2+x+1}dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{6}\int\left(\frac{2x+1}{x^2+x+1} + \frac{3}{x^2+x+1}\right)dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{6}\int \frac{2x+1}{x^2+x+1}dx - \frac{3}{6}\int \frac{1}{x^2+x+1}dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{6}\ln(x^2+x+1) - \left(\frac{1}{2}\right)\int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{6}\ln(x^2+x+1) - \left(\frac{1}{2}\right)\int \frac{1}{\left(x+1/2\right)^2+\left(\sqrt{3/4}\right)^2}dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{6}\ln(x^2+x+1) - \left(\frac{1}{2}\right)\frac{1}{\sqrt{3/4}}\tan^{-1}\left[\frac{x+1/2}{\sqrt{3/4}}\right]$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right)\right) + c$$

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e) $\int \frac{x^4 - x^2 + 2}{x^2(x-1)} dx$

The given ration function $\frac{x^4 - x^2 + 2}{x^2(x-1)}$ is not a proper function so, we first make it proper RF.

Division:

$$x^3 - x^2 \;\Big|\; \begin{array}{l} x+1 \\ \overline{x^4 - x^2 + 2} \\ \pm x^4 \mp x^3 \\ \overline{x^3 - x^2 + 2} \\ \pm x^3 \mp x^2 \\ \overline{2} \end{array}$$

Thus $\frac{x^4 - x^2 + 2}{x^2(x-1)} = (x+1) + \frac{2}{x^2(x-1)} \rightarrow (i)$

Let $\frac{2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$ Multiply by $x^2(x-1)$

$2 = A(x)(x-1) + B(x-1) + Cx^2 \rightarrow (ii)$

To find B, put x=0 in (ii)

$2 = A(0)(0-1) + B(0-1) + C(0)^2 \Rightarrow -B = 2$

B=–2

To find C, put x–1=0 $\Rightarrow$ x=1 in (ii)

$2 = A(1)(1-1) + B(1-1) + C(1)^2 \Rightarrow$ **C=2**

To find A, expand (ii) i.e.

$2 = Ax^2 - Ax + Bx - B + Cx^2$

$2 = (A+C)x^2 + (B-A)x - B$

Equating coefficient on both sides

A+C=0 put C=2

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$A+2=0 \quad \Rightarrow \quad \mathbf{A=-2}$

Putting value of A, B and C in (i)

$$\because \frac{2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$\frac{x^4-x^2+2}{x^2(x-1)} = x+1-\frac{2}{x}-\frac{2}{x^2}+\frac{2}{x-1}$$

Taking integral

$$\int \frac{x^4-x^2+2}{x^2(x-1)}dx = \int\left(x+1-\frac{2}{x}-2x^{-2}+\frac{2}{x-1}\right)dx$$

$$\int \frac{x^4-x^2+2}{x^2(x-1)}dx = \int (x+1)dx - \int \frac{2}{x}dx - \int 2x^{-2}dx + \int \frac{2}{x-1}dx$$

$$\int \frac{x^4-x^2+2}{x^2(x-1)}dx = \frac{x^2}{2}+x-2\ln x - 2\frac{x^{-1}}{-1}+2\ln(x-1)+c$$

$$\int \frac{x^4-x^2+2}{x^2(x-1)}dx = \frac{x^2}{2}+x-2\ln x + \frac{2}{x}+2\ln(x-1)+c$$

$$\int \frac{\mathbf{x^4-x^2+2}}{\mathbf{x^2(x-1)}}\mathbf{dx} = \left[\frac{\mathbf{x^2}}{\mathbf{2}}+\mathbf{x}+\mathbf{2\ln(x-1)}-\mathbf{2\ln(x)}\right]$$

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Q.2 Evaluate the following integrals through partial fractions decompositions.

a) $\int \frac{dx}{x^2-1}$

Solution: Given the fraction is $\frac{1}{x^2-1}$ which is in proper form.

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1}+\frac{B}{x+1} \rightarrow (i)$$

Multiplying (i) by (x–1)(x+1)

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$1=A(x+1)+B(x-1) \rightarrow$ (ii)

To find A, put $x-1=0 \Rightarrow x=1$ in (ii)

$1=A(1+1)+B(1-1)$

$2A=1 \Rightarrow \mathbf{A=\frac{1}{2}}$

To find B, put $x+1=0 \Rightarrow x=-1$ in (ii)

$1=A(-1+1)+B(-1-1)$

$-2B=1 \Rightarrow \mathbf{B=\frac{-1}{2}}$

Putting values of A and B in (i)

$$\frac{1}{x^2-1}=\frac{1/2}{x-1}+\frac{-1/2}{x+1}$$

Taking integration of both sides

$$\int\frac{1}{x^2-1}dx=\int\left[\frac{1}{2(x-1)}-\frac{1}{2(x+1)}\right]dx$$

$$\int\frac{1}{x^2-1}dx=\frac{1}{2}\int\frac{1}{x-1}dx-\frac{1}{2}\int\frac{1}{x+1}dx$$

$$\int\frac{1}{x^2-1}dx=\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+c$$

$$\mathbf{\int\frac{1}{x^2-1}dx=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+c}$$

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b) $\mathbf{\int\frac{3x+5}{x^2+2x-3}dx}$

Solution: Given the fraction is $\frac{3x+5}{x^2+2x-3}$, which is in proper form.

(By factorization method)

$$\mathbf{\frac{3x+5}{x^2+2x-3}=\frac{3x+5}{(x+3)(x-1)}=\frac{A}{x+3}+\frac{B}{x-1}} \rightarrow \text{(i)}$$

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Multiplying (i) by $(x+3)(x-1)$

$$3x+5=A(x-1)+B(x+3) \rightarrow \text{(ii)}$$

To find A, put $x+3=0 \Rightarrow x=-3$ in (ii)

$$3(-3)+5=A(-3-1)+B(-3+3)$$

$$-4=-4A \quad \Rightarrow \quad \mathbf{A=1}$$

To find B, put $x-1=0 \Rightarrow x=1$ in (ii)

$$3(1)+5=A(1-1)+B(1+3)$$

$$4B=8 \Rightarrow \quad \mathbf{B=2}$$

Putting values of A and B in (i)

$$\frac{3x+5}{x^2+2x-3}=\frac{3x+5}{(x+3)(x-1)}=\frac{A}{x+3}+\frac{B}{x-1}$$

$$\frac{3x+5}{x^2+2x-3}=\frac{1}{x+3}+\frac{2}{x-1}$$

Taking integration of both sides

$$\int\frac{(3x+5)}{x^2+2x-3}dx=\int\left(\frac{1}{x+3}+\frac{2}{x-1}\right)dx$$

$$\int\frac{(3x+5)}{x^2+2x-3}dx=\int\frac{1}{x+3}dx+2\int\frac{1}{x-1}dx$$

$$\int\frac{(3x+5)}{x^2+2x-3}dx=\ln|x+3|+2\ln|x-1|+c$$

$$\int\frac{(3x+5)}{x^2+2x-3}dx=\ln|x+3|+\ln|x-1|^2+c$$

$$\mathbf{\int\frac{(3x+5)}{x^2+2x-3}dx=\ln(x+3)(x-1)^2+c}$$

c) $\mathbf{\int\frac{-x-3}{2x^2-x-1}dx}$

Solution: Given the fraction is in proper form. (By factorization method)

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$$\frac{-x-3}{2x^2-x-1}=\frac{-x-3}{2x^2-2x+x-1}=\frac{-x-3}{2x(x-1)+1(x-1)}$$

$$\frac{-x-3}{2x^2-x-1}=\frac{-x-3}{(2x+1)(x-1)}=\frac{A}{2x+1}+\frac{B}{x-1}\rightarrow(i)$$

Multiplying (i) by $(2x+1)(x-1)$

$$-x-3=A(x-1)+B(2x+1)\rightarrow(ii)$$

To find A, put $2x+1=0\Rightarrow x=\frac{-1}{2}$ in (ii)

$$-\left(\frac{-1}{2}\right)-3=A\left(\frac{-1}{2}-1\right)+B\left(2\left(\frac{-1}{2}\right)+1\right)$$

$$\frac{-5}{2}=\frac{-3}{2}A\Rightarrow \mathbf{A=\frac{5}{3}}$$

To find B, put $x-1=0\Rightarrow x=1$ in (ii)

$$-1-3=A(1-1)+B(2(1)+1)$$

$$-4=3B\Rightarrow \mathbf{B=\frac{-4}{3}}$$

Putting values of A and B in (i)

$$\frac{-x-3}{2x^2-x-1}=\frac{5/3}{2x+1}+\frac{-4/3}{x-1}$$

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Taking integration of both sides

$$\int\frac{-x-3}{2x^2-x-1}dx=\int\left(\frac{5/3}{2x+1}+\frac{-4/3}{x-1}\right)dx$$

$$\int\frac{-x-3}{2x^2-x-1}dx=\frac{5}{3}\int\frac{1}{2x+1}dx-\frac{4}{3}\int\frac{1}{x-1}dx$$

$$\int\frac{-x-3}{2x^2-x-1}dx=\frac{5}{3}\cdot\frac{1}{2}\ln|2x+1|-\frac{4}{3}\ln|x-1|+c$$

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$$\int \frac{-x-3}{2x^2-x-1}dx = \frac{5}{6}\ln|2x+1| - \frac{4}{3}\ln|x-1| + c$$

$$\int \frac{-x-3}{2x^2-x-1}dx = \ln|2x+1|^{\frac{5}{6}} - \ln|x-1|^{\frac{4}{3}} + c$$

$$\int \frac{-x-3}{2x^2-x-1}dx = \ln\left|\frac{(2x+1)^{\frac{5}{6}}}{(x-1)^{\frac{4}{3}}}\right| + c$$

d) $\int \frac{x^2-1}{x^2-2x-15}dx$

Solution: Given the fraction $\frac{x^2-1}{x^2-2x-15}$ is not a proper rational function, so by division.

$$x^2-2x-15 \;\overline{\Big|\; \begin{array}{l} 1 \\ x^2-1 \\ \pm x^2 \mp 15 \mp 2x \\ \hline 2x+14 \end{array}}$$

$$\frac{x^2-1}{x^2-2x-15} = 1 + \frac{2x+14}{x^2-2x-15} \rightarrow \quad \text{(i)} \qquad \text{(By factorization method)}$$

Now, $\frac{2x+14}{x^2-2x-15} = \frac{2x+14}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3} \rightarrow$ (ii)

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Multiplying (i) by (x–5)(x+3)

$2x+14=A(x+3)+B(x-5) \rightarrow$ (iii)

To find A, put $x-5=0 \Rightarrow x=5$ in (iii)

$2(5)+14=A(5+3)+B(5-5)$

$24=8A \quad \Rightarrow \quad \mathbf{A=3}$

To find B, put $x+3=0 \Rightarrow x=-3$ in (iii)

$2(-3)+14=A(-3+3)+B(-3-5)$

$8=-8B \quad \Rightarrow \quad \mathbf{B=-1}$

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Putting value of A and B in (ii)

$$\frac{2x+14}{x^2-2x-15}=\frac{3}{x-5}-\frac{1}{x+3} \text{ putting in (i)}$$

Taking integration of both sides

$$\int\frac{2x+14}{x^2-2x-15}dx=\int\left(1+\frac{3}{x-5}-\frac{1}{x+3}\right)dx$$

$$\int\frac{2x+14}{x^2-2x-15}dx=\int 1dx+\int\frac{3}{x-5}dx-\int\frac{1}{x+3}dx$$

$$\int\frac{2x+14}{x^2-2x-15}dx=x+3\ln|x-5|-\ln|x+3|+c$$

$$\int\frac{2x+14}{x^2-2x-15}dx=x+\ln|x-5|^3-\ln|x+3|+c$$

$$\int\frac{2x+14}{x^2-2x-15}dx=x+\ln\left|\frac{(x-5)^3}{(x+3)}\right|+c$$

e) $\int\frac{x^2}{(x+1)^3}dx$

Solution: Given the fraction is $\frac{x^2}{(x+1)^3}$, which is in proper form.

$$\frac{x^2}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}\rightarrow(i)$$

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Multiplying (i) by $(x+1)^3$

$x^2=A(x+1)^2+B(x+1)+C$

$x^2=Ax^2+2Ax+A+Bx+B+C$

$x^2=Ax^2+(2A+B)x+(A+B+C)$

Comparing coefficients on both sides of x^2, x, x^o

For x^2: A=1

For x: 2A+B=0 put **A=1**

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For x^o: $2(1)+B=0 \quad B=-2$

$A+B+C=0$ put $A=1, B=2$

$1-2+C=0 \quad \Rightarrow \quad C=1$

Putting values of A,B and C in (i)

$$\frac{x^2}{(x+1)^3}=\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3} \rightarrow (i)$$

Taking integration of both sides

$$\int\frac{x^2}{(x+1)^3}dx=\int\left(\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}\right)dx=\int\frac{dx}{x+1}-\int\frac{2dx}{(x+1)^2}+\int\frac{dx}{(x+1)^3}$$

$$\int\frac{x^2}{(x+1)^3}dx=\ln|x+1|-2\frac{(x+1)^{-2+1}}{-2+1}+\frac{(x+1)^{-3+1}}{-3+1}+c$$

$$\int\frac{x^2}{(x+1)^3}dx=\mathbf{\ln|x+1|+\frac{2}{x+1}-\frac{1}{2(x+1)^2}+c}$$

f) $\int\frac{x}{(x+1)(x^2+1)}dx$

Solution: Given the fraction is $\frac{x}{(x+1)(x^2+1)}$, which is in proper form.

$$\frac{x}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1} \rightarrow (i)$$

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Multiplying (i) by $(x+1)(x^2+1)$

$x=A(x^2+1)+(Bx+C)(x+1) \rightarrow (ii)$

To find A, put $x+1=0 \Rightarrow x=-1$ in (ii)

