SOME IMPORTANT POINTS OF CHAPTER NO: 4

The following points will enable you to solve the problems of this chapter

(i) In this chapter we discuss 3 types of sequences.

(a) Arithmetic Progression (A.P): $a_1, a_1+d, a_1+2d, a_1+3d, \ldots$

(b) Geometric Progression (G.P): $a_1, a_1r, a_1r^2, a_1r^3, \ldots$

(c) Hormonic Progression (H.P): Its reciprocals form A.P.

(ii) nth formula for A.P: $\underbrace{a_n}_{\text{nth term}} = \underbrace{a_1}_{\text{First term}} + (\underbrace{n}_{\text{number of terms}}-1)\underbrace{d}_{\text{common difference}}$, where $d = a_2 - a_1 = a_3 - a_2 = \ldots$

(iii) nth formula for G.P: $\underbrace{a_n}_{\text{nth term}} = \underbrace{a_1}_{\text{First term}} \underbrace{r}_{\text{common ratio}}^{\overbrace{n}^{\text{number of terms}}-1}$, where $r = \frac{r_2}{r_1} = \frac{r_3}{r_2} = \ldots$

(iv) Sum of first n terms of A.P. (i) $S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$

(ii) $S_n = \frac{n}{2}\{a_1 + a_n\}$ or $S_n = \frac{n}{2}(a_1 + \underbrace{l}_{\text{last term}})$

(v) Sum of first n terms of G.P.

(i) $S_n = a_1\frac{(r^n-1)}{r-1}, |r|>1$

$= a_1\frac{(1-r^n)}{1-r}, |r|<1$

(ii) $S_n = \frac{lr-a_1}{r-1}, |r|>1$

$= \frac{a_1 - lr}{1-r}, |r|<1$

(vi) Sum of infinite Geometric Series: $S_\infty = \frac{a_1}{1-r}, |r|<1$

(vii) $A.M = \frac{a+b}{2}$, $G.M = \pm\sqrt{ab}$, $H.M = \frac{2ab}{a+b}$

(viii) (i) $A > G > H$ (ii) $G^2 = AH$

(ix) To insert n A.Ms b/w a&b: **Procedure:** Let $A_1, A_2, A_3, \ldots, A_n$ be n A.Ms b/w a & b.
$\Rightarrow a_1, A_1, A_2, \ldots, A_n, b$ is A.P. (First find d & then use $A_n = a_1 + nd$).

(x) To insert n G.Ms b/w a&b: **Procedure:** Let $G_1, G_2, G_3, \ldots, G_n$ be n G.Ms b/w a & b.
$\Rightarrow a, G_1, G_2, G_3, \ldots, G_n, b$ is G.P (First find r & then use $G_n = a_1 r^n$)

(xi) For H.P find every thing from corresponding A.P.

EXERCISE 4.1

Q1: Classify the following into finite and infinite sequences.

(i) $2, 4, 6, 8, \ldots, 50$

Sol: Finite **Ans**

(ii) $1, 0, 1, 0, 1, \ldots$

Sol: Infinite **Ans**

(iii) $\ldots, -4, 0, 4, 8, \ldots, 60$

Sol: Infinite **Ans**

(iv) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots, -\frac{1}{2187}$

Sol: Finite **Ans**

Q.2. Find the first four terms of a sequence with the given general terms:

(i) $\frac{n(n+1)}{2}$

Sol: Let $a_n = \frac{n(n+1)}{2}$

Put $n = 1, 2, 3, 4$

$n=1: \quad a_1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$

$n=2: \quad a_2 = \frac{2(2+1)}{2} = \frac{2(3)}{2} = 3$

$n=3: \quad a_3 = \frac{3(3+1)}{2} = \frac{3(4)}{2} = 6$

$n=4: \quad a_4 = \frac{4(4+1)}{2} = \frac{4(5)}{2} = 10$

Thus the first four terms are

$\boxed{1, 3, 6, 10}$ **Ans**

(ii) $(-1)^{n-1} 2^{n+1}$

Sol: Let $a_n = (-1)^{n-1} 2^{n+1}$

Put $n = 1, 2, 3, 4$

$n=1: \quad a_1 = (-1)^{1-1} 2^{1+1} = (-1)^0 2^2 = 1 \cdot 4 = 4$

$n=2: \quad a_2 = (-1)^{2-1} 2^{2+1} = (-1)^1 2^3 = -8$

$n=3: \quad a_3 = (-1)^{3-1} 2^{3+1} = (-1)^2 2^4 = 2^4 = 16$

$n=4: \quad a_4 = (-1)^{4-1} 2^{4+1} = (-1)^3 2^5 = -2^5 = -32$

Thus the first four terms are

$\boxed{4, -8, 16, -32}$ **Ans**

(iii) $\left(\frac{1}{3}\right)^n$

Sol: Let $a_n = \left(\frac{1}{3}\right)^n$

Put $n = 1, 2, 3, 4$

$n=1$: $a_1 = \left(\frac{1}{3}\right)^1 = \frac{1}{3}$

$n=2$: $a_2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

$n=3$: $a_3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$

$n=4$: $a_4 = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$

Thus the first four terms are

$\boxed{\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}}$ Ans

(iv) $\frac{n(n-1)(n-2)}{6}$

Sol: Let $a_n = \frac{n(n-1)(n-2)}{6}$

Put $n = 1, 2, 3, 4$

$n=1$: $a_1 = \frac{1(1-1)(1-2)}{6} = 0$

$n=2$: $a_2 = \frac{2(2-1)(2-2)}{6} = 0$

$n=3$: $a_3 = \frac{3(3-1)(3-2)}{6} = \frac{3(2)(1)}{6} = 1$

$n=4$: $a_4 = \frac{4(4-1)(4-2)}{6} = \frac{4(3)(2)}{6} = 4$

Thus the first four terms are

$\boxed{0, 0, 1, 4}$ Ans

Q.3: Write down the nth term of each sequence as suggested by the pattern.

(i) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Sol: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Here $a_1 = \frac{1}{2} = \frac{1}{1+1}$

$a_2 = \frac{2}{3} = \frac{2}{2+1}$

$a_3 = \frac{3}{4} = \frac{3}{3+1}$

$a_4 = \frac{4}{5} = \frac{4}{4+1}$

$\vdots$

Hence $a_n = \frac{n}{n+1}$ Ans

(ii) $2, -4, 6, -8, 10, \ldots$

Sol: $2, -4, 6, -8, 10, \ldots$

Here $a_1 = 2 = (-1)^{1+1} 2\cdot1$

$a_2 = -4 = (-1)^{2+1} 2\cdot2$

$a_3 = 6 = (-1)^{3+1} 2\cdot3$

$a_4 = -8 = (-1)^{4+1} 2\cdot4$

$a_5 = 10 = (-1)^{5+1} 2\cdot5$

$\vdots$

Hence $a_n = (-1)^{n+1} 2n$ Ans

(iii) $1, -1, 1, -1, \ldots$

Sol: $1, -1, 1, -1, \ldots$

Here $a_1 = 1 = (-1)^{1+1}$

$a_2 = -1 = (-1)^{2+1}$

$a_3 = 1 = (-1)^{3+1}$

$a_4 = -1 = (-1)^{4+1}$

$\vdots$

Hence $a_n = (-1)^{n+1}$ Ans

Q.4 Write down the first five terms of each sequence defined recursively.

(i) $a_1=3,\ a_{n+1}=5-a_n$

Sol: Put $n=1,2,3,4$

$n=1:\ a_2=5-a_1=5-3=2$

$n=2:\ a_3=5-a_2=5-2=3$

$n=3:\ a_4=5-a_3=5-3=2$

$n=4:\ a_5=5-a_4=5-2=3$

Thus the first five terms are

$\boxed{3,2,3,2,3}$ Ans

(ii) $a_1=3,\ a_{n+1}=\frac{a_n}{n}$

Sol: Put $n=1,2,3,4$

$n=1:\ a_2=\frac{a_1}{1}=\frac{3}{1}=3$

$n=2:\ a_3=\frac{a_2}{2}=\frac{3}{2}$

$n=3:\ a_4=\frac{a_3}{3}=\frac{\frac{3}{2}}{3}=\frac{1}{2}$

$n=4:\ a_5=\frac{a_4}{4}=\frac{\frac{1}{2}}{4}=\frac{1}{8}$

Thus the first five terms are

$\boxed{3,3,\frac{3}{2},\frac{1}{2},\frac{1}{8}}$ Ans

Q.5 Write down the following series in expanded form.

(i) $\sum_{j=1}^{6}(2j-3)$

Sol: $\sum_{j=1}^{6}(2j-3)=(2\cdot1-3)+(2\cdot2-3)+(2\cdot3-3)+(2\cdot4-3)+(2\cdot5-3)+(2\cdot6-3)$

$=(2-3)+(4-3)+(6-3)+(8-3)+(10-3)+(12-3)$

$=\boxed{-1+1+3+5+7+9}$ Ans

(ii) $\sum_{k=1}^{5}(-1)^k2^{k-1}$

Sol: $\sum_{k=1}^{5}(-1)^k2^{k-1}=(-1)^1 2^{1-1}+(-1)^2 2^{2-1}+(-1)^3 2^{3-1}+(-1)^4 2^{4-1}+(-1)^5 2^{5-1}$

$=-2^0+2^1-2^2+2^3-2^4$

$=\boxed{-1+2-4+8-16}$ Ans

(iii) $\sum_{j=1}^{\infty}\frac{1}{2^j}$

Sol: $\sum_{j=1}^{\infty} \frac{1}{2^j} = \boxed{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots}$ Ans

(iv) $\sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^k$

Sol: $\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k = \left(\frac{3}{2}\right)^0+\left(\frac{3}{2}\right)^1+\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^3+\cdots$

$= \boxed{1+\frac{3}{2}+\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^3+\cdots}$ Ans

Q.6 Find the Pascal sequences for:

(i) $n=5$

Sol: Pascal sequence is given by

$\binom{n}{0}=1$ and $\binom{n}{r+1}=\frac{n-r}{r+1}\binom{n}{r}, r=0,1,2,3,\ldots$

Put $n=5$

$\binom{5}{0}=1$ and $\binom{5}{r+1}=\frac{5-r}{r+1}\binom{5}{r}, r=0,1,2,3,\ldots$

$r=0$: $\binom{5}{1}=\frac{5-0}{0+1}\binom{5}{0}=5\cdot 1=5$

$r=1$: $\binom{5}{2}=\frac{5-1}{1+1}\binom{5}{1}=\frac{4}{2}(5)=10$

$r=2$: $\binom{5}{3}=\frac{5-2}{2+1}\binom{5}{2}=\frac{3}{3}(10)=10$

$r=3$: $\binom{5}{4}=\frac{5-3}{3+1}\binom{5}{3}=\frac{2}{4}(10)=5$

$r=4$: $\binom{5}{5}=\frac{5-4}{4+1}\binom{5}{4}=\frac{1}{5}(5)=1$

Thus the required squence is $\boxed{1,5,10,10,5,1}$ Ans

(ii) $n = 6$

Sol: Pascal sequence is given by

$$\binom{n}{0} = 1 \text{ and } \binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}, r = 0, 1, 2, 3, \ldots$$

Put $n = 6$

$$\binom{6}{0} = 1 \text{ and } \binom{6}{r+1} = \frac{6-r}{r+1}\binom{6}{r}, r = 0, 1, 2, 3, \ldots$$

$r = 0$: $\binom{6}{1} = \frac{6-0}{0+1}\binom{6}{0} = \frac{6}{1}(1) = 6$

$r = 1$: $\binom{6}{2} = \frac{6-1}{1+1}\binom{6}{1} = \frac{5}{2}(6) = 15$

$r = 2$: $\binom{6}{3} = \frac{6-2}{2+1}\binom{6}{2} = \frac{4}{3}(15) = 20$

$r = 3$: $\binom{6}{4} = \frac{6-3}{3+1}\binom{6}{3} = \frac{3}{4}(20) = 15$

$r = 4$: $\binom{6}{5} = \frac{6-4}{4+1}\binom{6}{4} = \frac{2}{5}(15) = 6$

$r = 5$: $\binom{6}{6} = \frac{6-5}{5+1}\binom{6}{5} = \frac{1}{6}(6) = 1$

Thus the required sequence is $\boxed{1, 6, 15, 20, 15, 6, 1}$. Ans

(iii) $n = 7$

Sol: Pascal sequence is given by

$$\binom{n}{0} = 1 \text{ and } \binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}, r = 0, 1, 2, 3, \ldots$$

Put $n = 7$

$$\binom{7}{0} = 1 \text{ and } \binom{7}{r+1} = \frac{7-r}{r+1}\binom{7}{r}, r = 0, 1, 2, 3, \ldots$$

$r=0:$ $\quad \binom{7}{1} = \frac{7-0}{0+1}\binom{7}{0} = \frac{7}{1}(1) = 7$

$r=1:$ $\quad \binom{7}{2} = \frac{7-1}{1+1}\binom{7}{1} = \frac{6}{2}(7) = 21$

$r=2:$ $\quad \binom{7}{3} = \frac{7-2}{2+1}\binom{7}{2} = \frac{5}{3}(21) = 35$

$r=3:$ $\quad \binom{7}{4} = \frac{7-3}{3+1}\binom{7}{3} = \frac{4}{4}(35) = 35$

$r=4:$ $\quad \binom{7}{5} = \frac{7-4}{4+1}\binom{7}{4} = \frac{3}{5}(35) = 21$

$r=5:$ $\quad \binom{7}{6} = \frac{7-5}{5+1}\binom{7}{5} = \frac{2}{6}(21) = 7$

$r=6:$ $\quad \binom{7}{7} = \frac{7-6}{6+1}\binom{7}{6} = \frac{1}{7}(7) = 1$

Thus the required sequence is $\boxed{1, 7, 21, 35, 35, 21, 7, 1}$ Ans

EXERCISE 4.2

Q.1: Find the 15th term of the arithmetic sequence $2, 5, 8, \ldots$

Sol: $2, 5, 8, \ldots$

Here $a=2$, $d=5-2=3$, $n=15$

$a_n = a+(n-1)d$

$a_{15} = 2+(15-1)3$

$= 2+(14)3 = 2+42 = \boxed{44}$ Ans

Q.2: The 1^{st} term of an arithmetic sequence is 8 and the 21^{st} term is 108. Find the 7th term.