$-1=A((-1)^2+1)+(B(-1)+C)(-1+1)$

$-1=2A \quad \Rightarrow \quad \mathbf{A=-\frac{1}{2}}$

Expanding (ii)

$x=Ax^2+A+Bx^2+Bx+Cx+C$

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$x=(A+B)\,x^2+(B+C)x+(A+C)$ Comparing coefficient on both sides

For x^2:

$A+B=0 \qquad$ put $A=\dfrac{-1}{2}$

$\dfrac{-1}{2}+B=0 \quad \Rightarrow \quad \mathbf{B=\dfrac{1}{2}}$

Compare coefficient of x:

$B+C=1 \quad \Rightarrow \quad B=\dfrac{1}{2}$

$\dfrac{1}{2}+C=1 \quad \Rightarrow \quad \mathbf{C=\dfrac{1}{2}}$

Put values of A,B and C in (i) and take integration both sides

$$\int\frac{x}{(x+1)(x^2+1)}dx=\int\left(\frac{-1/2}{x+1}+\frac{1/2\,x+1/2}{x^2+1}\right)dx$$

$$\int\frac{x}{(x+1)(x^2+1)}dx=-\frac{1}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{x}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x^2+1}dx$$

$$\int\frac{x}{(x+1)(x^2+1)}dx=-\frac{1}{2}\ln|x+1|+\frac{1}{2}\cdot\frac{1}{2}\ln|x^2+1|+\frac{1}{2}\tan^{-1}x+c$$

$$\int\frac{x}{(x+1)(x^2+1)}dx=-\frac{1}{2}\ln|x+1|+\frac{1}{4}\ln|x^2+1|+\frac{1}{2}\tan^{-1}x+c$$

$$\int\frac{x}{(x+1)(x^2+1)}dx=\frac{1}{2}\tan^{-1}x+\ln(x^2+1)^{\frac{1}{4}}-\ln(x+1)^{\frac{1}{2}}+c$$

$$\int\frac{x}{(x+1)(x^2+1)}dx=\mathbf{\frac{1}{2}\tan^{-1}x+\ln\left[\frac{(x^2+1)^{\frac{1}{4}}}{(x+1)^{\frac{1}{2}}}\right]+c}$$

g) $\displaystyle\int\frac{x^2+2}{(x^2+1)^2}dx$

Solution: Hence it is proper, irreducible and repeated form

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$$\frac{x^2+2}{(x^2+1)^2}=\frac{Ax+B}{(x^2+1)}+\frac{Cx+D}{(x^2+1)^2}\rightarrow(i)$$

Multiplying (i) by $(x^2+1)^2$

$x^2+2=(Ax+B)(x^2+1)+Cx+D$

$x^2+2=Ax^3+Ax+Bx^2+B+Cx+D$

$x^2+2=Ax^3+Bx^2+(A+C)x+(B+D)$

Comparing coefficient on both sides

For x^3:

A=0

For x^2:

B=1

For x:

A+C=0 put A=0

0+C=0 $\Rightarrow$ C=0

B+D=2 $\Rightarrow$ 1+D=2 $\Rightarrow$ **D=1**

Putting values of A,B,C and D in (i)

$$\int\frac{x^2+2}{(x^2+1)^2}dx=\frac{Ax+B}{(x^2+1)}+\frac{Cx+D}{(x^2+1)^2}$$

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$$\int\frac{x^2+2}{(x^2+1)^2}dx=\int\left(\frac{(0)x+1}{x^2+1}+\frac{(0)x+1}{(x^2+1)^2}\right)dx$$

$$\int\frac{x^2+2}{(x^2+1)^2}dx=\int\frac{1}{x^2+1}dx+\int\frac{1}{(x^2+1)^2}dx$$

$$\int\frac{x^2+2}{(x^2+1)^2}dx=\tan^{-1}x+\int\frac{1}{(x^2+1)^2}dx\rightarrow(ii)$$

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$$\int \frac{1}{(x^2+1)^2} dx \Rightarrow \text{(iii)}$$

Now integration by substitution methods

Let $\mathbf{x=\tan\theta}$

Diff w.r.t x

$$\frac{dx}{d\theta} = \sec^2\theta$$

$$dx = \sec^2\theta d\theta$$

Substituting in (iii)

$$\int \frac{\sec^2\theta d\theta}{(1+\tan^2\theta)^2} = \int \frac{\sec^2\theta}{(\sec^2\theta)^2} = \int \frac{1}{\sec^2\theta} d\theta$$

$$= \int \cos^2\theta d\theta = \int \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c \text{ substituting } \theta = \tan^{-1} x$$

$$= \frac{1}{2}\left[\tan^{-1} x + \frac{\sin\left(2\tan^{-1} x\right)}{2}\right] + c$$

Putting in (ii)

$$\int \frac{x^2+2}{(x^2+1)^2} dx = \tan^{-1} x + \frac{1}{2}\left[\tan^{-1} x + \frac{\sin\left(2\tan^{-1} x\right)}{2}\right] + c$$

$$\int \frac{x^2+2}{(x^2+1)^2} dx = \mathbf{\frac{3}{2}\tan^{-1} x + \frac{1}{4}\sin\left(2\tan^{-1} x\right) + c}$$

h) $\mathbf{\int \frac{x^4+1}{x^4-1} dx}$

Solution: Given function $\frac{x^4+1}{x^4-1}$ is improper, so we make it proper by division.

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$$x^4-1 \;\overline{)\; \begin{array}{r} 1 \\ x^4+1 \\ \pm x^4 \mp 1 \\ \hline 2 \end{array}}$$

Thus; $\dfrac{x^4+1}{x^4-1}=1+\dfrac{2}{x^4-1} \rightarrow (i)$

Now to find the partial fractions

$$\frac{2}{x^4-1}=\frac{2}{(x-1)(x+1)(x^2+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1} \rightarrow (ii)$$

Multiplying (i) by $(x-1)(x+1)(x^2+1)$

$2=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x-1)(x+1) \rightarrow$ (iii)

To find A, put $x-1=0 \Rightarrow x=1$ in (iii)

$2=A(1+1)((1)^2+1)+B(1-1)((1)^2+1)+(Cx+D)(1-1)(1+1)$

$2=4A \quad \Rightarrow \quad \mathbf{A=\frac{1}{2}}$

To find B, put $x+1=0 \Rightarrow x=-1$ in (iii)

$2=A(-1+1)((-1)^2+1)+B(-1-1)((-1)^2+1)+(C(-1)+D)(-1-1)(-1+1)$

$2=-4B \quad \Rightarrow \quad \mathbf{B=\frac{-1}{2}}$

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Expanding (iii)

$2=Ax^3+Ax+Ax^2+A+Bx^3+Bx-Bx^2-B+Cx^3-Cx+Dx^2-D$

$2=(A+B+C)x^3+(A-B+D)x^2+(A+B-C)x+(A-B-D)$

Comparing coefficients

$A+B+C=0$, putting $A=\frac{1}{2}, B=\frac{-1}{2}$

$\frac{1}{2}+\left(\frac{-1}{2}\right)+C=0, \quad \Rightarrow \quad \mathbf{C=0}$

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$A - B + D = 0,$ Putting $A = \frac{1}{2}$ and $B = \frac{-1}{2}$

$\frac{1}{2} - \left(\frac{-1}{2}\right) + D = 0 \quad \Rightarrow \quad \mathbf{D = -1}$

Putting values of A,B,C and D in (ii)

$$\int \frac{x^4+1}{x^4-1}dx = \int \left(1 + \frac{1/2}{x-1} + \frac{-1/2}{x+1} + \frac{0x-1}{x^2+1}\right)dx$$

$$\int \frac{x^4+1}{x^4-1}dx = \int 1dx + \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{1}{x+1}dx - \int \frac{1}{1+x^2}dx$$

$$\int \frac{x^4+1}{x^4-1}dx = \mathbf{x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| - \tan^{-1}x + c}$$

Q.3 Use integral by part to evaluate the following integrals.

a) $\int \mathbf{xe^x dx}$

Solution: For two function u and v of x by part integral is given by:

$$\int uvdx = u\int vdx - \int\left(\int vdx \cdot \frac{d}{dx}(u)\right)dx$$

Here for; $\int xe^x dx$, $u = x$, $v = e^x$

So, $\int \underset{I}{x}\,\underset{II}{e^x}dx = x \cdot \int e^x dx - \int\left(\frac{d}{dx}(x)\int e^x dx\right)dx$

$= x \cdot e^x - \int 1 \cdot e^x dx$

$\mathbf{= xe^x - e^x + c}$

b) $\int \mathbf{xsinxdx}$

Solution: Using integration by parts method

$$\int uvdx = u \cdot \int vdx - \int\left(\int \frac{d}{dx}u \cdot \int vdx\right)dx$$

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Here for; $\int x\sin x dx, u = x, v = \sin x$

Then, $\int \underset{I}{x}\,\underset{II}{\sin x}\,dx = x\cdot\int \sin x dx - \int\left(\frac{d}{dx}(x)\int \sin x dx\right)dx$

$= x(-\cos x) - \int 1\cdot(-\cos x)dx$

$= -x\cos x + \int \cos x dx + c$

$$= -x\cos x + \sin x + c$$

c) $\int \tan^{-1}x dx$

Solution: Taking 1 in product

Here for $\int \tan x^{-1}dx = \int \tan x^{-1}\cdot 1 dx$

$u = \tan^{-1}x,\ v = 1$

$$\int \frac{\tan^{-1}x}{I}\cdot\frac{1}{II}dx = \tan^{-1}x\cdot\int 1dx - \int\left(\frac{d}{dx}\left(\tan^{-1}x\right)\cdot\int 1dx\right)dx$$

$$= \tan^{-1}x.(x) - \int \frac{1}{1+x^2}\cdot x dx$$

Multiplying and Dividing by 2

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{x^2+1}dx$$

$$= x\tan^{-1}x - \frac{1}{2}\ln(x^2+1) + c$$

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d) $\int \sin^{-1}x dx$

Solution: Taking 1 in product

Here for, $\int \sin^{-1}x\cdot dx = \int \sin^{-1}x\cdot 1 dx$

$u = \sin^{-1}x,\ v = 1, \text{So}$

$$\int \frac{\sin^{-1}}{I}x\cdot\frac{1}{II}dx = \sin^{-1}x\cdot\int 1dx - \int\left(\frac{d}{dx}\left(\sin^{-1}x\right)\cdot\int 1dx\right)dx$$

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$$= x\sin^{-1}x - \int \frac{1}{\sqrt{1-x^2}}.xdx$$

$$= x\sin^{-1}x + \int \frac{-x}{\sqrt{1-x^2}}dx$$

$$= x\sin^{-1}x - \int (1-x^2)^{\frac{-1}{2}}.xdx$$

Multiplying and Dividing by 2

$$= x\sin^{-1}x + \frac{1}{2}\int (1-x^2)^{\frac{-1}{2}}.(-2x)dx$$

According to rule

$$\int [f(x)]^n [f'(x)]dx = \frac{f(x)^{n+1}}{n+1}$$

$$= x\sin^{-1}x + \frac{1}{2}\frac{(1-x^2)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + c$$

$$= x\sin^{-1}x + \frac{1}{2}.\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \mathbf{x\sin^{-1}x + \sqrt{1-x^2} + c}$$

e) $\int x^2(x-3)^{11}dx$

Solution: Here for, $u = x^2, v = (x-3)^{11}$, so

$$\int x^2(x-3)^{11} = x^2\int (x-3)^{11}dx - \int\left(\frac{d}{dx}(x^2)\cdot\int (x-3)^{11}dx\right)dx$$

$$= x^2\frac{(x-3)^{12}}{12} - \int 2x\cdot\frac{(x-3)^{12}}{12}dx$$

$$= \frac{x^2}{12}(x-3)^{12} - \frac{1}{6}\int \underset{I}{x}\,\underset{II}{(x-3)^{12}}dx$$

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Again using by parts formula

$$= \frac{x^2(x-3)^{12}}{12} - \frac{1}{6}\left[x\int(x-3)^{12}dx - \int\left(\frac{d}{dx}(x)\int(x-3)^{12}dx\right)dx\right]$$

$$= \frac{x^2(x-3)^{12}}{12} - \frac{1}{6}(x)\frac{(x-3)^{13}}{13} + \frac{1}{6}\int\frac{(x-3)^{13}}{13}dx$$

$$= \frac{1}{12}x^2(x-3)^{12} - \frac{1}{78}x(x-3)^{13} + \frac{1}{78}\cdot\frac{(x-3)^{14}}{14} + c$$

$$\mathbf{= \frac{x^2(x-3)^{12}}{12} - \frac{x(x-3)^{13}}{78} + \frac{(x-3)^{14}}{1092} + c}$$

f) $\mathbf{I = \int e^x \cos x\, dx}$

Solution: Here for, $u = e^x$, $v = \cos x$

$$I = \int e^x \cos dx$$

$$I = e^x\int\cos x\,dx - \int\left(\frac{d}{dx}(e^x)\int\cos x\,dx\right)dx$$

$$I = e^x\sin x - \int\frac{e^x}{I}\frac{\sin x}{II}dx$$

Again using by parts formula

$$I = e^x\sin x - \left[e^x\int\sin x\,dx - \int\frac{d}{dx}(e^x)\int\sin x\,dx\right]dx$$

$$I = e^x\sin x - \left[e^x(-\cos x) - \int e^x(-\cos x)dx\right] = e^x\sin x + e^x\cos x - \int e^x\cos x dx$$

$$I = e^x\sin x + e^x\cos x - I + c$$

$$2I = e^x\sin x + e^x\cos x + c$$

$$I = \frac{1}{2}\left(e^x\sin x + e^x\cos x\right) + \frac{c}{2}$$

$$I = \frac{e^x}{2}\left(\sin x + \cos x\right) + \frac{c}{2}$$

$$\mathbf{I = \frac{e^x}{2}\left(\sin x + \cos x\right) + c}$$

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g) $\int (x+\sin x)^2\,dx = \int (x^2+\sin^2 x+2x\sin x)\,dx$

Solution:

$$= \int x^2 dx + 2\int \underset{I}{x}\,\underset{II}{\sin x}\,dx + \int \sin^2 x\,dx$$

Using integral by parts

$$= \frac{x^3}{3} + 2\left[x\int \sin x dx - \int\left(\frac{d}{dx}(x)\int \sin x\,dx\right)dx\right] + \int \frac{1-\cos 2x}{2}dx$$

$$= \frac{x^3}{3} + 2\left[x(-\cos x) - \int 1(-\cos x)dx\right] + \frac{1}{2}\int (1-\cos 2x)dx$$

$$= \frac{x^3}{3} - 2x\cos x + 2\sin x + \frac{1}{2}x - \frac{1}{2}\cdot\frac{\sin 2x}{2} + c$$

$$\mathbf{= \frac{x^3}{3} - 2x\cos x + 2\sin x + \frac{x}{2} - \frac{1}{4}\sin 2x + c}$$

h) $\int e^{2x}\sqrt{1-e^x}\,dx \rightarrow$ **(A)**

Solution:

First using integration by substitution

Let $t = 1-e^x \rightarrow (i)$

$e^x = 1-t \rightarrow (ii)$

Diff w.r.t x

$$\frac{d}{dx}e^x = \frac{d}{dx}(i) - \frac{d}{dx}(t)$$

$$\frac{dt}{dx} = -e^x$$

$$\frac{dt}{dx} = -(1-t)$$

$$\frac{dt}{-(1-t)} = dx \rightarrow (iii)$$

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Substituting (i)(ii) and (iii) in (A)

$$\int (1-t)^2\sqrt{t}.\left(\frac{-dt}{(1-t)}\right)$$

$$= -\int (1-t)\sqrt{t}\,dt = -\int (1-t)t^{\frac{1}{2}}dt$$

$$= -\int t^{1/2}dt + \int t^{3/2}dt = -\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1}$$

$$= -\int (t^{1/2} - t^{3/2})\,dt = -\left[\frac{t^{3/2}}{3/2} - \frac{t^{5/2}}{5/2}\right] + c$$

$$\mathbf{= -\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + c}$$

Put $t = 1 - e^x$

$$= -\frac{2}{3}(1-e^x)^{3/2} + \frac{2}{5}(1-e^x)^{5/2} + c$$

$$= (1-e^x)^{3/2}\left[-\frac{2}{3} + \frac{2}{5}(1-e^x)\right] + c$$

$$= (1-e^x)^{3/2}\left[\frac{-10+6-6e^x}{15}\right] + c$$

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$$\mathbf{= -\frac{2}{15}(1-e^x)^{3/2}\left[3e^x + 2\right] + c}$$

i) $\int \mathbf{x\sin x\cos x\,dx}$

Solution:

$$\frac{1}{2}\int x(2\sin x\cos x)\,dx = \frac{1}{2}\int \frac{x}{I}\cdot\frac{\sin 2x}{II}\,dx$$

As we know that; $2\sin x\cos x = \sin 2x$

Using integral by parts

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$$= \frac{1}{2}\left[x \cdot \int \sin 2x\, dx - \int \left(\frac{d}{dx}(x) \int \sin 2x\, dx \right) dx \right]$$

$$= \frac{1}{2}\left[x\left(\frac{-\cos 2x}{2} \right) - \int 1 \cdot \left(\frac{-\cos 2x}{2} \right) dx \right]$$

$$= \frac{-x\cos 2x}{4} + \frac{1}{4}\int \cos 2x\, dx$$

$$= \mathbf{\frac{-x}{4}\cos 2x + \frac{1}{8}\sin 2x + c}$$

j) $\int x^2 \ln x\, dx = \int \underset{\mathbf{I}}{\ln x}\ \underset{\mathbf{II}}{x^2} dx$

Solution:

Using integral by parts

$$= \ln x \cdot \int x^2 dx - \int \left(\frac{d}{dx}(\ln x) \cdot \int x^2 dx \right) dx$$

$$= \frac{x^3}{3}\ln x - \int \frac{1}{x} \cdot \frac{x^3}{3} dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3} \cdot \frac{x^3}{3} + c$$

$$= \mathbf{\frac{x^3}{3}\ln x - \frac{x^3}{9} + c}$$

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Q.4 Integrate the following integrals by part integration through appropriate substitution.

a) $\int \frac{\ln x . \sin(\ln x)}{x} dx$

Solution:

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First we use integration by substitution

Let $t=\ln x \rightarrow (i)$

$$\frac{dt}{dx}=\frac{1}{x}\Rightarrow dt=1/x\,dx \rightarrow (ii)$$

Substitute (i) and (ii) is given integral $\int \frac{\ln x \cdot \sin(\ln x)}{x}dx$

$$\int t.\sin t\,dt$$

Using by part integration

$$\int \frac{t}{I}\frac{\sin t}{II}dt = t\int \sin t\,dt - \int\left(\frac{d}{dt}(t)\int \sin t\,dt\right)dt$$

$$= t(-\cos t) - \int 1\cdot(-\cos t)dt$$

$$= -t\cos t + \sin t + c$$

Substitute value of t from (i)

$$\int \frac{\ln x \sin(\ln x)}{x}dt = -\ln x.\cos(\ln x)+\sin(\ln x)+c$$

b) $\int \sin 2x \ln(\cos x)\,dx$

Solution: $\int 2\sin x\cos x \ln(\cos x)dx \rightarrow (A) \quad \because \sin 2x = 2\sin x\cos x$

By substitution method

Let $t=\cos x \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx}=-\sin x$$

$$-dt=\sin x dx \rightarrow (ii)$$

Substituting (i) and (ii) in (A)

$$\int 2t\ln t(-dt) = -2\int \frac{t}{II}\frac{\ln t}{I}dt$$

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Using integration by parts

$$-2\int \frac{\ln t}{I}\cdot\frac{t}{II}dt = -2\left[\ln t\cdot\int tdt - \int\left(\frac{d}{dt}(\ln t)\cdot\int tdt\right)dt\right]$$

$$= -2\left[\ln t\cdot\frac{t^2}{2} - \int\frac{1}{t}\cdot\frac{t^2}{2}dt\right]$$

$$= -t^2\ln t + \frac{2}{2}\int tdt$$

$$= -t^2\ln t + 1\cdot\frac{t^2}{2} + c$$

$$= \frac{t^2}{2}(1 - 2\ln t) + c$$

Substituting value of t from (i)

$$= \frac{\cos^2 x}{2}(1 - 2\ln(\cos x)) + c$$

c) $\int e^{2x}\sin e^x\,dx = \int e^x\cdot\sin e^x\cdot e^x dx \rightarrow (A)$

Solution:

By substitution method

Let $t = e^x \rightarrow (i)$

Diff w.r.t x

$$\frac{dt}{dx} = e^x$$

$$dt = e^x dx \rightarrow (ii)$$

Substituting (i) and (ii) in (A)

$$\int t\sin t.dt$$

Using integration by parts

$$\int\frac{t}{I}\cdot\frac{\sin t}{II}dt = t\int\sin t\,dt - \int\left(\frac{d}{dt}(t)\int\sin t\,dt\right)dt$$

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$$= t(-\cos t) - \int 1(-\cos t)dt$$

$$= -t\cos t + \int \cos t\, dt$$

$$= -t\cos t + \sin t + c$$

Substituting value of t from (i) $t = e^x$

$$\int e^{2x}\sin e^x dx = -e^x\cos e^x + \sin e^x + c$$

Q.5 The rate at which the body eliminates a drug (in millimeters per hour) is given by.

$$\frac{R(t)}{dt} = \frac{60t}{(t+1)^2(t+2)}$$

Where t is the number of hours since the drug was administered. If $R(0)=0$ is the current drug elimination, how much of the drug is eliminated during the first hour after it was administered?

The fourth hour, after it was administered?

Solution: Given $\frac{R(t)}{dt} = \frac{60t}{(t+1)^2(t+2)}$ and $R(0) = 0$

First we find R(t) from given $\frac{dR(t)}{dt}$

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$$\frac{dR(t)}{dt} = \frac{60t}{(t+1)^2(t+2)}$$

$$dR(t) = \frac{60t}{(t+1)^2(t+2)}dt$$

Taking integration of both sides

$$\int dR(t) = \int \frac{60t}{(t+1)^2(t+2)}dt$$

$$R(t) = \int \frac{60t}{(t+1)^2(t+2)}dt \rightarrow (A)$$

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We will use partial fraction method for integration

$$\frac{60t}{(t+1)^2(t+2)} = \frac{A}{(t+1)} + \frac{B}{(t+1)^2} + \frac{C}{(t+2)} \rightarrow (i)$$

Multiplying both sides by $(t+1)^2(t+2)$

$$60t = A(t+1)(t+2) + B(t+2) + C(t+1)^2 \rightarrow (ii)$$

To find C, put $t+2=0$

$t=-2$ in (ii)

$$60(-2) = A(-2+1)(-2+2) + B(-2+2) + C(-2+1)^2$$

$$-120 = 0 + 0 + C$$

$$\mathbf{C = -120}$$

To find B, put $t+1=0$

$t=-1$ in (ii)

$$60(-1) = A(-1+1)(-1+2) + B(-1+2) + C(-1+1)^2$$

$$-60 = 0 + B + 0$$

$$\mathbf{B = -60}$$

To find A we expand eq(ii)

$$60t = A(t^2 + 3t + 2) + Bt + 2B + C(t^2 + 2t + 1)$$

$$60t = At^2 + 3At + 2A + Bt + 2B + Ct^2 + 2Ct + C$$

$$60t = (A + C)t^2 + (3A + B + 2C)t + (2A + 2B + C)$$

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Comparing coefficients

$$A + C = 0$$

$C=-120$

$A+(-120)=0$

$$\mathbf{A=120}$$

Putting values of A, B, and C in (1)

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$$R(t) = \int \frac{120}{t+1}dt + \int \frac{-60}{(t+1)^2}dt + \int \frac{-120}{t+2}dt$$

$$R(t) = 120\int \frac{1}{t+1}dt - 60\int (t+1)^{-2}dt - 120\int \frac{1}{t+2}dt$$

$$R(t) = 120\ln(t+1) - \frac{60(t+1)^{-1}}{-1} - 120\ln(t+2) + c$$

$$R(t) = 120\left[\ln(t+1) - \ln(t+2)\right] + \frac{60}{(t+1)} + c$$

$$R(t) = 120\ln\left(\frac{t+1}{t+2}\right) + \frac{60}{t+1} + c \rightarrow (iii)$$

To find c, put $R(0) = 0$

$$0 = 120\ln\left(\frac{0+1}{0+2}\right) + \frac{60}{0+1} + c$$

$$0 = 120\ln\left(\frac{1}{2}\right) + 60 + c$$

$$0 = 120(-0.69347) + 60 + c$$

c = 23.18

Putting in (iii)

$$R(t) = 120\ln\left(\frac{t+1}{t+2}\right) + \frac{60}{t+1} + 23.18$$

Now

$$R(1) = 120\ln\left(\frac{1+1}{1+2}\right) + \frac{60}{1+1} + 23.18$$

$$R(1) = 120\ln\left(\frac{2}{3}\right) + 30 + 23.18$$

$$R(1) = 120(-0.40546) + 30 + 23.18$$

$$R(1) = -48.6558 + 30 + 23.17766$$

R(1) = 4.52ml ———————— first hour

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$$R(4) = 120\ln\left(\frac{4+1}{4+2}\right) + \frac{60}{4+1} + 23.18$$

R(4) = 13.30443

$$R(3) = 120\ln\left(\frac{3+1}{3+2}\right) + \frac{60}{3+1} + 23.18$$

R(3) = 11.4031

$$R(4) - R(3) = 13.30443 - 11.4031$$

R(4) – R(3) = 1.89ml ———————— fourth hour

Q.6 The rate of change of the voting population of a city with respect to time t (in years) is estimated to be,

$$\frac{dN}{dt} = \frac{100t}{\left(1+t^2\right)^2}$$

Where N(t) is in thousands. If N(0) is the current voting population, then how much will this population increase during the next 3 years.