Sol: 1^{st} term $= a_1 = 8$

21^{st} term $= a_{21} = 108$

$\because a_n = a_1 + (n-1)d$

$\therefore a_{21} = a_1 + (21-1)d \Rightarrow 108 = 8 + 20d$

$\Rightarrow d = \frac{108-8}{20} = \frac{100}{20} = 5$

Now $a_7 = 8 + (7-1)5$

$= 8 + 30 = \boxed{38}$ Ans

Q.3: Find the number of terms in the arithmetic progression 6, 9, 12, ..., 78.

Sol: 6, 9, 12, ..., 78.

Here $a_1 = 6$, $d = 9-6 = 3$, $a_n = 78$, $n = ?$

$\because a_n = a_1 + (n-1)d$

$\therefore 78 = 6 + (n-1)3$

$78 = 6 + 3n - 3 \Rightarrow 78 = 3n + 3$

$\Rightarrow 3n = 78 - 3$

$\Rightarrow n = \frac{75}{3} = \boxed{25}$ Ans

Q.4: The nth term of a sequence is given by $a_n = 2n+7$. Show that it is an A.P. Also, find its 7^{th} term.

Sol: $a_n = 2n+7$

$n=1: \quad a_1 = 2(1) + 7 = 2 + 7 = 9$

$n=2:\ a_2 = 2(2)+7 = 4+7 = 11$

$n=3:\ a_3 = 2(3)+7 = 6+7 = 13$

$\therefore$ the sequence is $9, 11, 13, \ldots$

$\because\ 11-9 = 13-11 = \cdots = 2$

$\therefore$ the sequence is A.P with $d=2$

$a_n = a_1 + (n-1)d$

$a_7 = a_1 + 6d$

$= 9 + 6(2) = 9 + 12 = \boxed{21}$ Ans

Q.5: Show that the sequence $\log a, \log(ab), \log(ab^2), \log(ab^3), \ldots$ is an A.P. Find its nth term.

Sol: $\log a, \log(ab), \log(ab^2), \log(ab^3), \ldots$

$\because\ \log(ab) - \log a = \log(ab^2) - \log(ab)$

$\log\left(\frac{ab}{a}\right) = \log\left(\frac{ab^2}{ab}\right)$

$\log b = \log(b)$ which is true.

$\therefore$ this is an A.P. with $a_1 = \log a$, $d = \log b$

$\because\ a_n = a_1 + (n-1)d$

$= \log a + (n-1)\log b$

$= \log a + \log b^{n-1}$

$a_n = \boxed{\log(ab^{n-1})}$ Ans

Q.6: Find the value of 'k' if $2k+7, 6k-2, 8k-4$ are in A.P. Also find the sequence.

Sol: $\because$ $2k+7, 6k-2, 8k-4$ are in A.P.

$$\Rightarrow (6k-2)-(2k+7) = (8k-4)-(6k-2)$$
$$6k-2-2k-7 = 8k-4-6k+2$$
$$4k-9 = 2k-2$$
$$4k-2k = -2+9$$
$$2k = 7 \Rightarrow k = \frac{7}{2}$$

$\therefore$
$$2k+7 = 2\left(\frac{7}{2}\right)+7 = 7+7 = 14$$
$$6k-2 = 6\left(\frac{7}{2}\right)-2 = 21-2 = 19$$
$$8k-4 = 8\left(\frac{7}{2}\right)-4 = 28-4 = 24$$

Hence the sequence is $\boxed{14, 19, 24, \ldots}$ Ans

Q.7: If $a_6+a_4 = 6$ and $a_6-a_4 = \frac{2}{3}$, find the arithmetic sequence.

Sol:

$$a_6+a_4 = 6 \longrightarrow (1)$$
$$a_6-a_4 = \frac{2}{3} \longrightarrow (2)$$

$$(1)+(2) \Rightarrow 2a_6 = 6+\frac{2}{3}$$
$$\Rightarrow 2a_6 = \frac{20}{3}$$
$$\Rightarrow a_6 = \frac{10}{3}$$

$$a_6+a_4 = 6 \longrightarrow (1)$$
$$\pm a_6 \mp a_4 = \pm\frac{2}{3} \longrightarrow (2)$$

$$(1)-(2) \Rightarrow 2a_4 = 6-\frac{2}{3}$$
$$2a_4 = \frac{16}{3}$$
$$\Rightarrow a_4 = \frac{8}{3}$$

Since $a_n = a_1+(n-1)d$

$a_4 = a_1 + 3d$

$\frac{8}{3} = a_1 + 3d$

$8 = 3a_1 + 9d \longrightarrow (3)$

$a_6 = a_1 + 5d$

$\frac{10}{3} = a_1 + 5d$

$10 = 3a_1 + 15d \longrightarrow (4)$

$$10 = \cancel{3a_1} + 15d$$
$$\underline{\underset{-}{8} = \underset{-}{\cancel{3a_1}} \underset{-}{+} 9d}$$

$(4) - (3) \Rightarrow \quad 2 = 6d \Rightarrow d = \frac{1}{3}$

$(3) \Rightarrow \quad 8 = 3a_1 + 9(\frac{1}{3}) \Rightarrow a_1 = \frac{5}{3}$

Now

$a_1 = \frac{5}{3}$

$a_2 = a_1 + d = \frac{5}{3} + \frac{1}{3} = 2$

$a_3 = a_1 + 2d = \frac{5}{3} + 2(\frac{1}{3}) = \frac{5}{3} + \frac{2}{3} = \frac{7}{3}$

Thus the A.P is $\boxed{\frac{5}{3}, 2, \frac{7}{3}, \ldots}$ Ans

Q.8. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P. then prove $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Sol: $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

$\Rightarrow \frac{b+c-a}{a}+2, \frac{c+a-b}{b}+2, \frac{a+b-c}{c}+2$ are in A.P. (Adding 2 to each term of A.P.)

$\Rightarrow \frac{b+c+a}{a}, \frac{c+a+b}{b}, \frac{a+b+c}{c}$ are in A.P.

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. Proved (Dividing each term by $(a+b+c)$)

Q.9: A ball rolling up an incline covered 24m during the first second, 21m during the second, 18m during the third second. Find how many meters it covered in the eighth second?

Sol: The distances covered by the ball are

$24, 21, 18, \ldots$ (This is an A.P)

Here $a_1 = 24$, $d = 21 - 24 = -3$

$\because a_n = a_1 + (n-1)d$

$\Rightarrow a_8 = a_1 + 7d = 24 + 7(-3) = 24 - 21 = 3$

Thus the distance covered in 8th second is $\boxed{3m.}$ Ans

Q.10: The population of a town is decreasing by 500 inhabitants each year. If its population in 1960 was 20135, what was its population in 1970?

Sol: $a_1 = 20135$, $d = -500$. $n = 11$

$\because a_n = a_1 + (n-1)d$

$a_{11} = a_1 + 10d = 20135 + 10(-500) = 20135 - 5000 = 15135$

Hence its population in 1970 was $\boxed{15135}$. Ans

Q.11: Ahmad and Akram can climb 1000 feet in the first hour and 100 feet in each succeeding hour. When will they reach the top of a 5400 feet hill?

Sol: Here $a_1 = 1000$ ft, $d = 100$, $a_n = 5400$, $n = ?$

$\because a_n = a_1 + (n-1)d$

$$\therefore\ 5400 = 1000 + (n-1)100$$
$$5400 - 1000 = (n-1)100$$
$$4400 = (n-1)100$$
$$\Rightarrow n-1 = \frac{4400}{100} = 44 \Rightarrow n = 45$$

Thus they will reach the top of the hill in $\boxed{45 \text{ hours}}$ Ans

Q.12: A man earned \$3500 the first year he worked. If he recieved a raise of \$750 at the end of each year for 20 years, what was his salary during his twenty-first year of work?

Sol: $a_1 = 3500$, $d = 750$, $a_{21} = ?$

$$\because a_n = a_1 + (n-1)d$$
$$a_{21} = a_1 + 20d = 3500 + 20(750) = 3500 + 15000 = \boxed{\$18500}$$ Ans

Q.13 Find the arithmetic mean between the given numbers:

(i) 12, 18

Sol: Let $a = 12$, $b = 18$

$$A = \frac{a+b}{2} = \frac{12+18}{2} = \frac{30}{2} = \boxed{15}$$ Ans

(ii) $\frac{1}{3}, \frac{1}{4}$

Sol: Let $a = \frac{1}{3}$, $b = \frac{1}{4}$

$$A = \frac{a+b}{2} = \frac{\frac{1}{3}+\frac{1}{4}}{2} = \frac{1}{2}\left(\frac{4+3}{12}\right) = \boxed{\frac{7}{24}}$$ Ans

(iii) $-6, -216$

Sol: Let $a = -6$, $b = -216$

$$A = \frac{a+b}{2} = \frac{-6-216}{2} = \frac{-222}{2} = \boxed{-111}$$ Ans

(iv) $(a+b)^2, (a-b)^2$

Sol: Let $a_1 = (a+b)^2$, $b_1 = (a-b)^2$

$$A = \frac{a_1+b_1}{2} = \frac{(a+b)^2+(a-b)^2}{2}$$
$$= \frac{a^2+b^2+\cancel{2ab}+a^2+b^2-\cancel{2ab}}{2} = \frac{2a^2+2b^2}{2}$$
$$= \boxed{a^2+b^2}$$ Ans

Q.14: Insert: (i) Three arithmetic means between 6 and 41.

Sol: Let A_1, A_2, A_3 be three A.Ms between 6 and 41.

$\Rightarrow$ $6, A_1, A_2, A_3, 41$ is an A.P.

Here $a_1 = 6, a_5 = 41, n = 5$

First we find d.

$\because a_n = a_1 + (n-1)d$

$a_5 = a_1 + 4d$

$41 = 6 + 4d \Rightarrow d = \frac{35}{4}$

Now $A_n = a_1 + nd$

$A_1 = a_1 + d = 6 + \frac{35}{4} = \frac{24+35}{4} = \frac{59}{4} = 14\frac{3}{4}$

$A_2 = a_1 + 2d = 6 + 2\left(\frac{35}{4}\right) = \frac{12+35}{2} = \frac{47}{2} = 23\frac{1}{2}$

$A_3 = a_1 + 3d = 6 + 3\left(\frac{35}{4}\right) = \frac{24+105}{4} = \frac{129}{4} = 32\frac{1}{4}$

Thus the three A.Ms are $\boxed{14\frac{3}{4}, 23\frac{1}{2}, 32\frac{1}{4}}$ Ans.

(ii) Four arithmetic means between 17 and 32.

Sol: Let A_1, A_2, A_3, A_4 be the four A.Ms between 17 and 32.

$\Rightarrow$ $17, A_1, A_2, A_3, A_4, 32$ is an A.P.

Here $a = 17, n = 6, a_6 = 32$

First we find d.

$\because a_n = a + (n-1)d$

$\therefore \quad a_6 = a + 5d$

$32 = 17 + 5d \Rightarrow d = \frac{32-17}{5} = \frac{15}{5} = 3$

Now $A_n = a + nd$

$A_1 = a + d = 17 + 3 = 20$

$A_2 = a + 2d = 17 + 2(3) = 17 + 6 = 23$

$A_3 = a + 3d = 17 + 3(3) = 17 + 9 = 26$

$A_4 = a + 4d = 17 + 4(3) = 17 + 12 = 29$

Thus the four A.Ms are $\boxed{20, 23, 26, 29.}$ Ans

Q.15 For what value of n, $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the arithmetic mean between a and b?

Sol: Let $\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = A.M$

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \frac{a+b}{2}$$

$$2a^{n+1} + 2b^{n+1} = (a^n + b^n)(a+b)$$

$$2a^{n+1} + 2b^{n+1} = a^{n+1} + a^n b + ab^n + b^{n+1}$$

$$2a^{n+1} + 2b^{n+1} - a^{n+1} - b^{n+1} = a^n b + ab^n$$

$$a^{n+1} + b^{n+1} = a^n b + ab^n$$

$$a^{n+1} - a^n b = ab^n - b^{n+1}$$

$$a^n a - a^n b = ab^n - b^n b$$
$$a^n\cancel{(a-b)} = b^n\cancel{(a-b)}$$
$$\frac{a^n}{b^n} = 1$$
$$\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0 \Rightarrow \boxed{n=0} \text{ Ans}$$

Q.16. Insert five arithmetic means between 5 and 8 and show that their sum is five times the arithmetic mean between 5 and 8.

Sol: Let A_1, A_2, A_3, A_4, A_5 be the five A.Ms between 5 and 8.

$\Rightarrow$ $5, A_1, A_2, A_3, A_4, A_5, 8$ is an A.P.

Here $a=5$, $n=7$, $a_7=8$

We first find d.

$$\because a_n = a+(n-1)d$$
$$\therefore a_7 = a+6d$$
$$8 = 5+6d \Rightarrow d = \frac{8-5}{6} = \frac{3}{6} = \boxed{\frac{1}{2}}$$

Now $A_n = a+nd$

$$A_1 = a+d = 5+\frac{1}{2} = \frac{10+1}{2} = \frac{11}{2}$$
$$A_2 = a+2d = 5+2\left(\frac{1}{2}\right) = 5+1 = 6$$
$$A_3 = a+3d = 5+3\left(\frac{1}{2}\right) = 5+\frac{3}{2} = \frac{10+3}{2} = \frac{13}{2}$$
$$A_4 = a+4d = 5+4\left(\frac{1}{2}\right) = 5+2 = 7$$
$$A_5 = a+5d = 5+5\left(\frac{1}{2}\right) = 5+\frac{5}{2} = \frac{10+5}{2} = \frac{15}{2}$$

A.M between 5 and 8 $= \frac{5+8}{2} = \frac{13}{2}$

Sum of the five A.Ms between 5 and 8 $= A_1 + A_2 + A_3 + A_4 + A_5$

$= \frac{11}{2} + 6 + \frac{13}{2} + 7 + \frac{15}{2}$

$= \frac{11+12+13+14+15}{2}$

$= \frac{65}{2}$

$= 5\left(\frac{13}{2}\right)$

$= 5$ (A.M between 5 and 8)

Proved.

Q.17 There are n arithmetic means between 5 and 32 such that the ratio of the 3rd and 7th means is 7:13. Find the value of n.

Sol: Let $A_1, A_2, A_3, \ldots, A_n$ be the n A.Ms between 5 and 32.

$\Rightarrow 5, A_1, A_2, A_3, \ldots, A_n, 32$ is an A.P.

Here $a = 5$, $n = n+2$, $a_{n+2} = 32$

We first find d.

$\because a_n = a + (n-1)d$

$a_{n+2} = a + (n+2-1)d$

$a_{n+2} = a + (n+1)d$

$32 = 5 + (n+1)d \Rightarrow d = \frac{32-5}{n+1} = \frac{27}{n+1}$

Now $A_n = a + nd$

$A_3 = a + 3d = 5 + 3\left(\frac{27}{n+1}\right) = \frac{5n+5+81}{n+1} = \frac{5n+86}{n+1}$

$A_7 = a+7d = 5+7\left(\frac{27}{n+1}\right) = \frac{5n+5+189}{n+1} = \frac{5n+194}{n+1}$

$A_3 : A_7 = 7:13$

$\Rightarrow 13A_3 = 7A_7$

$13\left(\frac{5n+86}{\cancel{n+1}}\right) = 7\left(\frac{5n+194}{\cancel{n+1}}\right)$

$65n + 1118 = 35n+1358$

$65n - 35n = 1358-1118$

$30n = 240$

$\Rightarrow n = \frac{240}{30} \Rightarrow \boxed{n=8}$ Ans

EXERCISE 4.3

Q.1 Find the indicated term and the sum of the indicated number of terms in case of each of the following arithmetic sequence.

(i) 9, 7, 5, 3, ...; 20th term; 20 terms

Sol: 9, 7, 5, 3, ...

Here $a = 9$, $d = 7-9 = -2$, $n = 20$, a_{20}, S_{20}?

For a_{20} $\quad\because a_n = a+(n-1)d$

$a_{20} = a+19d = 9+19(-2) = 9-38 = -29$

For S_{20} $\quad\because S_n = \frac{n}{2}[2a+(n-1)d]$

$\therefore S_{20} = \frac{20}{2}[2a+19d] = 10[2(9)+19(-2)] = 10(18-38)$
$= 10(-20) = \boxed{-200}$ Ans

(ii) $3, \frac{8}{3}, \frac{7}{3}, 2, \ldots;$ 11^{th} term; 11 terms

Sol: Here $a=3$, $d = \frac{8}{3} - 3 = \frac{8-9}{3} = -\frac{1}{3}$, $n=11$, a_{11}, S_{11}?

For a_{11} $\because a_n = a+(n-1)d$

$a_{11} = a+10d = 3+10(-\frac{1}{3}) = \frac{9-10}{3} = -\frac{1}{3}$

For S_{11} $\because S_n = \frac{n}{2}(2a+(n-1)d)$

$\therefore S_{11} = \frac{11}{2}(2a+10d) = \frac{11}{2}[2(3)+10(-\frac{1}{3})] = \frac{11}{2}(6-\frac{10}{3})$

$= \frac{11}{2}(\frac{18-10}{3}) = \frac{11}{2}(\frac{8}{3}) = \boxed{\frac{44}{3}}$ Ans

Q.2: Some of the components a_1, a_n, n, d and S_n are given. Find the ones that are missing:

(i) $a_1 = 2$, $n = 17$, $d = 3$

Sol: a_{17}, S_{17}?

For a_{17} $\because a_n = a_1 + (n-1)d$

$\therefore a_{17} = a_1 + 16d = 2 + 16(3) = 2 + 48 = 50$ Ans

For S_{17} $\because S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\therefore S_{17} = \frac{17}{2}[2a_1 + 16d] = \frac{17}{2}[2(2)+16(3)] = \frac{17}{2}(4+48)$

$= \frac{17}{2}(52) = 17(26) = \boxed{442}$ Ans

(ii) $a_1 = -40$, $S_{21} = 210$

Sol: a_{21}, d?

For a_{21} $\because S_n = \frac{n}{2}(a_1 + a_n)$

$S_{21} = \frac{21}{2}(a_1 + a_{21})$

$210 = \frac{21}{2}(-40 + a_{21})$

$20 = -40 + a_{21} \Rightarrow a_{21} = 60$

For d $\because a_n = a_1 + (n-1)d$

$\therefore a_{21} = a_1 + 20d$

$60 = -40 + 20d \Rightarrow d = \frac{100}{20} = \boxed{5}$ Ans

(iii) $a_1 = -7, d = 8, S_n = 225$

Sol: n, a_n?

For n $\because S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$225 = \frac{n}{2}[2(-7) + (n-1)(8)]$

$450 = n[-14 + 8n - 8]$

$450 = n(8n - 22)$

$\Rightarrow 8n^2 - 22n - 450 = 0$

$\Rightarrow 4n^2 - 11n - 225 = 0$ $(\div \text{ by } 2)$

4×225
$4 \times 45 \times 5$
$4 \times 9 \times 5 \times 5$
36×25

$4n^2 - 36n + 25n - 225 = 0$

$4n(n-9) + 25(n-9) = 0$

$(n-9)(4n+25) = 0$

$n - 9 = 0 \Rightarrow n = 9$ | $4n + 25 = 0$, $n = \frac{-25}{4}$ (Not possible)

$\therefore \boxed{n = 9}$

For a_n $\quad \because a_n = a_1 + (n-1)d$

$$a_9 = a_1 + 8d$$
$$= -7 + 8(8)$$
$$= 57 \text{ Ans}$$

(iv) $a_n = 4,\ S_{15} = 30,\ n = 15$

Sol: d, a_1?

For d $\quad \because a_n = a_1 + (n-1)d$

$$4 = a_1 + 14d \longrightarrow (1)$$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$
$$S_{15} = \frac{15}{2}[2a_1 + 14d]$$
$$30 = \frac{15}{2} \times 2[a_1 + 7d]$$
$$30 = 15(a_1 + 7d)$$
$$2 = a_1 + 7d \longrightarrow (2)$$

$$\cancel{a_1} + 14d = 4 \longrightarrow (1)$$
$$\cancel{\pm a_1} \pm 7d = \pm 2 \longrightarrow (2)$$

$(1)-(2) \Rightarrow \quad 7d = 2 \Rightarrow \boxed{d = \frac{2}{7}}$ Ans

For a_1 $\quad$ Put $d = \frac{2}{7}$ in (1)

$$4 = a_1 + 14\left(\frac{2}{7}\right)$$
$$4 = a_1 + 4 \Rightarrow \boxed{a_1 = 0} \text{ Ans}$$

Q.3: Find the sum of all the numbers divisible by 5 from 25 through 350.

Sol: The numbers which are divisible by 5 from 25 through 350 are

25, 30, 35, ..., 350 (This is an A.P)

Here $a_1 = 25$, $d = 30 - 25 = 5$, $a_n = 350$, S_n?

We first find n.

$$\because a_n = a_1 + (n-1)d$$
$$350 = 25 + (n-1)5$$
$$350 - 25 = (n-1)5$$
$$325 = (n-1)5 \Rightarrow n-1 = \frac{325}{5}$$
$$\Rightarrow n-1 = 65 \Rightarrow \boxed{n = 66}$$

Now $S_n = 25 + 30 + 35 + \cdots + 350$

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_n = \frac{66}{2}(25 + 350)$$
$$= 33(375) = \boxed{12375}$$ Ans

Q.4: The sum of three numbers in an arithmetic sequence is 36 and the sum of their cubes is 6336. Find them. [Hint: suppose the numbers are a-d, a, a+d].

Sol: Suppose the numbers are $a-d, a, a+d$.

According to the Ist condition;

$$(a - \cancel{d}) + a + (a + \cancel{d}) = 36$$

$$3a = 36$$

$$\Rightarrow a = 6$$

<u>According to the 2nd condition:</u>

$$(a-d)^3 + a^3 + (a+d)^3 = 6336$$

$$a^3 - 3a^2d + 3ad^2 - d^3 + a^3 + a^3 + 3a^2d + 3ad^2 + d^3 = 6336$$

$$3a^3 + 6ad^2 = 6336$$

$$a^3 + 2ad^2 = 2112 \qquad (\div \text{ by } 3)$$

$$(12)^3 + 3(12)d^2 = 2112 \qquad (\because a = 12)$$

$$1728 + 24d^2 = 2112$$

$$\Rightarrow 24d^2 = 2112 - 1728$$

$$\Rightarrow 24d^2 = 384 \Rightarrow d^2 = \frac{384}{24} = 16$$

$$\Rightarrow d = \pm 4$$

$d = 4$	$d = -4$
$\therefore a - d = 12 - 4 = 8$	$\therefore a - d = 12 + 4 = 16$
$a = 12$	$a = 12$
$a + d = 12 + 4 = 16$	$a + d = 12 - 4 = 8$

Hence the numbers are $\boxed{8, 12, 16 \text{ or } 16, 12, 8}$ Ans

Q.5 Find four numbers in arithmetic sequence, whose sum is 20 and the sum of whose squares is 120.
<u>[Hint: Suppose the numbers are $a-3d, a-d, a+d, a+3d$].</u>

Sol: Suppose the numbers are $a-3d, a-d, a+d, a+3d$.

<u>According to the Ist condition:</u>

$$(a \cancel{-3d}) + (a \cancel{-d}) + (a \cancel{+d}) + (a \cancel{+3d}) = 20$$

$$4a = 20$$
$$\Rightarrow a = 5$$

According to the second condition:

$$(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2 = 120$$
$$a^2-6\cancel{ad}+9d^2+a^2-\cancel{2ad}+d^2+a^2+\cancel{2ad}+d^2+a^2+\cancel{6ad}+9d^2=120$$
$$4a^2+20d^2=120$$
$$\Rightarrow a^2+5d^2=30 \quad (\div \text{ by } 4)$$
$$\Rightarrow 5^2+5d^2=30 \quad (\because a=5)$$
$$\Rightarrow 25+5d^2=30$$
$$\Rightarrow 5d^2=30-25 \Rightarrow 5d^2=5 \Rightarrow d^2=1 \Rightarrow d=\pm 1$$

$d=1$	$d=-1$
$\therefore a-3d = 5-3(1) = 5-3 = 2$	$\therefore a-3d = 5-3(-1) = 5+3 = 8$
$a-d = 5-1 = 4$	$a-d = 5-(-1) = 5+1 = 6$
$a+d = 5+1 = 6$	$a+d = 5-1 = 4$
$a+3d = 5+3(1) = 5+3 = 8$	$a+3d = 5+3(-1) = 5-3 = 2$

Hence the numbers are $\boxed{2, 4, 6, 8 \text{ or } 8, 6, 4, 2}$ Ans

Q.6: $x_1, x_2, x_3, \ldots$ are in A.P. If $x_1+x_7+x_{10}=-6$ and $x_3+x_8+x_{12}=-11$. Find $x_3+x_8+x_{22}$.

Sol: Let the common difference be d.

$x_1+x_7+x_{10}=-6$	$x_3+x_8+x_{12}=-11$
$x_1+x_1+6d+x_1+9d=-6$	$x_1+2d+x_1+7d+x_1+11d=-11$
$3x_1+15d=-6 \longrightarrow (i)$	$3x_1+20d=-11 \longrightarrow (ii)$

(i) − (ii) $\Rightarrow$

$$\begin{array}{r} \cancel{3x_1}+15d=-6 \\ \underline{\pm\cancel{3x_1}\pm 20d=\mp 11} \\ -5d=5 \end{array} \Rightarrow \boxed{d=-1}$$

(i) $\Rightarrow$ $3x_1+15(-1)=-6 \Rightarrow 3x_1=-6+15$

$\Rightarrow 3x_1=9 \Rightarrow x_1=3$

Now $x_3+x_8+x_{22}=x_1+2d+x_1+7d+x_1+21d$

$=3x_1+3d$

$=3(3)+3(-1)$

$=9-30=\boxed{-21}$ Ans.

Q.7: Find: $1+3-5+7+9-11+13+15-17+\cdots$ up to $3n$ terms.