Solution: Given that $\frac{dN}{dt} = \frac{100t}{\left(1+t^2\right)^2}$

$$dN = \frac{100t}{\left(1+t^2\right)^2}dt$$

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Taking integration of both sides

$$\int dN = \int \frac{100t}{\left(1+t^2\right)^2}dt = 50\int \frac{2t}{\left(1+t^2\right)^2}dt \quad \rightarrow (A)$$

Let $u = 1+t^2 \rightarrow (i)$

$$\frac{du}{dt} = 2t \qquad \Rightarrow \qquad du = 2tdt \qquad \rightarrow (ii)$$

Substituting (i) and (ii) in (A)

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$$\int dN = 50\int \frac{du}{u^2} = 50\int u^{-2}du$$

$$N = 50\frac{u^{-1}}{-1} + c = \frac{-50}{u} + c \qquad \text{Substituting the values of u from (i)}$$

$$N(t) = \frac{-50}{(1+t^2)} + c \quad \rightarrow \qquad (B)$$

To find c, put N=N(0) and t=0

$$N(0) = \frac{-50}{(1+0^2)} + c$$

$$N(0) = -50 + c \qquad \Rightarrow \qquad c = N(0) + 50 \qquad \text{Putting in (B)}$$

$$N(t) = \frac{-50}{(1+t^2)} + N(0) + 50 \quad \rightarrow (C)$$

Now at t=3

$$N(3) = \frac{-50}{(1+3^2)} + N(0) + 50$$

$$N(3) = -5 + N(0) + 50 = N(0) + 45$$

Increase in population, $\Delta N = N(3) - N(0)$

$\Delta N = N(0) + 45 - N(0)$

ΔN=45 thousand

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Q.7 An oil tanker aground on a reef is losing oil and producing an oil stick that is radiating onward at a rate approximated by

$$\frac{dr}{dt} = \frac{100t}{\sqrt{t^2+9}} \qquad t \geq 0$$

Where r is radius (feet) of the circular slick after t minutes. Find the radius of the slick after 4 minutes if the radius r=0. When t=0

Solution: Given that $\frac{dr}{dt} = \frac{100}{\sqrt{t^2+9}}$

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$$dr = \frac{100}{\sqrt{t^2+9}}dt$$ Taking integration of both sides

$$\int dr = \int \frac{100}{\sqrt{t^2+9}}dt \quad \rightarrow (A)$$ Let $t=3\tan\theta \rightarrow (i)$

$$\frac{dt}{d\theta} = 3\sec^2\theta \Rightarrow dt = 3\sec^2\theta d\theta \quad \rightarrow (ii)$$ Substituting (i) and (ii) in (A)

$$\int dr = \int \frac{(100)3\sec^2\theta d\theta}{\sqrt{9\tan^2\theta+9}} = 300\int \frac{\sec^2\theta d\theta}{\sqrt{9(\tan^2\theta+1)}}$$

$$r = \frac{300}{3}\int \frac{\sec^2\theta d\theta}{\sqrt{\sec^2\theta}} = 100\int \frac{\sec^2\theta}{\sec\theta}d\theta$$

$$r = 100\int \sec\theta d\theta = 100\int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}d\theta$$

$$r = 100\int \frac{(\sec^2\theta+\sec\theta\tan\theta)}{\sec\theta+\tan\theta}d\theta$$

$$r = 100\ln|\sec\theta+\tan\theta| + c$$

$$r = 100\ln\left|\sqrt{\tan^2\theta+1}+\tan\theta\right| + c$$

Substituting value of $\tan\theta$ from (i)

$$\tan\theta = \frac{t}{3}$$

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$$r(t) = 100\ln\left|\sqrt{\frac{t^2}{9}+1}+\frac{t}{3}\right| + c \rightarrow (B)$$ To find c, put t=0 and r=0

$$0 = 100\ln\left|\sqrt{\frac{0^2}{9}+1}+\frac{0}{3}\right| + c \Rightarrow c = 0,$$ put in (B)

$$r(t) = 100\ln\left|\sqrt{\frac{t^2}{9}+1}+\frac{t}{3}\right| \rightarrow (C)$$ Now radius r after 4 minutes will be

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$$r(4) = 100\ln\left|\sqrt{\frac{4^2}{9}+1}+\frac{4}{3}\right| = 100\ln\left|\frac{4}{3}+\frac{5}{3}\right|$$

r(4) = 109.86 feet

Q.8 After a person takes a pill, the drug contained in the pill in assimilated into the blood stream. The rate of assimilation t minutes after taking the pill is,

$$\frac{dr}{dt} = te^{-0.2t},\ r(0) = 0$$

Find the total amount of the drug that is assimilated into the bloodstream during the first 10 minutes after the pill is taken.

Solution: Given that $\frac{dr}{dt} = te^{-0.2t}$

$dr = te^{-0.2t}dt$ Taking integration of both sides

$\int dr = \int te^{-0.2t}dt$ Taking integration by parts

$$r = t\int e^{-0.2t}dt - \int \frac{d}{dt}(t)\left(\int e^{-0.2t}\,dt\right)dt$$

$$r = t\cdot\frac{e^{-0.2t}}{-0.2} - \int\left(1\cdot\frac{e^{-0.2t}}{(-0.2)}\right)dt$$

$$r = -5te^{-0.2t} + 5\frac{e^{-0.2t}}{(-0.2)} + c$$

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$r = -5te^{-0.2t} - 25e^{-0.2t} + c \quad \rightarrow (A)$ Putting r=0 and t=0

$0 = -5(0)\ e^{-0.2(0)} - 25e^{-0.2(0)} + c$

$0 = -25 + c \quad \Rightarrow \quad c = 25$ Putting in (A)

$r(t) = -5te^{-0.2t} - 25e^{-0.2t} + 25 \quad \rightarrow (B)$ Now for r(10), put t=10, in B

$r(10) = -5(10)e^{-0.2(10)} - 25e^{-0.2(10)} + 25$

$r(10) = -50e^{-0.2(10)} - 25e^{-0.2(10)} + 25$

$r(10) = -6.77 - 3.375 + 25$

r(10) = 14.86

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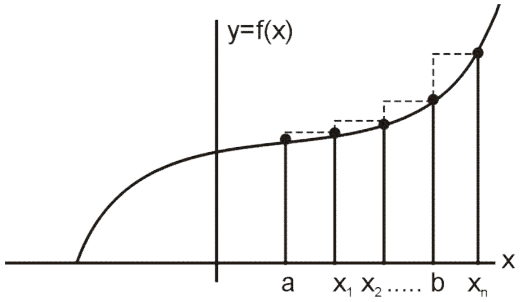
DEFINITE INTEGRAL:

Area under a curve:

The area under a curve f(x) between x=a to x=b (shown in figure) can be obtained by dividing the interval (a,b) into subintervals and approximating the area by the sum of rectangles shown, below;



For n subintervals $\Delta x = \frac{b-a}{n}$

$$\text{Area} = A \cong f(x_1)\Delta x + f(x_2)\Delta x + \ldots\ldots\ldots + f(x_n)\Delta x \quad \rightarrow (i)$$

The limiting value of (i) when $\Delta x \rightarrow 0$ becomes the actual area under f(x), under the limiting condition $\Delta x \rightarrow 0$ ($n \rightarrow \infty$), the value on right side of (i) becomes;

$$\lim_{\Delta x \rightarrow 0} \Delta x\left[f(x_1) + f(x_2) + \ldots\ldots\ldots + f(x_n)\right] = \int_a^b f(x)\,dx \quad \rightarrow (ii)$$

The right hand side of (ii) is called the definite integral of f(x) from a to b and gives the area under curve f(x) from a to b.

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EXERCISE 5.4

Q.1 In each case, determine the approximate area of the region bounded by f(x)=2x+1, x=a and x=b for n subintervals.

a) n=2, a=0, b=2

Solution: Here a=0, b=2 and n=2, f(x)=2x+1

So, the width of subinterval $\Delta x = \frac{b-a}{n} = \frac{2-0}{2} = 1$

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The right hand sides of the two subintervals are

$x_1=a+\Delta x=0+1=1$ $f(x) = 2x + 1$

$x_2=a+2\Delta x=0+2(1)=2$ $f(x_1) = f(1) = 2(1) + 1 = 3$

Thus, the approximate area is $f(x_1) = 3$

$A=f(x_1)\Delta x + f(x_2)\Delta x$ putting values $f(x) = 2x + 1$

$A=3+5$ $f(x_2) = f(2) = 2(2) + 1 = 5$

A=8 units $f(x_2) = 5$

b) n=4, a=0, b=2

Solution: Here a=0, b=2 and n=4, and **f(x)=2x+1**

So, the width of subinterval $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

The right hand sides of four subintervals are $f(x) = 2x + 1$

$x_1=a+\Delta x=0+0.5=0.5$ $f(x_1) = 2(0.5) + 1 = 2$

$x_2=a+2\Delta x=0+2(0.5)=1$ $f(x_2) = 2(1) + 1 = 3$

$x_3=a+3\Delta x=0+3(0.5)=1.5$ $f(x_3) = 2(1.5) + 1 = 4$

$x_4=a+4\Delta x=0+4(0.5)=2$ $f(x_4) = 2(2) + 1 = 5$

The approximate area is thus

$A=f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$ putting the values

$A = \Delta x[f(x_1) + f(x_2) + f(x_3) + f(x_4)]$

$A = \frac{1}{2}[2+3+4+5]$

A=7

c) n=8, a=0, b=2

Solution: Here a=0, b=2 and n=8, f(x)=2x+1

So, the width of subinterval $\Delta x = \frac{b-a}{n} = \frac{2-0}{8} = 0.25$

The right hand sides of four subintervals are $f(x)=2x+1$

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$x_1=a+\Delta x=0+0.25=0.25$ $f(x_1) = 2(0.25)+1 = 1.5$

$x_2=a+2\Delta x=0+2(0.25)=0.5$ $f(x_2) = 2(0.50)+1 = 2$

$x_3=a+3\Delta x=0+3(0.25)=0.75$ $f(x_3) = 2(0.75)+1 = 2.5$

$x_4=a+4\Delta x=0+4(0.25)=1$ $f(x_4) = 2(1.0)+1 = 3$

$x_5=a+5\Delta x=0+5(0.25)=1.25$ $f(x_5) = 2(1.25)+1 = 3.5$

$x_6=a+6\Delta x=0+6(0.25)=1.5$ $f(x_6) = 2(1.50)+1 = 4$

$x_7=a+7\Delta x=0+7(0.25)=1.75$ $f(x_7) = 2(1.75)+1 = 4.5$

$x_8=a+8\Delta x=0+8(0.25)=2$ $f(x_8) = 2(2)+1 = 5$

The approximate area is thus

$A = \Delta x[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7) + f(x_8)]$

$A = 0.25[1.5 + 2 + 2.5 + 3 + 3.5 + 4 + 4.5 + 5]$

$A = 0.25[26]$

A=6.5 units

d) n=2, a=1, b=5

Solution: Here a=1, b=5 and n=2, f(x)=2x+1

So, the width of subinterval $\Delta x = \frac{b-a}{n} = \frac{5-1}{5} = 2$

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The right hand sides of two subintervals are

$x_1=a+\Delta x=1+2=3$ $f(x_1)= 2(3) + 1 = 7$

$x_2=a+2\Delta x=1+2(2)=5$ $f(x_2)= 2(5) + 1 = 10 + 1 = 11$

The approximate are is thus

$A=f(x_1)\Delta x + f(x_2)\Delta x$

$A = \Delta x[f(x_1)+f(x_2)]$

$A = 2[7 + 11] = 2 \times 18$

A=36

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Q.2 In each case, find the approximate area of the region bounded by $f(x)=x^2+1$, $x=a$ and $x=b$ for n subintervals.

a) n=2, a=0, b=2

Solution: Here a=0, b=2 and n=2, $f(x)=x^2+1$

So, the width of subinterval, $\Delta x=\frac{b-a}{n}=\frac{2-0}{2}=1$

The right hand sides of each subinterval are

$x_1=a+\Delta x=0+1=1$ $\quad f(x_1)=f(1)=(1)^2+1=2$

$x_2=a+2\Delta x=0+2(1)=2$ $\quad f(x_2)=f(2)=(2)^2+1=5$

The approximate area is thus

$A=f(x_1)\Delta x+f(x_2)\Delta x$, putting values

$A=\Delta x[f(x_1)+f(x_2)]$

$A=1[2+5]$

$A=2+5$

$A=7$

b) n=4, a=0, b=2

Solution: Here a=0, b=2 and n=4 and $f(x)=x^2+1$

So, the width of subinterval, $\Delta x=\frac{b-a}{n}=\frac{2-0}{4}=0.5$

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The right hand sides of 4 subintervals are

$x_1=a+\Delta x=0+0.5=0.5$ $\quad f(x_1)=f(0.5)=(0.5)^2+1=1.25$

$x_2=a+2\Delta x=0+2(0.5)=1$ $\quad f(x_2)=f(1)=(1)^2+1=2$

$x_3=a+3\Delta x=0+3(0.5)=1.5$ $\quad f(x_3)=f(1.5)=(1.5)^2+1=3.25$

$x_4=a+4\Delta x=0+4(0.5)=2$ $\quad f(x_4)=f(2)=(2)^2+1=5$

The approximate area is thus

$A=f(x_1)\Delta x+f(x_2)\Delta x+f(x_3)\Delta x+f(x_4)\Delta x$

$A=\Delta x[f(x_1)+f(x_2)+f(x_3)+f(x_4)]$

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$A = 0.5[1.25+2+3.25+5]$

A=5.75

c) n=8, a=0, b=2

Solution: Here a=0, b=2 and n=8, f(x)=2x+1

So, the width of subinterval $\Delta x = \frac{b-a}{n} = \frac{2-0}{8} = 0.25$

The right hand sides of four subintervals are

$x_1 = a+\Delta x = 0+0.25 = 0.25$ $f(x_1) = f(0.25) = (0.25)^2 + 1 = 1.0625$

$x_2 = a+2\Delta x = 0+2(0.25) = 0.5$ $f(x_2) = f(0.5) = (0.5)^2 + 1 = 1.25$

$x_3 = a+3\Delta x = 0+3(0.25) = 0.75$ $f(x_3) = f(0.75) = (0.75)^2 + 1 = 1.5625$

$x_4 = a+4\Delta x = 0+4(0.25) = 1$ $f(x_4) = f(1) = (1)^2 + 1 = 2$

$x_5 = a+5\Delta x = 0+5(0.25) = 1.25$ $f(x_5) = f(1.25) = (1.25)^2 + 1 = 2.5625$

$x_6 = a+6\Delta x = 0+6(0.25) = 1.5$ $f(x_6) = f(1.5) = (1.5)^2 + 1 = 3.25$

$x_7 = a+7\Delta x = 0+7(0.25) = 1.75$ $f(x_7) = f(1.75) = (1.75)^2 + 1 = 4.0625$

$x_8 = a+8\Delta x = 0+8(0.25) = 2$ $f(x_8) = f(2) = (2)^2 + 1 = 5$

The approximate are is thus

$A = \Delta x[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7) + f(x_8)]$

$A = 0.25[1.0625 + 1.25 + 1.5625 + 2 + 2.5625 + 3.25 + 4.0625 + 5]$

A=5.1875

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d) n=2, a=1, b=5

Solution: Here a=1, b=5 and n=2, $f(x)=x^2+1$

So, the width of subinterval $\Delta x = \frac{b-a}{n} = \frac{5-1}{2} = 2$

The right hand sides of two subintervals are

$x_1 = a+\Delta x = 1+2 = 3$

$x_2 = a+2\Delta x = 1+2(2) = 5$

The approximate are is thus

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$\mathbf{A=f(x_1)\Delta x + f(x_2)\Delta x}$