Sol: $S_{3n}=1+3-5+7+9-11+13+15-17+\cdots$ up to $3n$ terms.

Method I The given series is not Arithmetic series but we can write it as the sum of three Arithmetic series.

$S_{3n}=(1+7+13+\cdots$ up to n terms$)+(3+9+15+\cdots$ up to n terms$)$

$-(5+11+17+\cdots$ up to n terms$)$

$=\frac{n}{2}\{2(1)+(n-1)6\}+\frac{n}{2}\{2(3)+(n-1)6\}-\frac{n}{2}\{2(5)+(n-1)6\}$

$=\frac{n}{2}(2+6n-6)+\frac{n}{2}(\cancel{6}+6n-\cancel{6})-\frac{n}{2}(10+6n-6)$

$=\frac{n}{2}(6n-4)+\frac{n}{2}(6n)-\frac{n}{2}(6n+4)$

$=\frac{n}{2}[6n-4+\cancel{6n}-\cancel{6n}-4]=\frac{n}{2}(6n-8)=\boxed{3n^2-4n}$ Ans

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Method II

$S_{3n} = 1+3-5+7+9-11+13+15-17+\ldots$ up to $3n$ terms

$= (1+3-5)+(7+9-11)+(13+15-17)+\ldots$ up to n terms

$S_{3n} = -1 + 5 + 11 + \ldots$ up to n terms (This is an A.P)

$= \frac{n}{2}[2(-1)+(n-1)6]$ (Using $S_n = \frac{n}{2}[2a+(n-1)d]$ with $a=-1$, $d=5-(-1)=6$)

$= \frac{n}{2}[-2+6n-6]$

$= \frac{n}{2}(6n-8)$

$= \boxed{3n^2-4n}$ Ans

Q.8: Show that the sum of the first n positive odd integers is n^2.

Sol: Let $S_n = 1+3+5+\cdots$ up to n Terms. (This is an A.P)

$= \frac{n}{2}[2a_1+(n-1)d]$ $\because a_1=1$, $d=3-1=2$

$= \frac{n}{2}[2(1)+(n-1)2]$

$= \frac{n}{2}[\cancel{2}+2n-\cancel{2}]$

$= \boxed{n^2}$ Proved

Q.9: Find the sum of all multiples of 9 between 300 and 700.

Sol: The numbers which are multiples of 9 between 300 and 700 are

$306, 315, 324, \ldots, 693$ (This is an A.P)

Here $a_1 = 306$, $d = 315-306 = 9$, $a_n = 693$, S_n?

We first find n.

$\because a_n = a_1 + (n-1)d$

$693 = 306 + (n-1)9$

$n-1 = \frac{693-306}{9} = \frac{387}{9} = 43$

$\Rightarrow n = 43+1 = 44$

Now $S_n = 306 + 315 + 324 + \dots + 693$

$= \frac{n}{2}(a_1 + a_n)$

$= \frac{44}{2}(306 + 693)$

$= 22(999) = \boxed{21978}$ Ans

Q.10: The sum of Rs.1000 is distributed among four people so that each person after the first recieves Rs.20 less than the preceding person. How much does each person recieve?

Sol: Here $S_n = 1000$, $n = 4$, $d = -20$

We first find a_1.

$\because S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$1000 = \frac{4}{2}[2a_1 + (4-1)(-20)]$

$1000 = 2[2a_1 - 60]$

$500 = 2a_1 - 60$

$500 + 60 = 2a_1 \Rightarrow 2a_1 = 560 \Rightarrow a_1 = 280$

$a_1 = 280$

$\therefore a_2 = a_1 + d = 280 - 20 = 260$

$a_3 = a_1+2d = 280+2(-20) = 280-40 = 240$

$a_4 = a_1+3d = 280+3(-20) = 280-60 = 220$

Thus

Ist person	recieves	=	Rs.280
2nd	"	" =	Rs.260
3rd	"	" =	Rs. 240
4th	"	" =	Rs. 220

Ans

Q.11: The distance which an object dropped from a cliff will fall 16ft the first second, 48ft the next second, 80ft the third second and so on. What is the total distance the object will fall in six seconds?

Sol: The distances are

16, 48, 80,... (This is an A.P)

$a_1 = 16,\ d = 48-16 = 32,\ n = 6$

$S_n = 16+48+80+\cdots$ to 6 terms

$\because S_n = \frac{n}{2}\{2a_1+(n-1)d\}$

$S_6 = \frac{6}{2}\{2a_1+5d\} = 3\{2(16)+5(32)\} = 3\{32+160\} = 3(192) = \boxed{576ft}$ Ans

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Q.12: Afzal Khan saves Rs.1 the first day, Rs.2 the second, Rs.3 the third and Rs.N on the nth day for thirty days. How much does he save at the end of the thirtieth day?

Sol: $S_{30} = 1+2+3+\cdots+30$

$= \frac{30}{2}\{1+30\} = 15(31) = \boxed{465}$ Ans

Q.13: A theater has 40 rows with 20 seats in the first row, 23 in the second row, 26 in the third row and so forth. How many seats are in the theater?

Sol: No of seats in the rows are

20, 23, 26, ... (This is an A.P)

$a_1 = 20$, $d = 23 - 20 = 3$, $n = 40$

$S_{40} = 20 + 23 + 26 + \dots$ up to 40 terms

$= \frac{40}{2}[2(20) + (40-1)3]$ using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$= 20(40 + 117)$

$= 20(157)$

$= \boxed{3140}$ Ans

Q.14: Insert enough arithmetic means between 1 and 50 so that the sum of the resulting series will be 459.

Sol: Let $A_1, A_2, A_3, \dots, A_n$ be n A.Ms b/w 1 and 50.

$\Rightarrow 1, A_1, A_2, A_3, \dots, A_n, 50$ is an A.P.

Here $a_1 = 1$, $n = n+2$, $a_{n+2} = 50$

We first find d.

$\because a_n = a_1 + (n-1)d$

$a_{n+2} = a_1 + (n+2-1)d$

$50 = 1 + (n+1)d$

$49 = (n+1)d$

$\Rightarrow d = \frac{49}{n+1}$

Now we find n.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{n+2} = \frac{n+2}{2}(a_1 + a_{n+2})$$

$$459 = \frac{n+2}{2}(1+50)$$

$$918 = (n+2)(51)$$

$$\frac{918}{51} = n+2$$

$$18 = n+2 \Rightarrow n = 16$$

$\because$ $$d = \frac{49}{n+1} = \frac{49}{16+1} = \frac{49}{17}$$

Now $A_n = a_1 + nd$

$$A_1 = a_1 + d = 1 + \frac{49}{17} = \frac{17+49}{17} = \frac{66}{17}$$

$$A_2 = a_1 + 2d = 1 + 2\left(\frac{49}{17}\right) = \frac{17+98}{17} = \frac{115}{17}$$

$$A_3 = a_1 + 3d = 1 + 3\left(\frac{49}{17}\right) = \frac{17+147}{17} = \frac{164}{17}$$

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$$A_{16} = a_1 + 16d = 1 + 16\left(\frac{49}{17}\right) = \frac{17+784}{17} = \frac{801}{17}$$

Hence the required A.Ms are

$$\boxed{\frac{66}{17}, \frac{115}{17}, \frac{164}{17}, \ldots, \frac{801}{17}}$$ Ans

EXERCISE 4.4

Q.1. Write the first five terms of a geometric sequence given that:

(i) $a_1 = 5$; $r = 3$

Sol: $\because a_n = a_1 r^{n-1}$

Put $n = 2, 3, 4, 5$

$n=2$: $a_2 = a_1 r^{2-1} = a_1 r = (5)(3) = 15$

$n=3$: $a_3 = a_1 r^{3-1} = a_1 r^2 = (5)(3)^2 = 5\times 9 = 45$

$n=4$: $a_4 = a_1 r^{4-1} = a_1 r^3 = (5)(3)^3 = 5\times 27 = 135$

$n=5$: $a_5 = a_1 r^{5-1} = a_1 r^4 = (5)(3)^4 = 5\times 81 = 405$

Hence the required five terms are

$\boxed{5, 15, 45, 135, 405}$ Ans

(ii) $a_1 = 8$, $r = -\frac{1}{2}$

Sol: $\because a_n = a_1 r^{n-1}$

Put $n = 2, 3, 4, 5$

$n=2$: $a_2 = a_1 r^{2-1} = a_1 r = (8)(-\frac{1}{2}) = -4$

$n=3$: $a_3 = a_1 r^{3-1} = a_1 r^2 = (8)(-\frac{1}{2})^2 = (8)(\frac{1}{4}) = 2$

$n=4$: $a_4 = a_1 r^{4-1} = a_1 r^3 = (8)(-\frac{1}{2})^3 = 8(-\frac{1}{8}) = -1$

$n=5$: $a_5 = a_1 r^{5-1} = a_1 r^4 = (8)(-\frac{1}{2})^4 = 8(\frac{1}{16}) = \frac{1}{2}$

Hence the required five terms are

$\boxed{8, -4, 2, -1, \frac{1}{2}}$ Ans

(iii) $a_1 = -\frac{9}{16}$; $r = -\frac{2}{3}$

Sol: $\because a_n = a_1 r^{n-1}$

Put $n = 2, 3, 4, 5$

$n=2$: $a_2 = a_1 r^{2-1} = a_1 r = \left(-\frac{9}{16}\right)\left(-\frac{2}{3}\right) = \frac{3}{8}$

$n=3$: $a_3 = a_1 r^{3-1} = a_1 r^2 = \left(-\frac{9}{16}\right)\left(-\frac{2}{3}\right)^2 = \left(-\frac{9}{16}\right)\left(\frac{4}{9}\right) = -\frac{1}{4}$

$n=4$: $a_4 = a_1 r^{4-1} = a r^3 = \left(-\frac{9}{16}\right)\left(-\frac{2}{3}\right)^3 = \left(-\frac{9}{16}\right)\left(-\frac{8}{27}\right) = \frac{1}{6}$

$n=5$: $a_5 = a_1 r^{5-1} = a_1 r^4 = \left(-\frac{9}{16}\right)\left(-\frac{2}{3}\right)^4 = \left(-\frac{9}{16}\right)\left(\frac{16}{81}\right) = -\frac{1}{9}$

Thus the required Ist five terms are

$\boxed{-\frac{9}{16}, \frac{3}{8}, -\frac{1}{4}, \frac{1}{6}, -\frac{1}{9}}$ Ans

(iv) $a_1 = \frac{x}{y}$, $r = -\frac{y}{x}$

Sol: $\because a_n = a_1 r^{n-1}$

Put $n = 2, 3, 4, 5$

$n=2$: $a_2 = a_1 r^{2-1} = a_1 r = \left(\frac{x}{y}\right)\left(-\frac{y}{x}\right) = -1$

$n=3$: $a_3 = a_1 r^{3-1} = a_1 r^2 = \left(\frac{x}{y}\right)\left(-\frac{y}{x}\right)^2 = \left(\frac{x}{y}\right)\left(\frac{y^2}{x^2}\right) = \frac{y}{x}$

$n=4$: $a_4 = a_1 r^{4-1} = a_1 r^3 = \left(\frac{x}{y}\right)\left(-\frac{y}{x}\right)^3 = \left(\frac{x}{y}\right)\left(-\frac{y^3}{x^3}\right) = -\frac{y^2}{x^2}$

$n=5$: $a_5 = r^{5-1} = a_1 r^4 = \left(\frac{x}{y}\right)\left(-\frac{y}{x}\right)^4 = \left(\frac{x}{y}\right)\left(\frac{y^4}{x^4}\right) = \frac{y^3}{x^3}$

Thus the required first five terms are

$\boxed{\frac{x}{y}, -1, \frac{y}{x}, -\frac{y^2}{x^2}, \frac{y^3}{x^3}}$ Ans

Q.2: Suppose that the third term of a geometric sequence is 27 and the fifth term is 243. Find the first term and common ratio of the sequence.

Sol: Here $a_3 = 27$, $a_5 = 243$, a_1, r?

$a_n = a_1 r^{n-1}$

$\therefore\ a_3 = a_1 r^2 \Rightarrow 27 = a_1 r^2 \longrightarrow (i)$

$a_5 = a_1 r^4 \Rightarrow 243 = a_1 r^4 \longrightarrow (ii)$

$(ii) \div (i) \Rightarrow \dfrac{\cancelto{9}{243}}{\cancel{27}} = \dfrac{\cancel{a_1} r^4}{\cancel{a_1} r^2} \Rightarrow r^2 = 9 \Rightarrow \boxed{r = 3}$ Ans

$(i) \Rightarrow 27 = a_1(9) \Rightarrow \boxed{a_1 = 3}$ Ans

Q.3: Find the seventh term of a geometric sequence that has 2 and $-\sqrt{2}$ for its second and third terms respectively.

Sol: Here $a_2 = 2$, $a_3 = -\sqrt{2}$, a_7?

$\because\ a_n = a_1 r^{n-1}$

$\therefore\ a_2 = a_1 r^{2-1} = a_1 r \Rightarrow 2 = a_1 r \longrightarrow (i)$

$a_3 = a_1 r^{3-1} = a_1 r^2 \Rightarrow -\sqrt{2} = a_1 r^2 \longrightarrow (ii)$

$(ii) \div (i) \Rightarrow \dfrac{-\sqrt{2}}{2} = \dfrac{a_1 r^2}{a_1 r} \Rightarrow \boxed{r = -\dfrac{1}{\sqrt{2}}}$

$(i) \Rightarrow 2 = a_1\left(-\dfrac{1}{\sqrt{2}}\right) \Rightarrow \boxed{a_1 = -2\sqrt{2}}$

Now $a_7 = a_1 r^{7-1} = a_1 r^6 = (-2\sqrt{2})\left(-\frac{1}{\sqrt{2}}\right)^6$

$= -2^{\frac{3}{2}}\left[\left(-\frac{1}{\sqrt{2}}\right)^2\right]^3 = -2^{\frac{3}{2}}\left(+\frac{1}{2}\right)^3 = -2^{\frac{3}{2}}\frac{1}{2^3} = -2^{\frac{3}{2}-3} = \boxed{-2^{\frac{-3}{2}}}$ Ans

Q.4: How many terms are there in a geometric sequence in which the first and the last terms are 16 and $\frac{1}{64}$ respectively and $r = \frac{1}{2}$?

Sol: Here $a_1 = 16$, $a_n = \frac{1}{64}$, $r = \frac{1}{2}$, $n = ?$

$\because a_n = a_1 r^{n-1}$

$\therefore \frac{1}{64} = (16)\left(\frac{1}{2}\right)^{n-1} \Rightarrow \left(\frac{1}{2}\right)^{n-1} = \frac{1}{64 \times 16}$

$\Rightarrow \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^6 \times 2^4}$

$\Rightarrow \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{10} \Rightarrow n-1 = 10 \Rightarrow \boxed{n = 11}$ Ans

Q.5: Find x so that $x+7, x-3, x-8$ form a three term geometric sequence in the given order. Also give the sequence.

Sol: $\because x+7, x-3, x-8$ is a G.P.

$\Rightarrow (x-3)^2 = (x+7)(x-8)$

$\Rightarrow \cancel{x^2} - 6x + 9 = \cancel{x^2} - 8x + 7x - 56$

$\Rightarrow 6x - x = 9 + 56$

$\Rightarrow 5x = 65 \Rightarrow \boxed{x = 13}$ Ans

Thus the sequence is $x+7, x-3, x-8$

$$= 13+7,\ 13-3,\ 13-8$$

$$= \boxed{20,\ 10,\ 5}$$ Ans

Q.6: If $a_{10} = l$, $a_{13} = m$, $a_{16} = n$; show that $ln = m^2$.

Sol: $\because\ a_n = a_1 r^{n-1}$

$\therefore\ a_{10} = a_1 r^{10-1} \Rightarrow l = a_1 r^9 \longrightarrow (1)$

$a_{13} = a_1 r^{13-1} \Rightarrow m = a_1 r^{12} \longrightarrow (2)$

$a_{16} = a_1 r^{16-1} \Rightarrow n = a_1 r^{15} \longrightarrow (3)$

$ln = (a_1 r^9)(a_1 r^{15})$ (using (1) and (2))

$= a_1^2 r^{24}$

$= (a_1 r^{12})^2$

$= m^2$ (using (3))

Hence $ln = m^2$ Proved.

Q.7: Show that the reciprocals of the terms of a geometric sequence also form a geometric sequence.

Sol: Let $a_1, a_1 r, a_1 r^2, \ldots, a_1 r^{n-1}, \ldots$ be a G.P.

The reciprocal of the G.P is

$$\frac{1}{a_1}, \frac{1}{a_1 r}, \frac{1}{a_1 r^2}, \ldots, \frac{1}{a_1 r^{n-1}}, \ldots \longrightarrow (N)$$

Now $\frac{a_2}{a_1} = \frac{\frac{1}{a_1 r}}{\frac{1}{a_1}} = \frac{1}{r} \longrightarrow (1)$

$\frac{a_3}{a_2} = \frac{\frac{1}{a_1 r^2}}{\frac{1}{a_1 r}} = \frac{1}{a_1 r^2} \times a_1 r = \frac{1}{r} \longrightarrow (2)$

From (1) & (2), $\frac{a_2}{a_1} = \frac{a_3}{a_2} \Rightarrow (N)$ is a G.P.

Hence the reciprocals of the terms of a G.P is also a G.P.

Q.8: Find the geometric mean of the following:

(i) 3.14 and 2.71

Sol: Let $a = 3.14,\ b = 2.71$

$$G.M = \pm\sqrt{ab} = \pm\sqrt{(3.14)(2.71)} = \boxed{\pm 2.92} \text{ Ans}$$

(ii) -6 and -216

Sol: Let $a = -6,\ b = -216$

$$G.M = \pm\sqrt{ab} = \pm\sqrt{(-6)(-216)} = \boxed{\pm 36} \text{ Ans}$$

(iii) $x+y$ and $x-y$

Sol: Let $a = x+y,\ b = x-y$

$$G.M = \pm\sqrt{ab} = \pm\sqrt{(x+y)(x-y)} = \boxed{\pm\sqrt{x^2-y^2}} \text{ Ans}$$

(iv) $\sqrt{2}+3$ and $\sqrt{2}-3$

Sol: Let $a = \sqrt{2}+3,\ b = \sqrt{2}-3$

$$G.M = \pm\sqrt{ab} = \pm\sqrt{(\sqrt{2}+3)(\sqrt{2}-3)} = \pm\sqrt{2-9}$$

$= \pm\sqrt{-7}$ which is not real

Hence G.M does not exist. Ans

Q.9: (i) Insert 5 geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

Sol: Let G_1, G_2, G_3, G_4, G_5 be the 5 geometric means b/w $3\frac{5}{9}$ and $40\frac{1}{2}$.

$\Rightarrow 3\frac{5}{9}, G_1, G_2, G_3, G_4, G_5, 40\frac{1}{2}$ is a G.P.

Here $a_1 = 3\frac{5}{9} = \frac{32}{9}$, $n=7$, $a_7 = 40\frac{1}{2} = \frac{81}{2}$

We first find r.

$\because a_n = a_1 r^{n-1}$

$a_7 = a_1 r^6 \Rightarrow \frac{81}{2} = \frac{32}{9} r^6$

$\Rightarrow r^6 = \frac{81}{2} \times \frac{9}{32} \Rightarrow r^6 = \frac{3^4 \times 3^2}{2 \times 2^5}$

$\Rightarrow r^6 = \frac{3^6}{2^6} \Rightarrow r^6 = \left(\frac{3}{2}\right)^6 \Rightarrow \boxed{r = \frac{3}{2}}$

Now $G_n = a_1 r^n$

$G_1 = a_1 r = \left(\frac{\cancel{32}^{16}}{\cancel{9}_3}\right)\left(\frac{\cancel{3}}{\cancel{2}}\right) = \frac{16}{3}$

$G_2 = a_1 r^2 = \left(\frac{32}{9}\right)\left(\frac{3}{2}\right)^2 = \left(\frac{32}{9}\right)\left(\frac{9}{4}\right) = 8$

$G_3 = a_1 r^3 = \left(\frac{32}{9}\right)\left(\frac{3}{2}\right)^3 = \left(\frac{32}{9}\right)\left(\frac{27}{8}\right) = 12$

$G_4 = a_1 r^4 = \left(\frac{32}{9}\right)\left(\frac{3}{2}\right)^4 = \left(\frac{32}{9}\right)\left(\frac{81}{16}\right) = 18$

$G_5 = a_1 r^5 = \left(\frac{32}{9}\right)\left(\frac{3}{2}\right)^5 = \left(\frac{32}{9}\right)\left(\frac{243}{32}\right) = 27$

Thus the required 5 G.Ms are

$\boxed{\frac{16}{3}, 8, 12, 18, 27}$ Ans.

(ii) Insert 6 geometric means 14 and $-\frac{7}{64}$.

Sol: Let $G_1, G_2, G_3, G_4, G_5, G_6$ be the 6 G.Ms b/w 14 and $-\frac{7}{64}$.

$\Rightarrow 14, G_1, G_2, G_3, G_4, G_5, G_6, -\frac{7}{64}$ is a G.P.

Here: $a_1 = 14,\ n = 8,\ a_8 = -\frac{7}{64}$

We first find r.

$\because a_n = a_1 r^{n-1}$

$a_8 = a_1 r^7 \Rightarrow -\frac{7}{64} = 14 r^7 \Rightarrow \frac{-\cancel{7}}{64 \times \cancel{14}_{2}} = r^7$

$\Rightarrow -\frac{1}{2^7} = r^7 \Rightarrow r^7 = \left(-\frac{1}{2}\right)^7 \Rightarrow \boxed{r = -\frac{1}{2}}$

Now $G_n = a_1 r^n$

$G_1 = a_1 r = 14\left(-\frac{1}{2}\right) = -7$

$G_2 = a_1 r^2 = 14\left(-\frac{1}{2}\right)^2 = 14\left(\frac{1}{4}\right) = \frac{7}{2}$

$G_3 = a_1 r^3 = 14\left(-\frac{1}{2}\right)^3 = 14\left(-\frac{1}{8}\right) = -\frac{7}{4}$

$G_4 = a_1 r^4 = 14\left(-\frac{1}{2}\right)^4 = 14\left(\frac{1}{16}\right) = \frac{7}{8}$

$G_5 = a_1 r^5 = 14\left(-\frac{1}{2}\right)^5 = 14\left(-\frac{1}{32}\right) = -\frac{7}{16}$

$G_6 = a_1 r^6 = 14\left(-\frac{1}{2}\right)^6 = 14\left(\frac{1}{64}\right) = \frac{7}{32}$

Hence the required 6 G.Ms are

$\boxed{-7, \frac{7}{2}, -\frac{7}{4}, \frac{7}{8}, -\frac{7}{16}, \frac{7}{32}}$ Ans

Q.10: Find two numbers if the difference between them is 48 and their A.M exceeds their G.M by 18.

Sol: Let the two numbers be x and y.

According to the Ist condition:

$$x - y = 48 \longrightarrow (1)$$

According to the 2nd condition:

$$A.M - G.M = 18$$

$$\frac{x+y}{2} - \sqrt{xy} = 18$$

$$x + y - 2\sqrt{xy} = 36 \longrightarrow (2)$$

(1) + (2) gives

$$x - \cancel{y} = 48$$

$$x + \cancel{y} - 2\sqrt{xy} = 36$$

$$2x - 2\sqrt{xy} = 84$$

$$x - \sqrt{xy} = 42 \quad (\div \text{ by } 2)$$

$$x - 42 = \sqrt{xy}$$

squaring both sides

$$(x-42)^2 = (\sqrt{xy})^2$$

$$x^2 - 84x + 1764 = xy$$

$$x^2 - 84x + 1764 = x(x-48) \quad (\text{using}(1))$$

$$\cancel{x^2} - 84x + 1764 = \cancel{x^2} - 48x$$

$$84x - 48x = 1764$$

$$36x = 1764 \Rightarrow \boxed{x = 49}$$

$$(1) \Rightarrow 49 - y = 48 \Rightarrow 49 - 48 = y \Rightarrow \boxed{y = 1}$$

Thus the required numbers are $\boxed{49, 1}$ Ans

Q.12: For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the geometric mean between a and b.

Sol: Let $\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = G.M$

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \sqrt{ab}$$

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = a^{\frac{1}{2}}b^{\frac{1}{2}}$$

$$\Rightarrow a^{n+1}+b^{n+1} = (a^n+b^n)(a^{\frac{1}{2}}b^{\frac{1}{2}})$$

$$\Rightarrow a^{n+\frac{1}{2}}a^{\frac{1}{2}} + b^{n+\frac{1}{2}}b^{\frac{1}{2}} = a^{n+\frac{1}{2}}b^{\frac{1}{2}} + b^{n+\frac{1}{2}}a^{\frac{1}{2}}$$

$$\Rightarrow a^{n+\frac{1}{2}}a^{\frac{1}{2}} - a^{n+\frac{1}{2}}b^{\frac{1}{2}} = b^{n+\frac{1}{2}}a^{\frac{1}{2}} - b^{n+\frac{1}{2}}b^{\frac{1}{2}}$$

$$\Rightarrow a^{n+\frac{1}{2}}\cancel{(a^{\frac{1}{2}}-b^{\frac{1}{2}})} = b^{n+\frac{1}{2}}\cancel{(a^{\frac{1}{2}}-b^{\frac{1}{2}})}$$

$$\Rightarrow \frac{a^{n+\frac{1}{2}}}{b^{n+\frac{1}{2}}} = 1$$

$(\because x^0 = 1)$

$$\Rightarrow \left(\frac{a}{b}\right)^{n+\frac{1}{2}} = \left(\frac{a}{b}\right)^0$$

$$\Rightarrow n+\frac{1}{2} = 0 \Rightarrow \boxed{n = -\frac{1}{2}} \text{ Ans}$$

Q.11. Prove that the product of n geometric means between a and b is equal to the nth power of the single geometric mean between them.

Sol : Let $G_1, G_2, G_3, \ldots, G_n$ be the n G.Ms b/w a and b.

$\Rightarrow a, G_1, G_2, G_3, \ldots, G_n, b$ is a G.P

Here $a_1 = a$, $n = n+2$, $a_{n+2} = b$

We first find r.

$$\because a_n = a_1 r^{n-1}$$
$$a_{n+2} = a_1 r^{n+2-1}$$
$$a_{n+2} = a_1 r^{n+1}$$
$$b = a\, r^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Now $G_n = a_1 r^n$

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$
$$G_2 = ar^2 = a\left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$
$$G_3 = ar^3 = a\left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}$$
$$\vdots$$
$$G_n = ar^n = a\left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Consider $G_1 \cdot G_2 \cdot G_3 \cdots G_n = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}} \cdot a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} \cdot a\left(\frac{b}{a}\right)^{\frac{3}{n+1}} \cdots a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

$$= a^n \left(\frac{b}{a}\right)^{\frac{1}{n+1} + \frac{2}{n+1} + \frac{3}{n+1} + \cdots + \frac{n}{n+1}}$$
$$= a^n \left(\frac{b}{a}\right)^{\frac{1}{n+1}[1+2+3+\cdots+n]}$$
$$= a^n \left(\frac{b}{a}\right)^{\frac{1}{n+1}\left[\frac{n(n+1)}{2}\right]} = a^n \left(\frac{b}{a}\right)^{\frac{n}{2}}$$
$$= a^{n-\frac{n}{2}} b^{\frac{n}{2}} = a^{\frac{n}{2}} b^{\frac{n}{2}} = (\sqrt{a.b})^n = G^n$$

Proved.