$A=\left((3)^2+1\right)(2)+\left((5)^2+1\right)(2)$

$A=20+52$

$\mathbf{A=72}$

Q.3 In each case, determine the actual value of the integral using definition 5.6.1

a) $\int_{x=0}^{x=3} \mathbf{3xdx}$

Solution: Here a=0, b=3, f(x)=3x

For n subintervals, the width of each subinterval, Δx will be

$$\Delta x=\frac{b-a}{n}=\frac{3-0}{n}=\frac{3}{n}$$

The right hands of n subintervals are

$$x_1=a+\Delta x=0+\frac{3}{n}=\frac{3}{n}$$

$$x_2=a+2\Delta x=0+2(\frac{3}{n})=\frac{6}{n}$$

$$x_3=a+3\Delta x=0+3(\frac{3}{n})=\frac{9}{n}$$

$$x_n=a+n\Delta x=0+n(\frac{3}{n})=3$$

The approximate area is thus

$\mathbf{A=f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x+\ldots\ldots\ldots\ldots\ldots\ldots+ f(x_n)\Delta x}$

Putting values

$$A=3\left(\frac{3}{n}\right)\left(\frac{3}{n}\right)+3\left(\frac{6}{n}\right)\left(\frac{3}{n}\right)+3\left(\frac{9}{n}\right)\left(\frac{3}{n}\right)+\ldots\ldots\ldots\ldots\ldots+3(3)\left(\frac{3}{n}\right)$$

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The actual area is equal to the limit

$$\mathbf{A = \int_a^b f(x)dx = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x]}$$

$$\mathbf{= \lim_{n\to\infty}\Delta x[f(x_1) + f(x_2) + ... + f(x_n)]}$$

$$A = \lim_{n\to\infty}\left(\frac{3}{n}\right)\left[3\left(\frac{3}{n}\right) + 3\left(\frac{6}{n}\right) + ... + 3(3)\right]$$

$$A = \lim_{n\to\infty}\left(\frac{3}{n}\right)\left(\frac{9}{n}\right)[1 + 2 + 3 + ...n]$$

$$A = \lim_{n\to\infty}\frac{27}{n^2}\bullet\frac{n(n+1)}{2}$$

$$A = \lim_{n\to\infty}\frac{27n\bullet n\left(1 + 1/n\right)}{2n^2}$$

$$A = \frac{27}{2}\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)$$

$$A = \frac{27}{2}\left(1 + \frac{1}{\infty}\right)$$

$$\mathbf{A = 13.5}$$

b) $\mathbf{\int_{x=0}^{x=3}(2x-4)dx}$

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Solution: Here a=0, b=3, f(x)=2x–4

For 4 subintervals, the width of each subinterval, will be

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

The right hands of n subintervals are

$x_1 = a + \Delta x = 0 + \frac{3}{n} = \frac{3}{n}$

$x_2 = a + 2\Delta x = 0 + 2\ (\frac{3}{n}) = \frac{6}{n}$

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$x_3 = a + 3\Delta x = 0 + 3(\frac{3}{n}) = \frac{9}{n}$

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$x_n = a + n\Delta x = 0 + n(\frac{3}{n}) = \frac{3n}{n}$

The area is given by the limit

$$\mathbf{A = \int_{x=0}^{x=3} (2x-4)dx = \lim_{n\to\infty} \Delta x\left[f(x_1)+f(x_2)+f(x_3)+.....f(x_n)\right]}$$

$$A = \lim_{n\to\infty} \frac{3}{n}\left[\left(2(\frac{3}{n})-4\right)+\left(2(\frac{6}{n})-4\right)+\left(2(\frac{9}{n})-4\right)+.....\left(2(\frac{3n}{n})-4\right)\right]$$

$$A = \lim_{n\to\infty} \frac{3}{n}\left[\frac{6}{n}-4+\frac{12}{4}-4+\frac{18}{n}-4.....\frac{6n}{n}-4\right]$$

$$A = \lim_{n\to\infty} \frac{3}{n}\left[\frac{6}{n}+\frac{12}{n}+\frac{18}{n}+.....+\frac{6n}{n}-4n\right]$$

$$A = \lim_{n\to\infty} \frac{3}{n}\left[\frac{6}{n}(1+2+.....n)-4n\right]$$

$$A = \lim_{n\to\infty} \frac{3}{n}\left[\frac{6}{n}\frac{n(n+1)}{2}-4n\right]$$

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$$A = \lim_{n\to\infty} \frac{3}{n}[3(n+1)-4n] = \lim_{n\to\infty} \frac{3}{n}[3n+3-4n]$$

$$A = \lim_{n\to\infty} \frac{3}{n}[3-n] = \lim_{n\to\infty}\left(\frac{9}{n}-3\right)$$

A = –3

c) $\mathbf{\int_{x=0}^{x=2} x^2 dx}$

Solution: Here a=0, b=3, $f(x)=x^2$

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For n subintervals, the width of each subinterval, will be

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

The right hands of n subintervals are

$$x_1 = a + \Delta x = 0 + \frac{2}{n} = \frac{2}{n}$$

$$x_2 = a + 2\Delta x = 0 + 2\left(\frac{2}{n}\right) = \frac{4}{n}$$

$$x_3 = a + 3\Delta x = 0 + 3\left(\frac{2}{n}\right) = \frac{6}{n}$$

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$$x_n = a + n\Delta x = 0 + n\left(\frac{2}{n}\right) = \frac{2n}{n}$$

The area is given by the limit

$$\mathbf{A = \int_{x=0}^{x=2} x^2 dx = \lim_{n\to\infty} \Delta x\left[f(x_1) + f(x_2) + f(x_3) + ...f(x_n)\right]}$$

$$A = \lim_{n\to\infty} \frac{2}{n}\left[\left(\frac{2}{n}\right)^2 + \left(\frac{4}{n}\right)^2 + \left(\frac{6}{n}\right)^2 + ... + \left(\frac{2n}{n}\right)^2\right]$$

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$$A = \lim_{n\to\infty} \frac{2}{nn^2}\left[2^2 + 4^2 + 6^2 + ...(2n)^2\right]$$

$$A = \lim_{n\to\infty} \frac{2.2^2}{nn^2}\left[1^2 + 2^2 + 3^2 + 4^2.....n^2\right]$$

$$A = \lim_{n\to\infty} \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$A = \lim_{n\to\infty} \frac{4}{3n^2} n^2 \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

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$$A = \lim_{n\to\infty} \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$A = \lim_{n\to\infty} \frac{4}{3}(1+0)(2+0)$$

$$\mathbf{A = \frac{8}{3}}$$

d) $\int_{x=2}^{x=3} (x^2-4)\,dx$

Solution: Here a=2, b=3, $f(x)=x^2-4$

So, for n subintervals, the width of each subinterval, will be

$$\Delta x = \frac{b-a}{n} = \frac{3-2}{n} = \frac{1}{n}$$

The right hands of n subintervals are

$x_1 = a+\Delta x = 2+\frac{1}{n}$

$x_2 = a+2\Delta x = 2+2\,(\frac{1}{n}) = 2+\frac{2}{n}$

$x_3 = a+3\Delta x = 2+3(\frac{1}{n}) = 2+\frac{3}{n}$

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$$x_n = a + n\Delta x = 2 + n\left(\frac{1}{n}\right) = 2+\frac{n}{n}$$

The area is given by the limit

$$\mathbf{A = \int_{x=2}^{x=3} (x^2-4)dx = \lim_{n\to\infty} \Delta x\left[f(x_1)+f(x_2)+f(x_3)+.....f(x_n)\right]}$$

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$$A=\lim_{n\to\infty}\frac{1}{n}\left[\left((2+\frac{1}{n})^2-4\right)+\left((2+\frac{2}{n})^2-4\right)+\left((2+\frac{3}{n})^2-4\right)+...+\left((2+\frac{n}{n})^2-4\right)\right]$$

$$A=\lim_{n\to\infty}\frac{1}{n}\left[4+\frac{1}{n^2}+\frac{4}{n}-4+4+\frac{4}{n^2}+\frac{8}{n}-4+4+\frac{3^2}{n^2}+\frac{12}{n}-4+....4+\frac{n^2}{n^2}+4-4\right]$$

$$A=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1^2}{n^2}+\frac{2^2}{n^2}+\frac{3^2}{n^2}+.....\frac{n^2}{n^2}+\frac{4}{n}+\frac{8}{n}+\frac{12}{n}.....\frac{4n}{n}\right]$$

$$A=\lim_{n\to\infty}\left[\frac{1}{n}\bullet\frac{1}{n^2}\left(1^2+2^2+3^2+....n^2\right)+\frac{1}{n}\bullet\frac{4}{n}\left(1+2+3+....n\right)\right]$$

$$A=\lim_{n\to\infty}\left[\frac{1}{n^3}\bullet\frac{n(n+1)(2n+1)}{6}+\frac{4}{n^2}\bullet\frac{n(n+1)}{2}\right]$$

$$A=\lim_{n\to\infty}\left[\frac{n\bullet n\bullet n(1+\frac{1}{n})(2+\frac{1}{n})}{6n^3}+\frac{4\bullet n(1+\frac{1}{n})}{2n}\right]$$

$$A=\lim_{n\to\infty}\left[\frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6}+2\left(1+\frac{1}{n}\right)\right]$$

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$$A=\frac{(1+0)(2+0)}{6}+2(1+0)$$

$$A=\frac{1}{3}+2$$

$$\mathbf{A=\frac{7}{3}}$$

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Fundamental theorem of definite integral:

I a function f(x) is continuous on an interval [a,b], then the definite integral of f(x) in the interval [a,b] is given by;

$$\int_{x=a}^{x=b} f(x)dx = \left|F(x)\right|_{x=a}^{x=b} = F(b) - F(a)$$

Where $F'(x) = f(x)$ for all x is in [a,b]

Properties of definite integral:

Linearity: If f(x) and g(x) are integrable on interval [a,b], then for any constants r and s;

$$\int_a^b rf(x) + sg(x)dx = r\int_a^b f(x)dx + s\int_a^b g(x)dx$$

Equating rule: If

$$\int_a^b f(x)dx = \int_a^b g(y)dy$$

Then $f(x) = g(y)$

Subdivision rule:

For any number c such that a<c<b;

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Point rule:

If the two limits of definite integral are same i.e. a=b or b=a, then the interval [a,b] becomes a single point and definite integral;

$$\int_a^a f(x)dx = 0$$

Opposite rule:

Inter changing the limits of integration charges the sign of the definite integral i.e.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

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EXERCISE 5.5

Q.1 Evaluate the following definite integrals:

a) $\int_3^4 5dx$

Solution: $\int_3^4 5dx = 5\int_3^4 1dx = |5x|_3^4$

$\int_3^4 5dx = 5(4) - 5(3) = 20 - 15 = 5$

= 5

b) $\int_{12}^{20} dx$

Solution: $\int_{12}^{20} dx = |x|_{12}^{20} = 20 - 12 = 8$

= 8

c) $\int_1^2 (2x^{-2} - 3)dx$

Solution: $\int_1^2 (2x^{-2} - 3)dx = \left|2\frac{x^{-1}}{-1} - 3x\right|_1^2 = \left|\frac{-2}{x} - 3x\right|_1^2$

$= \left|\frac{-2}{2} - 3(2)\right| - \left|\frac{-2}{1} - 3(1)\right|$

$= -1 - 6 + 2 + 3 = -2$

= –2

d) $\int_1^4 3\sqrt{x}dx$

Solution: $\int_1^4 3\sqrt{x}dx = \int_1^4 3x^{1/2}dx = \left|\frac{3x^{1/2+1}}{1/2+1}\right|_1^4 = \left|\frac{3x^{3/2}}{3/2}\right|_1^4 = \left|2x^{3/2}\right|_1^4$

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$$\int_1^4 3\sqrt{x}dx = \left|2x^{3/2}\right|_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 2(2^2)^{3/2} - 2(1)^{3/2}$$

$$= 16 - 2 = 14$$

$$\mathbf{= 14}$$

e) $\int_2^3 12(x^2-4)^5 x\,dx$

Solution: $\int_2^3 12(x^2-4)^5 x\,dx = ?$

Integration by substitution method

Method I: Let $t = x^2 - 4 \rightarrow$ (i)

Diff w.r.t x

$$\frac{dt}{dx} = 2x$$

$$\frac{dt}{2} = xdx \rightarrow \text{(ii)}$$

Substituting (i) and (ii) in given integral

$$\int_2^3 12(x^2-4)^5 x\,dx = \int 12t^5 \frac{dt}{2} = 6\frac{t^6}{6} = t^6$$

Substituting t from (i)

$$\int_2^3 12\left(x^2-4^5\right)xdx = \left|\left(x^2-4\right)^6\right|_2^3$$

$$= \left((3)^2-4\right)^6 - \left((2)^2-4\right)^6 = 5^6 - 0$$

$$= 125 \times 125$$

$$\mathbf{= 15625}$$

Method II: $\int_2^3 12(x^2-4)^5 xdx$

Integration by substitution method

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Let $t=x^2-4 \rightarrow$ (i)

Diff w.r.t x

$$\frac{dt}{dx}=2x$$

$$\frac{dt}{2}=xdx \rightarrow \text{(ii)}$$

Limits: **When $x=2 \Rightarrow t=(2)^2-4=0$**

When $x=3 \Rightarrow t=(3)^2-4=5$

Substituting (i) and (ii) and corresponding limits

$$\int_2^3 12(x^2-4)^5 x dx = \int_0^5 12t^5 \frac{dt}{2} = 6\int_0^5 t^5 dt = \left|\frac{6t^6}{6}\right|_0^5$$

$$= \left|t^6\right|_0^5 = (5)^6-(0)^6$$

$$= (5)^6 = \mathbf{15625}$$

f) $\int_{-6}^{0} \sqrt{4-2x}\, dx$

Solution: $\int_{-6}^{0} \sqrt{4-2x}\, dx$

Integration by substitution method

Let $t=4-2x \rightarrow$ (i)