EXERCISE 4.5

Q.1 Compute the sum:

(i) $3+6+12+\cdots+3\cdot 2^9$

Sol: Let $S_n = 3+6+12+\cdots+3\cdot 2^9$

$= 3+3\cdot 2+3\cdot 2^2+\cdots+3\cdot 2^9$

$= 3\dfrac{[2^{10}-1]}{2-1}$ (Using $S_n = a_1\dfrac{(r^n-1)}{r-1}$ with $a_1=3, n=10$, $r=\dfrac{3\cdot 2}{3}=2>1$)

$= 3\dfrac{[1024-1]}{1}$

$= 3(1023) = \boxed{3069}$ Ans

(ii) $8+4+2+1+\cdots+\frac{1}{16}$

Sol: Here $a_1 = 8$, $r=\frac{4}{8}=\frac{1}{2}<1$, $a_n=\frac{1}{16}$

We first find n.

$\because a_n = a_1 r^{n-1}$

$\frac{1}{16} = 8\left(\frac{1}{2}\right)^{n-1} \Rightarrow \frac{1}{16\times 8} = \left(\frac{1}{2}\right)^{n-1} \Rightarrow \frac{1}{2^4\times 2^3} = \left(\frac{1}{2}\right)^{n-1}$

$\Rightarrow \frac{1}{2^7} = \left(\frac{1}{2}\right)^{n-1} \Rightarrow \left(\frac{1}{2}\right)^7 = \left(\frac{1}{2}\right)^{n-1}$

$\Rightarrow 7 = n-1 \Rightarrow \boxed{n=8}$

Now $S_n = a_1\left(\dfrac{1-r^n}{1-r}\right)$

$\Rightarrow S_8 = 8\left[\dfrac{1-\left(\frac{1}{2}\right)^8}{1-\frac{1}{2}}\right] = \dfrac{8\left[1-\frac{1}{2^8}\right]}{\frac{1}{2}} = 16\left[1-\frac{1}{256}\right] = \cancel{16}\left(\dfrac{255}{\cancel{256}_{16}}\right)$

$= \boxed{\dfrac{255}{16}}$ Ans

(iii) $2^4+2^5+2^6+\dots+2^{10}$

Sol:

Here $a_1=2^4$, $r=\frac{2^5}{2^4}=2>1$, $n=7$, $a_7=2^{10}$

$\because S_n=a_1\left(\frac{r^n-1}{r-1}\right)$

$S_7=2^4\left(\frac{2^7-1}{2-1}\right)$

$=16\left(\frac{128-1}{1}\right)$

$=16(127)$

$=\boxed{2032}$ Ans

(iv) $\frac{8}{5}, -1, \frac{5}{8}, \dots$

Sol:

Here $a_1=\frac{8}{5}$, $r=\frac{-1}{\frac{8}{5}}=-\frac{5}{8}$, $|r|=\frac{5}{8}<1$

$\therefore S_\infty=\frac{a_1}{1-r}$

$=\frac{8/5}{1-(-5/8)}$

$=\frac{8/5}{1+5/8}$

$=\frac{8/5}{13/8}$

$=\frac{8}{5}\times\frac{8}{13}$

$=\boxed{\frac{64}{65}}$ Ans

(v) $2, \frac{2}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, \frac{1}{2}$

Sol:

Here $a_1=2$, $r=\frac{\frac{2}{\sqrt{2}}}{2}=\frac{2}{\sqrt{2}}\times\frac{1}{2}=\frac{1}{\sqrt{2}}<1$

$S_\infty=\frac{a_1}{1-r}$

$=\frac{2}{1-\frac{1}{\sqrt{2}}}$

$=\frac{2}{\frac{\sqrt{2}-1}{\sqrt{2}}}=\frac{2\sqrt{2}}{\sqrt{2}-1}$

$=\frac{2\sqrt{2}}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}$

$=\frac{4+2\sqrt{2}}{2-1}=\boxed{4+2\sqrt{2}}$ Ans

(vi) $-\frac{1}{3}, \frac{1}{2}, -\frac{3}{4}, \dots$ to 7 terms

Sol:

Here $a_1=-\frac{1}{3}$, $r=\frac{\frac{1}{2}}{-\frac{1}{3}}=-\frac{3}{2}$, $|r|=\frac{3}{2}>1$, $n=7$

$S_n=a_1\left(\frac{r^n-1}{r-1}\right)$

$S_7=-\frac{1}{3}\left[\frac{(-\frac{3}{2})^7-1}{-\frac{3}{2}-1}\right]$

$=-\frac{1}{3}\left[\frac{-\frac{2187}{128}-1}{-\frac{5}{2}}\right]$

$=\left(-\frac{1}{3}\right)\left(-\frac{2}{5}\right)\left[\frac{-2315}{128}\right]$

$=\boxed{\frac{-463}{192}}$ Ans

Q.2: Some of the components a_1, a_n, n, r and S_n of a geometric sequence are given. Find the ones that are missing.

(i) $a_1 = 1,\ r = -2,\ a_n = 64$

Sol: n, S_n?

$\because a_n = a_1 r^{n-1}$

$64 = 1(-2)^{n-1}$

$(-2)^6 = (-2)^{n-1} \Rightarrow n-1 = 6 \Rightarrow n = 7$

$\because S_n = a_1\left(\frac{r^n - 1}{r-1}\right) \quad \because |r| = |-2| = 2 > 1$

$= 1\left[\frac{(-2)^7 - 1}{-2-1}\right] = \frac{-128-1}{-2-1} = \frac{-129}{-3} = \boxed{43}$ Ans

(ii) $r = \frac{1}{2}$ and $a_9 = 1$

Sol: a_1, S_9?

$\because a_n = a_1 r^{n-1}$

$a_9 = a_1 r^8$

$1 = a_1\left(\frac{1}{2}\right)^8$

$1 = a_1\left(\frac{1}{256}\right) \Rightarrow \boxed{a_1 = 256}$

$\because S_n = a_1\left(\frac{1-r^n}{1-r}\right)$

$S_9 = a_1\left(\frac{1-r^9}{1-r}\right)$

$= 256\left[\frac{1-\left(\frac{1}{2}\right)^9}{1-\frac{1}{2}}\right] = 256\left[\frac{1-\frac{1}{2^9}}{\frac{1}{2}}\right] = 256 \times 2\left(1 - \frac{1}{512}\right)$

$= \cancel{512}\left(\frac{512-1}{\cancel{512}}\right) = \boxed{511}$ Ans

(iii) $r=-2,\ S_n=-63,\ a_n=-96$

Sol: a_1, n?

$\because a_n = a_1 r^{n-1}$

$\therefore -96 = a_1(-2)^{n-1} \longrightarrow (1)$

$\because S_n = a_1\left(\frac{r^n-1}{r-1}\right)$

$\therefore -63 = a_1\left[\frac{(-2)^n-1}{-2-1}\right]$

$(-63)(-3) = a_1(-2)^n - a_1$

$189 = a_1(-2)^n - a_1$

$189 + a_1 = a_1(-2)^n \longrightarrow (2)$

$(1) \Rightarrow\ -96\times(-2) = a_1(-2)^n$

$\Rightarrow\ 192 = a_1(-2)^n \longrightarrow (3)$

From (2) and (3), we get

$189 + a_1 = 192$

$\Rightarrow a_1 = 192-189 = 3$ i.e. $\boxed{a_1 = 3}$ Ans

$(1) \Rightarrow\ -96 = 3(-2)^{n-1}$

$-32 = (-2)^{n-1}$

$(-2)^5 = (-2)^{n-1} \Rightarrow n-1 = 5 \Rightarrow \boxed{n=6}$ Ans

Q.3. Find the first five terms and the sum of an infinite geometric sequence having $a_2=2,\ a_3=1$.

Sol: $a_2 = 2,\ a_3 = 1$

We first find r and a_1.

$r = \frac{a_3}{a_2} = \boxed{\frac{1}{2}}$

Now $a_1 = \frac{a_2}{r} = \frac{2}{\frac{1}{2}} = \boxed{4}$

Now since $a_n = a_1 r^{n-1}$

$\therefore\ a_4 = a_1 r^{4-1} = a_1 r^3 = 4\left(\frac{1}{2}\right)^3 = 4\left(\frac{1}{8}\right) = \frac{1}{2}$

$a_5 = a_1 r^{5-1} = a_1 r^4 = 4\left(\frac{1}{2}\right)^4 = 4\left(\frac{1}{16}\right) = \frac{1}{4}$

Thus the first five terms are $\boxed{4; 2, 1, \frac{1}{2}, \frac{1}{4}}$ Ans

Now $S_\infty = \frac{a_1}{1-r} = \frac{4}{1-\frac{1}{2}} = \frac{4}{\frac{1}{2}} = \boxed{8}$ Ans

---

Q.4: Find the value of:

(i) $0.\overline{8}$

Sol: $0.\overline{8} = 0.888\ldots$

$= 0.8 + 0.08 + 0.008 + \ldots$ (This is an infinite geometric series with $a_1 = 0.8,\ r = \frac{0.08}{0.8} = \frac{1}{10} < 1$)

$= \frac{a_1}{1-r}$

$= \frac{0.8}{1-\frac{1}{10}} = \frac{0.8}{\frac{9}{10}} = \frac{8}{10} \times \frac{10}{9} = \boxed{\frac{8}{9}}$ Ans

(ii) $1.\overline{63}$

Sol: $1.\overline{63} = 1.636363\ldots$

$= 1 + 0.636363\ldots$

$= 1 + 0.63 + 0.0063 + 0.000063 + \ldots$

$= 1 + \dfrac{a_1}{1-r}$

This is infinite G. series with $a_1 = 0.63$ & $r = \dfrac{0.0063}{0.63} = \dfrac{1}{100} < 1$

$= 1 + \dfrac{0.63}{1 - \frac{1}{100}}$

$= 1 + \dfrac{0.63}{\frac{99}{100}}$

$= 1 + \dfrac{63}{100} \times \dfrac{100}{99}$

$= 1 + \dfrac{63}{99} = \dfrac{99+63}{99} = \boxed{\dfrac{162}{99}}$ Ans

(iii) $2.1\overline{5}$

Sol: $2.1\overline{5} = 2.1555\ldots$

$= 2.1 + 0.0555\ldots$

$= 2.1 + 0.05 + 0.005 + 0.0005 + \ldots$ (infinite G. series

$= 2.1 + \dfrac{a_1}{1-r}$

$a_1 = 0.05$
$r = \dfrac{0.005}{0.05} = \dfrac{1}{10} < 1$

$= 2.1 + \dfrac{0.05}{1 - \frac{1}{10}} = 2.1 + \dfrac{0.05}{\frac{9}{10}} = 2.1 + \dfrac{5}{100} \times \dfrac{10}{9}$

$= \dfrac{21}{10} + \dfrac{5}{90} = \dfrac{189+5}{90} = \dfrac{194}{90} = \boxed{\dfrac{97}{45}}$ Ans

(iv) $0.\overline{123}$

Sol: $0.\overline{123} = 0.123123123\ldots$

$= 0.123 + 0.000123 + 0.000000123 + \ldots$

$= \dfrac{a_1}{1-r}$

This is infinite G-series with $a_1 = 0.123$

$r = \dfrac{0.000123}{0.123} = 0.001$

$= \dfrac{0.123}{1-0.001}$

$= \dfrac{0.123}{0.999}$

$= \dfrac{\cancel{123}\ 41}{\cancel{999}\ 333}$

$= \boxed{\dfrac{41}{333}}$ Ans

Note: Q.4 Shows how Non-terminating Non-Repeating numbers Can be Written in the form of p/q where $p, q \in Z$.

e.g $0.\overline{123} = 0.123123123\text{------}$

$= \dfrac{41}{333}$. So by def: These are rational numbers.

Q.5: Find r such that: $S_{10} = 244 S_5$.

Sol: $\because S_{10} = 244 S_5$

$$\cancel{a_1}\left[\frac{1-r^{10}}{\cancel{1-r}}\right] = 244\cancel{a_1}\left[\frac{1-r^5}{\cancel{1-r}}\right] \quad \left(\text{Using } S_n = a_1\left(\frac{1-r^n}{1-r}\right)\right)$$

$$\Rightarrow 1-r^{10} = 244 - 244r^5$$

$$\Rightarrow r^{10} - 244r^5 + 243 = 0$$

$$\Rightarrow r^{10} - r^5 - 243r^5 + 243 = 0$$

$$\Rightarrow r^5(r^5-1) - 243(r^5-1) = 0$$

$$\Rightarrow (r^5-1)(r^5-243) = 0$$

$$\Rightarrow r^5 - 1 = 0 \quad | \quad r^5 - 243 = 0$$

$$\Rightarrow r^5 = 1 \quad | \quad r^5 = 243$$

$$\Rightarrow r = 1 \quad | \quad r^5 = 3^5$$

Not possible $\quad | \Rightarrow r = 3$

Thus $\boxed{r = 3}$ Ans

Q.6: Prove that: $S_n(S_{3n} - S_{2n}) = (S_n - S_{2n})^2$

Sol: $L.H.S = S_n(S_{3n} - S_{2n})$

$$= \frac{a_1(r^n-1)}{r-1}\left\{ a_1\frac{(r^{3n}-1)}{r-1} - a_1\frac{(r^{2n}-1)}{r-1}\right\}$$

$$= \frac{a_1^2(r^n-1)}{(r-1)^2}\left\{(r^{3n}-1) - (r^{2n}-1)\right\}$$

$$= \frac{a_1^2(r^n-1)}{(r-1)^2}\left\{r^{3n}\cancel{-1} - r^{2n}\cancel{+1}\right\} = \frac{a_1^2(r^n-1)}{(r-1)^2}(r^{3n} - r^{2n}) \longrightarrow (1)$$

$$R.H.S = (S_n - S_{2n})^2$$

$$= (S_{2n} - S_n)^2$$

$$= \left[\frac{a_1(r^{2n}-1)}{r-1} - \frac{a_1(r^n-1)}{r-1}\right]^2 = \left[\frac{a_1(r^{2n}-1) - a_1(r^n-1)}{r-1}\right]^2$$

$$= \left[\frac{a_1 r^{2n} - \cancel{a_1} - a_1 r^n + \cancel{a_1}}{r-1}\right]^2 = \left[\frac{a_1 r^{2n} - a_1 r^n}{r-1}\right]^2$$

$$= \left[\frac{a_1 r^n (r^n-1)}{r-1}\right]^2 = \frac{a_1 r^{2n}(r^n-1)^2}{(r-1)^2} = \frac{a_1(r^n-1)\, r^{2n}(r^n-1)}{(r-1)^2}$$

$$= \frac{a_1(r^n-1)(r^{3n}-r^{2n})}{(r-1)^2} \longrightarrow (2)$$

$\because$ From (1) and (2), $L.H.S = R.H.S$ Proved.

Q.7: Find the sum S_n of the first n terms of the sequence $\left\{\left(\frac{1}{2}\right)^n\right\}$

Sol: $\left\{\left(\frac{1}{2}\right)^n\right\} = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots\right\}$

let $S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$ (This is finite G. series with $a_1 = \frac{1}{2}$, $r = \frac{\frac{1}{2^2}}{\frac{1}{2}} = \frac{1}{2} < 1$)

$$= \frac{a_1(1-r^n)}{1-r}$$

$$= \frac{1}{2}\left[\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right] = \cancel{\frac{1}{2}}\left[\frac{1-\frac{1}{2^n}}{\cancel{\frac{1}{2}}}\right] = \boxed{1-\frac{1}{2^n}} \text{ Ans}$$

Q.8: The sum of three numbers in G.P. is 38, and their product is 1728, find them.

Sol: Suppose the three numbers are $\frac{a}{r}, a, ar$.

According to the Ist condition:

$$\frac{a}{r}+a+ar=38$$

or $a\left(\frac{1}{r}+1+r\right)=38 \longrightarrow (1)$

According to the 2nd condition:

$$\left(\frac{a}{\cancel{r}}\right)a(a\cancel{r})=1728$$

$$\Rightarrow a^3=1728$$

$$a^3=(12)^3 \Rightarrow a=12$$

so (1) becomes

$$12\left(\frac{1}{r}+1+r\right)=38$$

$$\Rightarrow \frac{1}{r}+1+r=\frac{38}{12}$$

$$\frac{1}{r}+1+r=\frac{19}{6}$$

$\times$ by $6r$, $6+6r+6r^2=19r$

$$6r^2-13r+6=0$$

$$6r^2-9r-4r+6=0$$

$$3r(2r-3)-2(2r-3)=0$$

$$(2r-3)(3r-2)=0$$

$2r-3=0$	$3r-2=0$
$\Rightarrow r=\frac{3}{2}$	$\Rightarrow r=\frac{2}{3}$
Now $\frac{a}{r}=\frac{12}{\frac{3}{2}}=\frac{12\times2}{3}=8$	$\frac{a}{r}=\frac{12}{\frac{2}{3}}=12\times\frac{3}{2}=18$
$a=12$	$a=12$

$ar = 12 \times \frac{3}{2} = 18$ | $ar = 12 \times \frac{2}{3} = 8$

Hence the numbers are $\boxed{8, 12, 18 \text{ or } 18, 12, 8.}$ Ans

---

Q.9: The sum of first 6 terms of a geometric series is 9 times the sum of its first three terms. Find the common ratio.

Sol: Sum of Ist 6 terms = 9 × Ist 3 terms

$$\Rightarrow S_6 = 9 S_3$$

$$\frac{\cancel{a_1}(r^6-1)}{\cancel{r-1}} = 9\frac{\cancel{a_1}(r^3-1)}{\cancel{r-1}}$$

$$\Rightarrow r^6-1 = 9(r^3-1)$$

$$\Rightarrow r^6-1 = 9r^3-9$$

$$\Rightarrow r^6-9r^3+8=0$$

$$\Rightarrow r^6-r^3-8r^3+8=0$$

$$\Rightarrow r^3(r^3-1)-8(r^3-1)=0$$

$$\Rightarrow (r^3-1)(r^3-8)=0$$

$r^3-1=0$	$r^3-8=0$
$r^3=1$	$r^3=8$
$r=1$	$r^3=2^3$
Not possible	$r=2$

Hence $\boxed{r=2}$ Ans

Q.10 How many terms of the series: $1+\sqrt{3}+3+\cdots$ be added to get the sum $40+13\sqrt{3}$.

Sol: $1+\sqrt{3}+3+\cdots$ to n terms $=40+13\sqrt{3}$

$$1\cdot\frac{(\sqrt{3})^n-1}{\sqrt{3}-1}=40+13\sqrt{3}\quad\left(\because a_1=1,\ r=\frac{\sqrt{3}}{1}=\sqrt{3}>1\right)$$

$$(\sqrt{3})^n-1=(40+13\sqrt{3})(\sqrt{3}-1)$$

$$(\sqrt{3})^n-1=40\sqrt{3}-40+13(\sqrt{3})^2-13\sqrt{3}$$

$$(\sqrt{3})^n-1=40\sqrt{3}-40+39-13\sqrt{3}$$

$$(\sqrt{3})^n\cancel{-1}=40\sqrt{3}\cancel{-1}-13\sqrt{3}$$

$$(\sqrt{3})^n=27\sqrt{3}$$

$$(3^{\frac{1}{2}})^n=3^3\cdot3^{\frac{1}{2}}$$

$$3^{\frac{n}{2}}=3^{\frac{7}{2}}\Rightarrow\frac{n}{2}=\frac{7}{2}\Rightarrow\boxed{n=7}$$

Ans

Q.11 If p^{th}, q^{th}, r^{th} terms of a G.P. be a, b, c respectively. Prove that $a^{q-r}b^{r-p}c^{p-q}=1$.

Sol: Let the common ratio of the G.P. be R

$\because a_n=a_1R^{n-1}$

p^{th} term $=a_p=a\Rightarrow a_1R^{p-1}=a\longrightarrow(1)$

q^{th} term $=a_q=b\Rightarrow a_1R^{q-1}=b\longrightarrow(2)$

r^{th} term $= a_r = c \Rightarrow a_1 R^{r-1} = c \longrightarrow (3)$

Now L.H.S $= a^{q-r} b^{r-p} c^{p-q}$

$= (a_1 R^{p-1})^{q-r} (a_1 R^{q-1})^{r-p} (a_1 R^{r-1})^{p-q}$

$= a_1^{q-r} R^{(p-1)(q-r)} a_1^{r-p} R^{(q-1)(r-p)} a_1^{p-q} R^{(r-1)(p-q)}$

$= a_1^{\cancel{q}-\cancel{r}+\cancel{r}-\cancel{p}+\cancel{p}-\cancel{q}} R^{(p-1)(q-1)+(q-1)(r-p)+(r-1)(p-q)}$

$= a_1^0 R^{\cancel{pq}-\cancel{pr}-\cancel{q}+\cancel{r}+\cancel{qr}-\cancel{qp}-\cancel{r}+\cancel{p}+\cancel{rp}-\cancel{rq}-\cancel{p}+\cancel{q}}$

$= a_1^0 R^0$

$= (1)(1)$

$= 1 =$ R.H.S Proved.

Q.12: Find an infinite geometric series whose sum is 6 and such that each term is four times the sum of all the terms that follow it.

Sol: According to the first condition:

$S_\infty = 6$

$\Rightarrow \frac{a_1}{1-r} = 6 \Rightarrow a_1 = 6(1-r) \longrightarrow (1)$

According to the second condition:

Each term $= 4$(Sum of all the terms that follow it)

$a_n = 4(a_{n+1} + a_{n+2} + a_{n+3} + \cdots)$

$a_n = 4(a_n r + a_n r^2 + a_n r^3 + \cdots)$

$\cancel{a_n} = 4\cancel{a_n}(r + r^2 + r^3 + \cdots)$

$1 = 4(r + r^2 + r^3 + \cdots)$

$1 = 4\left(\frac{r}{1-r}\right)$

$$1-r=4r \Rightarrow 1=5r \Rightarrow \boxed{r=\frac{1}{5}}$$

So (1) becomes, $a_1 = 6\left(1-\frac{1}{5}\right) = 6\left(\frac{4}{5}\right) = \frac{24}{5} \Rightarrow \boxed{a_1=\frac{24}{5}}$

Now

$$a_1 = \frac{24}{5}$$

$$a_2 = a_1 r = \left(\frac{24}{5}\right)\left(\frac{1}{5}\right) = \frac{24}{5^2}$$

$$a_3 = a_2 r = \left(\frac{24}{5^2}\right)\left(\frac{1}{5}\right) = \frac{24}{5^3}$$

. . . .

. . . .

Hence the required series is $\boxed{\frac{24}{5}+\frac{24}{5^2}+\frac{24}{5^3}+\cdots}$ Ans

Q.13 If $y = \frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\ldots$, where $0<x<3$, then show that $x = \frac{3y}{1+y}$.

Sol: $y = \frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\ldots$ (This is infinite Geometric series with $a_1 = \frac{x}{3}$, $r = \frac{x^2/3^2}{x/3} = \frac{x}{3} < 1$ $\because 0<x<3 \Rightarrow 0<\frac{x}{3}<1$)

$$= \frac{a_1}{1-r}$$

$$= \frac{\frac{x}{3}}{1-\frac{x}{3}}$$

$$= \frac{\frac{x}{3}}{\frac{3-x}{3}}$$

$$y = \frac{x}{3-x} \Rightarrow y(3-x)=x \Rightarrow 3y-xy=x$$

$$\Rightarrow x+xy=3y \Rightarrow x(1+y)=3y$$

$$\Rightarrow x=\frac{3y}{1+y} \quad \text{Proved.}$$

Q.14 A ball rebounds to half the height from which it is dropped. If it is dropped from 10 ft, how far does it travel from the moment it is dropped until the moment of its eighth bounce?

Sol: The required distance

$$= 2S_8 - 10$$

$$= 2\left[\frac{a_1(1-r^8)}{1-r}\right] - 10$$

$$= 2\left[\frac{10\left\{1-\left(\frac{1}{2}\right)^8\right\}}{1-\frac{1}{2}}\right] - 10$$

$$= 2\left[\frac{10\left(1-\frac{1}{256}\right)}{\frac{1}{2}}\right] - 10$$

$$= 2\left[20\left(1-\frac{1}{256}\right)\right] - 10$$

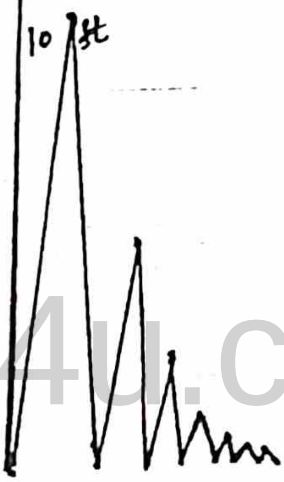
$$= 40\left(\frac{256-1}{256}\right) - 10$$

$$= \frac{5(255)}{32} - 10$$

$$= \frac{1275-320}{32}$$

$$= \frac{955}{32}$$

$$= \boxed{29\frac{27}{32}\text{ ft}} \text{ Ans}$$



Q.15: A man wishes to save money by setting aside Rs.1 the first day, Rs.2 the second day, Rs.4 the third day and so on, doubling the amount each day. If this continued, how much must be set aside on the 15th day? What is the total amount saved at the end of 30 days?

Sol: The sequence formed by the savings is

$1, 2, 4, \ldots$ (This is a G.P.)

Here $a_1 = 1$, $r = \frac{2}{1} = 2 > 1$

The saving on 15th day $= a_{15}$

$= a_1 r^{14}$

$= 1(2)^{14} = 16384$

Thus the saving on 15th day $=$ Rs. 16384

The total amount saved at the end of 30 days:

$S_{30} = 1+2+4+\cdots+$ to 30 terms

$= a_1\left(\frac{r^n-1}{r-1}\right)$

$= 1\left(\frac{2^{30}-1}{2-1}\right) = \frac{2^{30}-1}{1} = 2^{30}-1 = 1073741824-1$

$= \boxed{\text{Rs. } 1073741823}$ Ans

Q.16 The number of bacteria in a culture increased geometrically from 64000 to 729000 in 6 days. Find the daily rate of increase if the rate is assumed to be constant.