Diff w.r.t x

$$\frac{dt}{dx}=-2$$

$$\frac{dt}{-2}=dx \rightarrow \text{(ii)}$$

$$t = 4-2x$$

Limits: **When $x=-6 \Rightarrow t=4-2(-6)=16$**

When $x=0 \Rightarrow t=4-0(0)=4$

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$$\int_{-6}^{0}\sqrt{4-2x}dx=\int_{16}^{4}\sqrt{t}\left(\frac{dt}{-2}\right)=\frac{-1}{2}\int_{16}^{4}t^{1/2}\,dt=\frac{-1}{2}\left|\frac{t^{3/2}}{3/2}\right|_{16}^{4}=\frac{-1}{3}\left|t^{3/2}\right|_{16}^{4}$$

$$=\frac{-1}{3}(4)^{3/2}+\frac{1}{3}(16)^{3/2}=\frac{-1}{3}(8)+\frac{1}{3}(64)$$

$$=\frac{-8+64}{3}$$

$$=\frac{56}{3}=\mathbf{18.667}$$

g) $\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx$

Solution: $\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx$

Integration by substitution method

Let $t = x+2 \rightarrow$ (i)

$x = t-2 \rightarrow$ (ii)

Diff w.r.t x

$$\frac{dt}{dx}=1$$

$dt = dx \rightarrow$ (iii)

$x + 2 = t$

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Limits: **When x=–1⇒ t=–1+2=1**

When x=7⇒ t=7+2=9

Substituting (i)(ii) and (iii) and corresponding limits

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\int_{1}^{9}\frac{(t-2)}{\sqrt{t}}dt=\int_{1}^{9}\left(\frac{t}{\sqrt{t}}-\frac{2}{\sqrt{t}}\right)dt=\int_{1}^{9}\left(t^{1/2}-2t^{-1/2}\right)dt$$

$$=\left|\frac{t^{3/2}}{3/2}-\frac{2t^{1/2}}{1/2}\right|_{1}^{9}=\left|\frac{2}{3}t^{3/2}-4t^{1/2}\right|_{1}^{9}$$

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$$= \left|\frac{2}{3}(9)^{3/2} - 4(9)^{1/2}\right| - \left|\frac{2}{3}(1)^{3/2} - 4(1)^{1/2}\right|$$

$$= 18 - 12 - \frac{2}{3} + 4 = \frac{54 - 36 - 2 + 12}{3}$$

$$= \frac{28}{3} = \mathbf{9.33}$$

h) $\int_0^1 (e^{2x} - 2x)^2 (e^{2x} - 1)\,dx$

Solution: $\int_0^1 (e^{2x} - 2x)^2 (e^{2x} - 1)\,dx$

Integration by substitution method

Let $t = e^{2x} - 2x \rightarrow$ (i)

Diff w.r.t x

$$\frac{dt}{dx} = e^{2x}(2) - 2 = 2(e^{2x} - 1)$$

$$\frac{dt}{2} = (e^{2x} - 1)dx \rightarrow (ii)$$

Limits: **When $x = 0 \Rightarrow t = e^{2(0)} - 2(0) = 1$**

When $x = 1 \Rightarrow t = e^{2(1)} - 2(1) = e^2 - 2$

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Substituting (i) and (ii) and the limits

$$\int_1^{(e^2-2)} t^2 \frac{dt}{2} = \frac{1}{2}\int_1^{(e^2-2)} t^2 dt = \left|\frac{1}{2} \bullet \frac{t^3}{3}\right|_1^{(e^2-2)}$$

$$= \frac{1}{6}\left[(e^2 - 2)^3 - (1)^3\right]$$

$$= \frac{1}{6}\left[(7.38905 - 2)^3 - (1)^3\right] = \frac{1}{6}[155.508]$$

$$= \mathbf{25.918}$$

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Q.2 Evaluate the following definite integrals:

a) $\int_2^3 x\sqrt{2x^2-3}dx$

Solution: $\int_2^3 x\sqrt{2x^2-3}\,dx = \int_2^3 \sqrt{2x^2-3}\bullet xdx$

Integration by substitution method

Let $t=2x^2-3 \rightarrow$ (i)

$$\frac{dt}{dx}=4x$$

$$\frac{dt}{4}=xdx \rightarrow (ii)$$

$$t = 2x^2-3$$

Limits: **When $x=2 \Rightarrow t=2(2)^2-3=5$**

When $x=3 \Rightarrow t= 2(3)^2-3=15$

Substituting (i) and (ii) and corresponding limits

$$\int_2^3 \sqrt{2x^2-3}\,xdx = \int_5^{15}\sqrt{t}\bullet\frac{dt}{4}=\frac{1}{4}\int_5^{15} t^{1/2}dt = \left|\frac{1}{4}\frac{t^{3/2}}{3/2}\right|_5^{15}$$

$$=\left|\frac{1}{6}t^{3/2}\right|_5^{15}=\frac{1}{6}(15)^{3/2}-\frac{1}{6}(5)^{3/2}$$

=7.819

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b) $\int_0^1 x\sqrt{3x^2+2}dx$

Solution: $\int_0^1 x\sqrt{3x^2+2}\,dx = \int_0^1 \sqrt{3x^2+2}xdx$

Integration by substitution method

Let $t=3x^2+2 \rightarrow$ (i)

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Diff w.r.t x

$$\frac{dt}{dx} = 6x$$

$$\frac{dt}{6} = xdx \rightarrow (ii)$$

Limits: **When $x=0 \Rightarrow t=3(0)^2+2=2$**

When $x=1 \Rightarrow t=3(1)^2+2=5$

Substituting (i) and (ii) and corresponding limits

$$\int_0^1 x\sqrt{3x^2+2}\,dx = \int_2^5 \sqrt{t} \bullet \frac{dt}{6} = \frac{1}{6}\int_2^5 t^{1/2}dt = \left|\frac{t^{1/2+1}}{1/2+1}\right|_2^5 = \left|\frac{1}{6}\frac{t^{3/2}}{3/2}\right|_2^5$$

$$= \left|\frac{1}{9}t^{3/2}\right|_2^5 = \frac{1}{9}(5)^{3/2} - \frac{1}{9}(2)^{3/2}$$

$$= \frac{1}{9}(11.1803 - 2.82842) = \frac{1}{9}(8.35188)$$

=0.93

c) $\int_0^1 \frac{x-1}{x^2-2x+3}dx$

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Solution: $\int_0^1 \frac{x-1}{x^2-2x+3}dx$

Integration by substitution method

Let $t=x^2-2x+3 \rightarrow$ (i)

Diff w.r.t x

$$\frac{dt}{dx} = \frac{d}{dx}(x^2-2x+3)$$

$$\frac{dt}{dx} = 2x-2 = 2(x-1)$$

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$$\frac{dt}{2} = (x-1)dx \rightarrow (ii)$$

$$t = x^2 - 2x + 3$$

Limits: **When $x=0 \Rightarrow t=(0)^2-2(0)+3=3$**

When $x=1 \Rightarrow t=(1)^2-2(1)+3=2$

Substituting (i) and (ii) and corresponding limits

$$\int_0^1 \frac{(x-1)dx}{x^2-2x+3} = \int_3^2 \frac{1}{t}\frac{dt}{2} = \frac{1}{2}\int_3^2 \frac{1}{t}dt = \frac{1}{2}\left|\ln t\right|_3^2$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\ln 3 = \frac{1}{2}[\ln 2 - \ln 3] = \frac{1}{2}\ln\left(\frac{2}{3}\right)$$

$= -0.203$

d) $\int_{-1}^{1} \frac{e^{-x}-e^{x}}{(e^{-x}+e^{x})^2}dx$

Solution: Integration by substitution method

$t = e^{-x} + e^{x} \rightarrow$ (i)

Diff w.r.t x

$$\frac{dt}{dx} = \frac{d}{dx}(e^{-x}+e^{x})$$

$$\frac{dt}{dx} = -e^{-x} + e^{x} = -1(e^{-x} - e^{x})$$

$$\frac{dt}{-1} = (e^{-x} - e^{x})dx \rightarrow (ii)$$

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Limits: **When $x=-1 \Rightarrow t=e^{-(-1)}+e^{-1}=e-e^{-1}=e+\frac{1}{e}=\frac{e^2+1}{e}$**

When $x=1 \Rightarrow t=e^{-1}+e^{1}=\frac{1}{e}+e=\frac{1+e^2}{e}$

Substituting (i) and (ii) and corresponding limits

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$$\int_{-1}^{1} \frac{(e^{-x}-e^{x})}{(e^{-x}+e^{x})^2}dx = \int_{\frac{e^2+1}{e}}^{\frac{1+e^2}{e}} \frac{-dt}{t^2} = -\int_{\frac{e^2+1}{e}}^{\frac{1+e^2}{e}} t^{-2}dt = \left|\frac{-t^{-1}}{-1}\right|_{\frac{e^2+1}{e}}^{\frac{1+e^2}{e}}$$

$$\left|\frac{1}{t}\right|_{\frac{e^2+1}{e}}^{\frac{1+e^2}{e}} = \frac{1}{\left(\frac{1+e^2}{e}\right)} - \frac{1}{\left(\frac{e^2+1}{e}\right)} = \frac{e}{1+e^2} - \frac{e}{1+e^2} = 0$$

Q.3 Evaluate the following definite integrals:

a) $\int_{\pi/2}^{\pi} \cos\left(\frac{x}{2}+\pi\right)dx$

Solution: Integration by substitution method

Let $t = \frac{x}{2}+\pi \rightarrow$ (i)

Diff w.r.t x

$$\frac{dt}{dx} = \frac{d}{dx}\left(\frac{x}{2}+\pi\right)$$

$$\frac{dt}{dx} = \frac{1}{2}$$

$2dt = dx \rightarrow$ (ii)

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Limits: **When** $x = \frac{\pi}{2} \Rightarrow t = \frac{(\pi/2)}{2} + \pi = \frac{5\pi}{4}$

When $x = \pi \Rightarrow t = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$

Substituting (i) and (ii) and corresponding limits

$$\int_{\pi/2}^{\pi} \cos\left(\frac{x}{2}+\pi\right)dx = \int_{5\pi/4}^{3\pi/2} \cos t \bullet 2dt = |2\sin t|_{5\pi/4}^{3\pi/2}$$

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$$= 2\sin\left(\frac{3\pi}{2}\right) - 2\sin\left(\frac{5\pi}{4}\right) = 2(-1) - 2(\frac{-1}{\sqrt{2}}) = -2 + \frac{2}{\sqrt{2}}$$

$$= -2 + \sqrt{2}$$

b) $\int_{0.75}^{2.5} x\cos x^2 dx$

Solution: Integration by substitution method

Let $t = x^2 \rightarrow$ (i)

Diff w.r.t x

$$\frac{dt}{dx} = 2x \Rightarrow \frac{dt}{2} = xdx \rightarrow (ii)$$

$t = x^2$

Limits: **When** $x = 0.75 = \frac{3}{4} \Rightarrow t = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

When $x = 2.5 = \frac{5}{2} \Rightarrow t = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$

Substituting (i) and (ii) and limits

$$\int_{0.75}^{2.5} x\cos x^2 dx = \int_{9/16}^{25/4} \cos t \bullet \frac{dt}{2} = \frac{1}{2}\int_{9/16}^{25/4} \cos t dt = \left|\frac{1}{2}\sin t\right|_{9/16}^{25/4}$$

$$= \frac{1}{2}\sin\left(\frac{25}{4}\right) - \frac{1}{2}\sin\left(\frac{9}{16}\right)$$

$$= -0.28$$

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c) $\int_{0}^{\pi/4} \sec^2\theta d\theta$

Solution: $\int_{0}^{\pi/4} \sec^2\theta d\theta = \left|\tan\theta\right|_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0$

$= 1 - 0$

$= 1$

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d) $\int_0^{\pi/4} \tan 2\pi x\,dx = \int_0^{\pi/4} \frac{\sin 2\pi x}{\cos 2\pi x}\,dx$

Solution: Integration by substitution method

Let $t=\cos 2\pi x$

$$\frac{dt}{dx} = -\sin 2\pi x \bullet 2\pi \Rightarrow \frac{dt}{-2\pi} = \sin 2\pi x\,dx$$

Limits: **When $x=0 \Rightarrow t=\cos 2\pi(0)=1$**

When $x=\frac{\pi}{4} \Rightarrow t=\cos 2\pi\left(\frac{\pi}{4}\right)=0.22$

Substituting (i) and (ii) and limits

$$\int_0^{\pi/4} \tan 2\pi x\,dx = \int_0^{\pi/4} \frac{\sin 2\pi x}{\cos 2\pi x}\,dx = \int_1^{0.22} \frac{1}{t}\frac{dt}{(-2\pi)}$$

$$= \left|\frac{-1}{2\pi}\ln(t)\right|_1^{0.22} = \frac{-1}{2\pi}\ln(0.222) - \frac{1}{2\pi}\ln(1)$$

$=0.241$

e) $\int_0^1 \frac{1}{x^2+1}\,dx$ s

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Solution: $\int_0^1 \frac{1}{1+x^2}\,dx = \left|\tan^{-1}x\right|_0^1 = \tan^{-1}1 - \tan^{-1}0$

$$\frac{\pi}{4} - 0 = \frac{\pi}{4}$$

f) $\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1}}\,dx = \left|\sec^{-1}x\right|_{\sqrt{2}}^2$

Solution: $\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1}}\,dx = \left|\sec^{-1}x\right|_{\sqrt{2}}^2 = \sec^{-1}2 - \sec^{-1}\sqrt{2}$

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$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Q.4 Evaluate the following definite integrals:

a) $\int_1^2 \frac{5t^2 - 3t + 18}{t(9 - t^2)} dt$

Solution: By partial fraction

Let $\frac{5t^2 - 3t + 18}{t(9 - t^2)} = \frac{A}{t} + \frac{B}{3 - t} + \frac{C}{3 + t} \quad \rightarrow (i)$

Multiplying by $t(3-t)(3+t)$

$5t^2 - 3t + 18 = A(3-t)(3+t) + B(t)(3+t) + c(t)(3-t) \rightarrow (ii)$

To find A, put $t=0$ in (ii)

$5(0)^2 - 3(0) + 18 = A(3-0)(3+0) + B(0)(3+0) + c(0)(3-0)$

$18 = 9A \quad \Rightarrow \quad A = \frac{18}{9} \Rightarrow \mathbf{A = 2}$

To find B, put $3-t=0 \quad \Rightarrow \quad t=3$ in (ii)

$5(3)^2 - 3(3) + 18 = A(3-3)(3+3) + B(3)(3+3) + c(3)(3-3)$

$54 = 18B \quad \Rightarrow \quad \mathbf{B=3}$

To find A, put $3+t=0 \quad \Rightarrow \quad t=-3$, in (ii)

$5(-3)^2 - 3(-3) + 18 = A(3-(-3))(3+(-3)) + B(-3)\ (3+(-3)) + c(-3)\ (3-(-3))$

$72 = -18C \quad \Rightarrow \quad \mathbf{C=-4}$

Putting value of A, B and C in (i)

$$\frac{5t^2 - 3t + 18}{t(9 - t^2)} = \frac{2}{t} + \frac{3}{3 - t} - \frac{4}{3 + t}$$

Taking integral on both sides

$$\int_1^2 \frac{5t^2 - 3t + 18}{t(9 - t^2)} dt = \int_1^2 \left(\frac{2}{t} + \frac{3}{3 - t} - \frac{4}{3 + t} \right) dt$$

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$$= \left|2\ln t - 3\ln|t-3| - 4\ln|t+3|\right|_1^2$$

$$= \left|2\ln t\right|_1^2 - \left|3\ln(t-3)\right|_1^2 - \left|4\ln(t+3)\right|_1^2$$

$$= 2(\ln 2 - \ln 1) - 3(\ln(3-2)) - \ln(3-1) - 4(4\ln(2+3) - \ln(1+3))$$

$$= 1.386 + 2.079 + 0.892$$

$$\mathbf{= 4.357}$$

b) $\displaystyle\int_2^3 \frac{\mathbf{4x^5 - 3x^4 - 6x^3 + 4x^2 + 6x - 1}}{\mathbf{(x-1)(x^2-1)}}$

Solution:

The given function is not in proper form so, first to convert it into proper form by usual division;

$$\frac{4x^5 - 3x^4 - 6x^3 + 4x^2 + 6x - 1}{(x-1)(x^2-1)} = \frac{4x^5 - 3x^4 - 6x^3 + 4x^2 + 6x - 1}{x^3 - x^2 - x + 1}$$

$$x^3 - x^2 - x + 1 \overline{)\ 4x^5 - 3x^4 - 6x^3 + 4x^2 + 6x - 1}\ \ 4x^2 + x - 1$$

$$\underline{\pm 4x^5 \mp 4x^4 \mp 4x^3 \pm 4x^2}$$

$$x^4 - 2x^3 + 0 + 6x + 1$$

$$\underline{\pm x^4 \mp x^3 \mp x^2 \pm x}$$

$$-x^3 + x^2 + 5x + 1$$

$$\underline{\mp x^3 \pm x^2 \pm x \mp 1}$$

$$4x + 2$$

So, $\dfrac{4x^5 - 3x^4 - 6x^3 + 4x^2 + 6x - 1}{(x-1)(x^2-1)} = 4x^2 + x - 1 + \dfrac{4x+2}{(x-1)(x^2-1)} \rightarrow (i)$

Now using partial fractions

$$\frac{4x+2}{(x-1)(x^2-1)} = \frac{4x+2}{(x-1)(x-1)(x+1)} = \frac{4x+2}{(x-1)^2(x+1)}$$

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$$\frac{4x+2}{(x+1)(x-1)^2}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}\rightarrow(ii)$$

Multiplying by $(x+1)(x-1)^2$

$$4x+2=A(x-1)^2+B(x+1)(x-1)+C(x+1)\rightarrow(iii)$$

To find A, put $x+1=0 \Rightarrow x=-1$ in (iii)

$$4(-1)+2=A(-1-1)^2+B(-1+1)(-1-1)+C(-1+1)$$

$$-2=4A \qquad \Rightarrow \qquad \mathbf{A=\frac{-1}{2}}$$

To find C, put $x-1=0 \Rightarrow x=1$ in (iii)

$$4(1)+2=A(1-1)^2+B(1+1)(1-1)+C(1+1)$$

$$6=2C \qquad \Rightarrow \qquad \mathbf{C=3}$$

To find B, expand equation (iii)

$$4x+2=Ax^2-2Ax+A+Bx^2-B+Cx+c$$

$$4x+2=(A+B)x^2-2(A-C)x+A-B+C$$

Comparing coefficients on both sides of x^2, x and x^o.

$$A+B=0, \qquad \Rightarrow \qquad \text{put } A=\frac{-1}{2}$$

$$\frac{-1}{2}+B=0, \qquad \Rightarrow \qquad \mathbf{B=\frac{1}{2}}$$

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Putting values of A, B and C in (ii)

$$\frac{4x+2}{(x-1)^2(x+1)}=\frac{-1/2}{x+1}+\frac{1/2}{x-1}+\frac{3}{(x-1)^2} \qquad \text{Putting in (i)}$$

$$\int_2^3\frac{4x^5-3x^4-6x^3+4x^2+6x-1}{(x-1)(x^2-1)}dx=\int_2^3 4x^2+x-1-\frac{1}{2(x-1)}+\frac{1}{2(x+1)}+\frac{3}{(x-1)}dx$$

$$=\left|\frac{4x^3}{3}+\frac{x^2}{2}-x-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+3\frac{(x-1)^{-1}}{-1}\right|_2^3$$

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$$=\left|\frac{4x^3}{3}+\frac{x^2}{2}-x-\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|-\frac{3}{(x-1)}\right|_2^3$$

$$=\left[\frac{4(3)^3}{3}+\frac{(3)^2}{2}-3+\frac{1}{2}\ln\left|\frac{3-1}{3+1}\right|-\frac{3}{(3-1)}\right]-\left[\frac{4(2)^3}{3}+\frac{(2)^2}{2}-2+\frac{1}{2}\ln\left|\frac{2-1}{2+1}\right|-\frac{3}{(2-1)}\right]$$

$$=36-\frac{9}{2}-3+\frac{1}{2}\ln\left(\frac{1}{2}\right)-\frac{3}{2}-\frac{32}{3}-2+2-\frac{1}{2}\ln\left(\frac{1}{3}\right)+3$$

$$=28.5$$

c) $\displaystyle\int_3^5 \frac{x^2-2}{(x-2)^2}dx$

Solution:

The given function is not in proper form so, first to convert it into proper form by usual division;

$$\frac{x^2-2}{(x-2)^2}=\frac{x^2-2}{x^2-4x+4}$$

$$\begin{array}{r|l} & 1 \\ x^2-4x+4 & x^2-2 \\ & \underline{\pm x^2 \pm 4 \mp 4x} \\ & 4x-6 \end{array}$$

So, $\dfrac{x^2-2}{(x-2)^2}=1+\dfrac{4x-6}{(x-2)^2}\rightarrow$ (i)

Now by partial fraction

$$\frac{4x-6}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}\rightarrow \text{(ii)}$$

Multiplying by $(x-2)^2$

$4x-6=A(x-2)+B \quad \rightarrow$(iii)

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To find B, put $x-2=0 \Rightarrow x=2$ in (iii)

$4(2)-6=A(2-2)+B \Rightarrow \mathbf{B=2}$

To find A, expanding (ii)

$4x-6=Ax-2A+B$

Comparing coefficients

$\mathbf{A=4}$

Putting values of A and B in (ii)

$$\frac{4x-6}{(x-2)^2}=\frac{4}{(x-2)}+\frac{2}{(x-2)^2}$$

Putting in (i)

$$\int_3^5 \frac{x^2-2}{(x-2)^2}dx=\int_3^5\left[1+\frac{4}{x-2}+\frac{2}{(x-2)^2}\right]dx$$

$$\int_3^5 \frac{x^2-2}{(x-2)^2}dx=\left|x+4\ln|x-2|+2\frac{(x-2)^{-1}}{-1}\right|_3^5$$

$$\int_3^5 \frac{x^2-2}{(x-2)^2}dx=\left|x+4\ln|x-2|-\frac{2}{x-2}\right|_3^5$$

$$\int_3^5 \frac{x^2-2}{(x-2)^2}dx=\left[5+4\ln|5-2|-\frac{2}{5-2}\right]-\left[3+4\ln|3-2|-\frac{2}{3-2}\right]$$

$$\int_3^5 \frac{x^2-2}{(x-2)^2}dx=7.73$$

d) $\displaystyle\int_1^2 \frac{4}{t^3+4t}dt$

Solution:

Using partial fraction expression

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$$\frac{4}{t^3+4t}=\frac{4}{t(t^2+4)}=\frac{A}{t}+\frac{Bt+C}{t^2+4}\rightarrow(i)$$

Multiplying by $t(t^2+4)$

$4=A(t^2+4)+(Bt+C)(t)\rightarrow(ii)$

To find A, put t=0 in (ii)

$4=A(0^2+4)+(B(0)+c)(0)$

$4=4A$

A=1

To find B and C, expanding (ii)

$4=At^2+4A+Bt^2+Ct$

$4=(A+B)t^2+Ct+4A$

Comparing coefficients on both sides of x^2

$A+B=0$ putting A=1

$1+B=0$

B=–1 and **C=0**

Putting values of A, B and C in (i)

$$\frac{4}{t^3+4t}=\frac{1}{t}+\frac{-1t+0}{t^2+4}$$

$$\frac{4}{t^3+4t}=\frac{1}{t}-\frac{t}{t^2+4}$$

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Taking integral on both sides

$$\int_1^2\frac{4}{t^3+4t}dt=\int_1^2\left(\frac{1}{t}-\frac{2t}{2\left(t^2+4\right)}\right)dt$$

$$\int_1^2\frac{4}{t^3+4t}dt=\left|\ln t-\frac{1}{2}\ln\left|t^2+4\right|\right|_1^2$$

$$=\left[\ln 2-\frac{1}{2}\ln\left(2^2+4\right)\right]-\left[\ln 1-\frac{1}{2}\ln\left(1^2+4\right)\right]$$

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$$= \ln(2) - \frac{1}{2}\ln(8) - \ln(1) + \frac{1}{2}\ln(5)$$

$$= 0.693 - \frac{1}{2}(2.08) - 0 + \frac{1}{2}(1.61)$$

$$= \mathbf{0.458}$$

e) $\int_1^3 \ln(2x+1)\,dx$

Solution:

Using integration by parts method,

$$\int \underset{\text{I}}{\ln(2x+1)}\,\underset{\text{II}}{1}\,dx = \ln(2x+1)\cdot\int 1\,dx - \int\left(\frac{d}{dx}(2x+1)\int 1\,dx\right)dx$$

$$= \ln(2x+1)x - \int \frac{2}{2x+1}\cdot x\,dx$$

$$= x\ln(2x+1) - \int \frac{2x}{2x+1}\,dx$$

$$= x\ln(2x+1) - \int\left(\frac{2x+1}{2x+1} - \frac{1}{2x+1}\right)dx$$

$$= x\ln(2x+1) - \int\left(1 - \frac{1}{2x+1}\right)dx$$

$$= x\ln(2x+1) - x + \frac{1}{2}\ln(2x+1)$$

Applying limits

$$\int_1^3 \ln(2x+1)\,dx = \left| x\ln(2x+1) - x + \frac{1}{2}\ln(2x+1)\right|_1^3$$

$$= \left[3\ln(2(3)+1) - 3 + \frac{1}{2}\ln(2(3)+1)\right] - \left[1\ln(2(1)+1) - 1 + \frac{1}{2}\ln(2(1)+1)\right]$$

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$$= 3\ln(7) - 3 + \frac{1}{2}\ln(7) - \ln(3) + 1 - \frac{1}{2}\ln(3)$$

=3.16

f) $\int_1^4 \frac{\ln x}{x^3} dx$

Solution:

Using integration by parts method,

$$\int \frac{\ln x}{x^3} dx = \int \underset{\text{I}}{\ln x} \cdot \underset{\text{II}}{\frac{1}{x^3}} dx$$

$$= \ln x \int \frac{1}{x^3} dx - \int \left(\frac{d}{dx}(\ln x) \cdot \int \frac{1}{x^3} dx \right) dx$$

$$= \ln x \cdot \frac{x^{-2}}{-2} - \int \frac{1}{x} \cdot \frac{x^{-2}}{-2} dx$$

$$= \frac{-1}{2x^2} \ln x + \frac{1}{2} \int x^{-3} dx$$

$$= \frac{-1}{2x^2} \ln x + \frac{1}{2} \frac{x^{-2}}{(-2)} = \frac{1}{-2x^2} \ln x - \frac{1}{4x^2}$$

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Applying limits

$$\int_1^4 \frac{\ln x}{x^3} dx = \left| -\frac{1}{2x^2} \ln x - \frac{1}{4x^2} \right|_1^4$$

$$= \left[\frac{-\ln 4}{2(4)^2} - \frac{1}{4(4)^2} \right] - \left[\frac{-\ln 1}{2(1)^2} - \frac{1}{4(1)^2} \right]$$

$$= -\frac{\ln 4}{32} - \frac{1}{64} + \frac{\ln 1}{2} + \frac{1}{4}$$

=0.19

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g) $\int_1^2 x(x-1)^6\,dx$

Solution:

Integration by substitution method

Let $t=x-1 \quad \rightarrow(i)$

$x=t+1 \quad \rightarrow(ii)$

Diff w.r.t x

$$\frac{dt}{dx}=1$$

$$dt=dx$$

Limits: When x=1, t=1–1=0

When x=2, t=2–1=1

Substituting these values is given integral

$$\int_1^2 x(x-1)^6\,dx=\int_0^1 (t+1)t^6dt=\int_0^1 (t^7+t^6)dt$$

$$=\left|\frac{t^8}{8}+\frac{t^7}{7}\right|_0^1$$

$$=\left[\frac{(1)^8}{8}+\frac{(1)^7}{7}\right]-\left[\frac{(0)^8}{8}+\frac{(0)^7}{7}\right]$$

$$=\frac{1}{8}+\frac{1}{7}=\frac{7+8}{56}=\frac{\mathbf{15}}{\mathbf{56}}$$

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h) $\int_0^1 (x-3)e^x dx$

Solution:

Using integration by parts method,

$$\int \underset{I}{(x-3)}\underset{II}{e^x dx}=(x-3)\int e^x dx-\int\left(\frac{d}{dx}(x-3)\int e^x dx\right)dx$$

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$$=(x-3)e^x-\int 1\cdot e^x dx$$

$$=(x-3)e^x-e^x$$

Applying limits

$$\int_0^1 (x-3)e^x dx=\left|(x-3)e^x-e^x\right|_0^1$$

$$=\left[(1-3)e^1-e^1\right]-\left[(0-3)e^0-e^0\right]$$

$$=-2e-e+3+1$$

$$=-3e+4$$

$=-4.155$

Q.5 Use the definite integral to find out the area between the curve f(x) and the x-axis over the indicated interval [a, b].

a) $f(x)=4-x^2,[0,3]$

Solution: To find, if the function changes sign in the given interval, we check if the functions cross x-axis in given interval putting,

$$f(x)=0$$

$$4-x^2=0$$

$$4=x^2$$

$$x=\pm 2$$

$x=2$ and **$x=-2$**

Thus the functions crosses x-axis at x=–2 and x=2. To find the area, we divide the interval into two subinterval [0,2] and [2,3]

Thus the area is

$$A=\left|\int_0^2 f(x)dx\right|+\left|\int_2^3 f(x)dx\right|$$

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$$A = \left|\int_0^2 (4 - x^2)dx\right| + \left|\int_2^3 (4 - x^2)dx\right|$$

$$A = \left|4x - \frac{x^3}{3}\right|_0^2 + \left|4x - \frac{x^3}{3}\right|_2^3$$

$$A = \left|4(2) - \frac{(2)^3}{3} - 4(0) + \frac{(0)^3}{3}\right| + \left|4(3) - \frac{(3)^3}{3} - 4(2) + \frac{(2)^3}{3}\right|$$

$$A = \left|8 - \frac{8}{3}\right| + \left|12 - 9 - 8 + \frac{8}{3}\right|$$

$$A = \frac{16}{3} + \left|\frac{-7}{3}\right|$$

$$A = \frac{16}{3} + \frac{7}{3}$$

$$\mathbf{A = \frac{23}{3}}$$

b) **$f(x) = x^2 - 5x + 6$, [0,3]**

Solution: To find the point where the given function crosses x-axis, put $f(x) = 0$.

$$x^2 - 5x + 6 = 0$$

$$x^2 - 3x - 2x + 6 = 0$$

$$(x-3)(x-2) = 0$$

$x = 3, x = 2$

As f(x) crosses x-axis at x=2 and x=3, the given interval is divided into two subintervals [0,2] and [2,3]

Thus the area is

$$A = \left|\int_0^2 f(x)dx\right| + \left|\int_2^3 f(x)dx\right|$$

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$$A = \left|\int_0^2 (x^2 - 5x + 6)dx\right| + \left|\int_2^3 (x^2 - 5c + 6)dx\right|$$

$$A = \left|\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right|_0^2 + \left|\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right|_2^3$$

$$A = \left|\frac{(2)^3}{3} - \frac{5(2)^2}{2} + 6(2) - \frac{(0)^3}{3} + \frac{5(0)^2}{2} - 6(0)\right| + \left|\frac{(3)^3}{3} - \frac{5(3)^2}{2} + 6(3) - \frac{(2)^3}{3} + \frac{5(2)^2}{2} - 6(2)\right|$$

$$A = \left|\frac{14}{3}\right| + \left|\frac{-1}{6}\right| = \frac{14}{3} + \frac{1}{6}$$

$$\mathbf{A = \frac{29}{6}}$$

c) **$f(x) = x^2 - 6x + 8$, [0,4]**

Solution: To find the point where the given function crosses x-axis, put $f(x) = 0$.

$x^2 - 6x + 8 = 0$

$x^2 - 4x - 2x + 8 = 0$

$(x-4)(x-2) = 0$

$x = 2, x = 4$

As f(x) crosses x-axis at x=2 and x=4, the given interval is divided into two subintervals [0,2] and [2,4]

Thus the area is

$$A = \left|\int_0^2 f(x)dx\right| + \left|\int_2^4 f(x)dx\right|$$

$$A = \left|\int_0^2 (x^2 - 6x + 8)dx\right| + \left|\int_2^4 (x^2 - 6x + 8)dx\right|$$

$$A = \left|\frac{x^3}{3} - \frac{6x^2}{2} + 8x\right|_0^2 + \left|\frac{x^3}{3} - \frac{6x^2}{2} + 8x\right|_2^4$$

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$$A=\left|\frac{(2)^3}{3}-3(2)^2+8(2)-\frac{(0)^3}{3}+3(0)^2-8(0)\right|+\left|\frac{(4)^3}{3}-3(4)^2+8(4)-\frac{(2)^3}{3}+3(2)^2-8(2)\right|$$

$$A=\left|\frac{8}{3}-12+16\right|+\left|\frac{64}{3}-48+32-\frac{8}{3}+12-16\right|$$

$$A=\left|\frac{20}{3}\right|+\left|\frac{-4}{3}\right|=\frac{20}{3}+\frac{4}{3}$$

$$A=\frac{24}{3}$$

A=8

d) **$f(x)=5x-x^2$, [1,3]**

Solution: To find the point where the given function crosses x-axis, put $f(x)=0$.

$5x-x^2=0$

$x(5-x)=0$

x=0, x=5

Since the function does not cross x-axis in the given interval [1,3] the area will be

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$$A=\int_1^3(5x-x^2)dx=\left|\frac{5x^2}{2}-\frac{x^3}{3}\right|_1^3$$

$$A=\frac{5(3)^3}{2}-\frac{(3)^2}{3}-\frac{5(1)^2}{2}+\frac{(1)^3}{3}$$

$$A=\frac{45}{2}-9-\frac{5}{2}+\frac{1}{3}=\frac{135-54-15+2}{6}$$

$$\mathbf{A=\frac{68}{6}}$$

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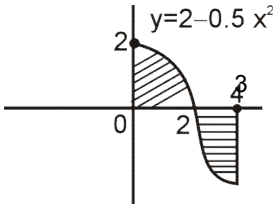
Q.6 Setup definite integrals in problems a to d that represent the indicated shaded area.

a) The area is;



Solution:

$$A = \left|\int_0^2 (2-0.5x^2)dx\right| - \left|\int_2^3 (2-0.5x^2)dx\right|$$

$$A = \left|2x - \frac{0.5x^3}{3}\right|_0^2 - \left|2x - \frac{0.5x^3}{3}\right|_2^3$$

$$A = \left|2(2) - \frac{0.5(2)^3}{3} - 2(0) + \frac{0.5(0)^3}{3}\right| - \left|2(3) - \frac{0.5(3)^3}{3} - 2(2) + \frac{0.5(2)^3}{3}\right|$$

$$A = \left|4 - \frac{4}{3}\right| - \left|6 - \frac{13.5}{3} - 4 + \frac{4}{3}\right|$$

$$A = \left|\frac{8}{3}\right| - \left|\frac{18 - 13.5 - 12 + 4}{3}\right|$$

$$A = \left|\frac{8}{3}\right| - \left|\frac{-3.5}{3}\right| = \frac{8}{3} + \frac{3.5}{3}$$

$$\mathbf{A = \frac{11.5}{3} = 3.83 \text{ square units}}$$

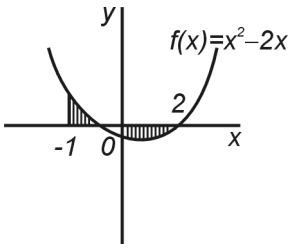
b) The area is;

Solution:

$$A = \left|\int_{-1}^0 (x^2 - 2x)dx\right| - \left|\int_0^2 (x^2 - 2x)dx\right|$$

$$A = \left|\frac{x^3}{3} - \frac{2x^2}{2}\right|_{-1}^0 - \left|\frac{x^3}{3} - \frac{2x^2}{2}\right|_0^2$$

$$A = \left|\frac{x^3}{3} - x^2\right|_{-1}^0 - \left|\frac{x^3}{3} - x^2\right|_0^2$$



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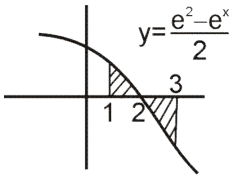
$$A=\left|\frac{(0)^3}{3}-(0)^2-\frac{(-1)^3}{3}+(-1)^2\right|-\left|\frac{(2)^3}{3}-(2)^2-\frac{(0)^3}{3}+(0)^2\right|$$

$$A=\left|\frac{1}{3}+1\right|-\left|\frac{8}{3}-4\right|=\left|\frac{4}{3}\right|-\left|\frac{-4}{3}\right|=\frac{4}{3}+\frac{4}{3}$$

$$\mathbf{A=\frac{8}{3}\ square\ units}$$

c) The area is;



Solution:

$$A=\left|\int_1^2\left(\frac{e^2-e^x}{2}\right)dx\right|-\left|\int_2^3\left(\frac{e^2-e^x}{2}\right)dx\right|$$

$$A=\left|\frac{e^2x-e^x}{2}\right|_1^2-\left|\frac{e^2x-e^x}{2}\right|_2^3$$

$$A=\left|\frac{e^2(2)-e^2}{2}-\frac{e^2(1)-e^1}{2}\right|-\left|\frac{e^2(3)-e^3}{2}-\frac{e^2(2)-e^2}{2}\right|$$

$$A=\left|\frac{e^2}{2}-\frac{e^2-e}{2}\right|-\left|\frac{3e^2-e^3}{2}-\frac{e^2}{2}\right|$$

$$A=\left|\frac{e}{2}\right|-\left|\frac{2e^2-e^3}{2}\right|=\frac{e}{2}+\frac{e^3-2e^2}{2}$$

$$A=\frac{e+e^3-2e^2}{2}=\frac{2.71828+20.08549-14.778}{2}=\frac{8.02577}{2}$$

A=4.01 square units

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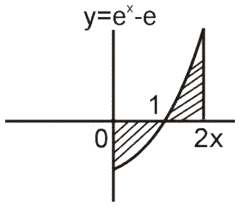
d) $y=e^x-e$

The area is;



Solution:

$$A=-\left|\int_0^1\left(e^x-e\right)dx\right|+\left|\int_1^2\left(e^x-e\right)dx\right|$$

$$A=-\left|e^x-ex\right|_0^1+\left|e^x-ex\right|_1^2$$

$$A=-\left|e^1-e(1)-e^0+e(0)\right|+\left|e^2-e(2)-e^1+e(1)\right|$$

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$$A = -|-1| + |e^2 - 2e| = 1 + e^2 - 2e$$

$$A = 1 + 7.3890 - 5.43656$$

A=2.953 square units

Q.7 An oil tanker is leaking oil at a rate given in barrels per hour by.

$$\frac{dL}{dt} = \frac{80\ln(t+1)}{t+1}$$

Where t is the time in hours after the tanker hits a hidden rock (when t=0)

a) Find the total number of barrels that the ship will leak on the first day.

Solution:

$$\frac{dL}{dt} = \frac{80\ln(t+1)}{t+1}$$

$$dL = \frac{80\ln(t+1)}{t+1}dt$$

The number of barrels leaking during first day (24 hours)

$$L_{1day} = \int_0^{24} \frac{80\ln(t+1)}{t+1}dt$$

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$$L_{1day} = \int_0^{24} \frac{80\ln(t+1)}{t+1}dt$$

Let $u = \ln(t+1) \rightarrow (i)$

$$\frac{du}{dt} = \frac{1}{t+1} \cdot 1$$

$$du = \frac{dt}{t+1} \rightarrow (ii)$$

Limits: **When $t=0 \Rightarrow u=\ln(0+1)=0$**

When $t=24 \Rightarrow u=\ln(25+1)=\ln 25$

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Substituting (i) and (ii), we get

$$L_{1day} = \int_0^{\ln 25} 80u \bullet du = 80\int_0^{\ln 25} udu = 80\left|\frac{u^2}{2}\right|_0^{\ln 25}$$

$$L_{1day} = 40\left[(\ln 25)^2 - (0)^2\right]$$

$$\mathbf{L_{1day} = 414.44 \text{ barrels}}$$

b) Find the total number of barrels that the ship will leak on the second day.

Solution: As $\frac{dL}{dt} = \frac{80\ln(t+1)}{t+1} \Rightarrow dL = \frac{80\ln(t+1)}{t+1}dt$

Leakage for second day is given by

$$L_{2day} = \int_{24}^{48} \frac{80\ln(t+1)}{t+1}dt$$

Let $u = \ln(t+1) \rightarrow (i)$

$$\frac{du}{dt} = \frac{1}{t+1} \Rightarrow du = \frac{1}{t+1}dt \rightarrow (ii)$$

Limits: **When t=24, u=ln (24+1)=ln25**

When t=48, u=ln (48+1)=ln49

Substituting (i)and (ii) we get

$$L_{2day} = \int_{\ln 25}^{\ln 49} 80udu = \left|80\frac{u^2}{2}\right|_{\ln 25}^{\ln 49} = 40\left[(\ln 49)^2 - (\ln 25)^2\right]$$

$$\mathbf{L_{2day} = 191.6 \text{ barrels}}$$

c) What is happening over long run to the amount of oil leakage per day?

Solution:

As is apparent from the amount of oil leaked on 1st and 2nd day, the number of barrel of oil leakage per day decreases with the passage of time.

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