Sol: Here $a_1=64000$, $a_n=729000$, $n=7$, r?

$\because a_n=a_1r^{n-1}$

$729000=a_1r^{7-1}$

$729000=64000r^6 \Rightarrow r^6=\frac{729000}{64000} \Rightarrow r^6=\frac{729}{64}$

$\Rightarrow r^6=\left(\frac{3}{2}\right)^6 \Rightarrow \boxed{r=\frac{3}{2}}$ Ans

EXERCISE 4.6

Q.1: Find the indicated term of each of the following harmonic progressions:

(i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots;$ 9^{th}

Sol: $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ (H.P)

The corresponding A.P is

$2, 5, 8, \ldots$

Here $a_1=3$, $d=6-3=3$, $n=9$

$\because a_n=a_1+(n-1)d$

$a_9=a_1+8d$

$=3+8(3)=3+24=29$

In A.P, $a_9=29$

In H.P, $\boxed{a_9=\frac{1}{29}}$ Ans.

(ii) $6, 2, \frac{6}{5}, \ldots;$ 20^{th} term

Sol: $6, 2, \frac{6}{5}, \ldots$ (H.P)

The corresponding A.P is

$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \ldots$

Here $a_1=\frac{1}{6}$, $d=\frac{1}{2}-\frac{1}{6}=\frac{3-1}{6}=\frac{2}{6}=\frac{1}{3}$, $n=20$

$\because a_n=a_1+(n-1)d$

$a_{20}=a_1+19d$

$=\frac{1}{6}+19\left(\frac{1}{3}\right)=\frac{1+38}{6}=\frac{39}{6}=\frac{13}{2}$

In A.P, $a_{20}=\frac{13}{2}$

In H.P, $\boxed{a_{20}=\frac{2}{13}}$ Ans

(iii) $5\frac{2}{3}, 3\frac{2}{5}, 2\frac{3}{7}, \ldots;$ 8^{th} term

Sol: $\frac{17}{3}, \frac{17}{5}, \frac{17}{7}, \ldots$ (H.P)

The corresponding A.P is

$\frac{3}{17}, \frac{5}{17}, \frac{7}{17}, \ldots$

Here $a_1=\frac{3}{17}$, $d=\frac{5}{17}-\frac{3}{17}=\frac{2}{17}$, $n=8$

$\because a_n=a_1+(n-1)d$

$a_8=a_1+7d=\frac{3}{17}+7\left(\frac{2}{17}\right)=\frac{17}{17}=1$

In A.P, $a_8=1$

In H.P, $\boxed{a_8=1}$ Ans

Q.2: Find five more terms of the H.P: $\frac{1}{3}, 1, -1, \ldots$

Sol: $\frac{1}{3}, 1, -1, \ldots$ (H.P)

The corresponding A.P is

$3, 1, -1, \ldots$

Here $a_1 = 3$, $d = 1 - 3 = -2$

$\because a_n = a_1 + (n-1)d$. Put $n = 4, 5, 6, 7, 8$.

$n=4: a_4 = a_1 + 3d = 3 + 3(-2) = 3 - 6 = -3$

$n=5: a_5 = a_1 + 4d = 3 + 4(-2) = 3 - 8 = -5$

$n=6: a_6 = a_1 + 5d = 3 + 5(-2) = 3 - 10 = -7$

$n=7: a_7 = a_1 + 6d = 3 + 6(-2) = 3 - 12 = -9$

$n=8: a_8 = a_1 + 7d = 3 + 7(-2) = 3 - 14 = -11$

Hence five more terms are

In A.P, $-3, -5, -7, -9, -11$

In H.P, $\boxed{-\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, -\frac{1}{9}, -\frac{1}{11}}$ Ans.

Q.3 The second term of an H.P is $\frac{1}{2}$ and the fifth term is $-\frac{1}{4}$. Find the 12^{th} term.

Sol: In H.P, $a_2 = \frac{1}{2}$, $a_5 = -\frac{1}{4}$

In A.P, $a_2 = 2$, $a_5 = -4$

$\because a_n = a_1 + (n-1)d$

$\therefore a_2 = a_1 + d \Rightarrow 2 = a_1 + d \longrightarrow (1)$

$a_5 = a_1 + 4d \Rightarrow -4 = a_1 + 4d \longrightarrow (2)$

$(1) - (2) \Rightarrow 6 = -3d \Rightarrow \boxed{d = -2}$

$(1) \Rightarrow 2 = a_1 - 2 \Rightarrow \boxed{a_1 = 4}$

$\therefore \quad a_{12} = a_1 + 11d = 4 + 11(-2) = 4 - 22 = -18$

$\therefore$ In A.P, $a_{12} = -18$

In H.P, $\boxed{a_{12} = -\frac{1}{18}}$ Ans.

Q.4 Find the arithmetic, harmonic and geometric means of each of the following. Also verify that $A \times H = G^2$.

(i) 3.14 and 2.71

Sol: let $a = 3.14$, $b = 2.71$

$$A = \frac{a+b}{2} = \frac{3.14 + 2.71}{2} = \frac{5.85}{2} = 2.925$$

$$G = \pm\sqrt{ab} = \pm\sqrt{(3.14)(2.71)} = \pm\sqrt{8.5094} = \pm 2.92$$

$$H = \frac{2ab}{a+b} = \frac{2(3.14)(2.71)}{3.14 + 2.71} = \frac{17.019}{5.9} = 2.91$$

Now $G^2 = (\pm 2.92)^2 = 8.52 \longrightarrow (1)$

$A \times H = 2.925 \times 2.91 = 8.5117 \simeq 8.52 \longrightarrow (2)$

From (1) & (2), $\boxed{G^2 \doteq A \times H}$

(ii) $-6, -216$

Sol: let $a = -6$, $b = -216$

$$A = \frac{a+b}{2} = \frac{-6-216}{2} = \frac{-222}{2} = -111$$

$$G = \pm\sqrt{ab} = \pm\sqrt{(-6)(-216)} = \pm\sqrt{1296} = \pm 36$$

$$H = \frac{2ab}{a+b} = \frac{2(-6)(-216)}{(-6)+(-216)} = \frac{2592}{-222} = -11.68$$

Now $G^2 = (\pm 36)^2 = 1296 \longrightarrow (1)$

$A \times H = (-111)(-11.68) = 1296 \longrightarrow (2)$

From (1) and (2), $\boxed{G^2 = A \times H}$

(iii) $x+y, x-y$

Sol: Let $a = x+y$, $b = x-y$

$A = \frac{a+b}{2} = \frac{x+y+x-y}{2} = \frac{2x}{2} = x$

$G = \pm\sqrt{ab} = \pm\sqrt{(x+y)(x-y)} = \pm\sqrt{x^2-y^2}$

$H = \frac{2ab}{a+b} = \frac{2(x+y)(x-y)}{\cancel{x+y}+x-\cancel{y}} = \frac{\cancel{2}(x^2-y^2)}{\cancel{2}x} = \frac{x^2-y^2}{x}$

Now $G^2 = (\pm\sqrt{x^2-y^2})^2 = x^2-y^2 \longrightarrow (1)$

$A \times H = x\left(\frac{x^2-y^2}{x}\right) = x^2-y^2 \longrightarrow (2)$

From (1) & (2), $\boxed{G^2 = A \times H}$

Q.5: For what value of n will $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ be the harmonic mean between a and b?

Sol. Let $\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = H$

$\Rightarrow \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{2ab}{a+b}$

$\Rightarrow (a^{n+1}+b^{n+1})(a+b) = (a^n+b^n)(2ab)$

$a^{n+2} + a^{n+1}b + ab^{n+1} + b^{n+2} = 2a^{n+1}b + 2ab^{n+1}$

$$a^{n+2}+b^{n+2}=2a^{n+1}b+2ab^{n+1}-a^{n+1}b-ab^{n+1}$$

$$a^{n+2}+b^{n+2}=a^{n+1}b+ab^{n+1}$$

$$a\,a^{n+1}-a^{n+1}b=ab^{n+1}-b\,b^{n+1}$$

$$a^{n+1}\cancel{(a-b)}=b^{n+1}\cancel{(a-b)}$$

$$a^{n+1}=b^{n+1}$$

$$\Rightarrow \frac{a^{n+1}}{b^{n+1}}=1$$

$$\Rightarrow \left(\frac{a}{b}\right)^{n+1}=\left(\frac{a}{b}\right)^{0} \Rightarrow n+1=0 \Rightarrow \boxed{n=-1}$$

Q.6: Insert two harmonic means between 12 and 48.

Sol: Let H_1, H_2 be two H.Ms b/w 12 and 48.

$\Rightarrow 12, H_1, H_2, 48$ is H.P.

$\Rightarrow \frac{1}{12}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{48}$ is an A.P.

$\Rightarrow \frac{1}{12}, A_1, A_2, \frac{1}{48}$ is an A.P.

Here $a_1=\frac{1}{12}, n=4, a_4=\frac{1}{48}$

We first find d.

$\because a_n=a_1+(n-1)d$

$\therefore a_4=a_1+3d$

$\frac{1}{48}=\frac{1}{12}+3d \Rightarrow 3d=\frac{1}{48}-\frac{1}{12}=\frac{1-4}{48}=\frac{-3}{48}=\frac{-1}{12}$

$\Rightarrow \boxed{d=-\frac{1}{48}}$

Now $A_n=a_1+nd$

$A_1=a_1+d$

$\frac{1}{H_1}=\frac{1}{12}-\frac{1}{48}=\frac{4-1}{48}=\frac{3}{48}=\frac{1}{16}$

$\Rightarrow \boxed{H_1 = 16}$.

$A_2 = a_1 + 2d$

$\frac{1}{H_2} = \frac{1}{12} + 2\left(-\frac{1}{48}\right) = \frac{1}{12} - \frac{1}{24} = \frac{2-1}{24} = \frac{1}{24}$

$\Rightarrow \boxed{H_2 = 24}$. Hence the required 2 H.Ms are $\boxed{16, 24}$.

Thus - $\boxed{12, 16, 24, 48.}$ Ans

Q.7 Insert four harmonic means between $\frac{7}{3}$ and $\frac{7}{11}$.

Sol: Let H_1, H_2, H_3, H_4 be the 4 H.Ms between $\frac{7}{3}$ and $\frac{7}{11}$.

$\Rightarrow \frac{7}{3}, H_1, H_2, H_3, H_4, \frac{7}{11}$ is an H.P.

$\Rightarrow \frac{3}{7}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{11}{7}$ is an A.P.

$\Rightarrow \frac{3}{7}, A_1, A_2, A_3, A_4, \frac{11}{7}$ is an A.P.

Here $a_1 = \frac{3}{7}$, $n = 6$, $a_6 = \frac{11}{7}$

We first find d.

$\because a_n = a_1 + (n-1)d$

$\therefore a_6 = a_1 + 5d$

$\frac{11}{7} = \frac{3}{7} + 5d \Rightarrow 5d = \frac{11}{7} - \frac{3}{7} \Rightarrow 5d = \frac{8}{7} \Rightarrow \boxed{d = \frac{8}{35}}$

Now $A_n = a_1 + nd$

$A_1 = a_1 + d$

$\frac{1}{H_1} = \frac{3}{7} + \frac{8}{35} \Rightarrow \frac{1}{H_1} = \frac{15+8}{35} = \frac{23}{35} \Rightarrow \boxed{H_1 = \frac{35}{23}}$

$A_2 = a_1 + 2d$

$\frac{1}{H_2} = \frac{3}{7} + 2\left(\frac{8}{35}\right) \Rightarrow \frac{1}{H_2} = \frac{15+16}{35} = \frac{31}{35} \Rightarrow \boxed{H_2 = \frac{35}{11}}$

$A_3 = a_1 + 3d$

$\frac{1}{H_3} = \frac{3}{7} + 3\left(\frac{8}{35}\right) \Rightarrow \frac{1}{H_3} = \frac{15+24}{35} = \frac{29}{35} \Rightarrow \boxed{H_3 = \frac{35}{29}}$

$A_4 = a_1 + 4d \Rightarrow$

$\frac{1}{H_4} = \frac{3}{7} + 4\left(\frac{8}{35}\right) \Rightarrow \frac{1}{H_4} = \frac{15+32}{35} = \frac{47}{35} \Rightarrow \boxed{H_4 = \frac{35}{47}}$

Hence the required 4 H.Ms are $\boxed{\frac{35}{23}, \frac{35}{11}, \frac{35}{29}, \frac{35}{47}}$

Thus $\boxed{\frac{7}{3}, \frac{35}{23}, \frac{35}{11}, \frac{35}{29}, \frac{35}{47}, \frac{7}{11}}$ Ans

Q.8. Prove that the square of the geometric mean of two numbers equals the product of the arithmetic mean and the harmonic mean of the two numbers.

Sol: Let a and b be two numbers.

Then $A = \frac{a+b}{2}$, $G = \pm\sqrt{ab}$, $H = \frac{2ab}{a+b}$

We have to prove that $G^2 = A \times H$.

$L.H.S = G^2 = (\pm\sqrt{ab})^2 = ab \longrightarrow (1)$

$R.H.S = A \times H = \left(\frac{\cancel{a+b}}{\cancel{2}}\right)\left(\frac{\cancel{2}ab}{\cancel{a+b}}\right) = ab \longrightarrow (2)$

From (1) and (2), we have

$\boxed{G^2 = A \times H}$ Proved.

Q.9 The arithmetic mean of two numbers is 8, and the harmonic mean is 6. What are the numbers?

Sol: Let the two numbers be a and b.

$$A=8, \quad H=6$$

$$A=8 \Rightarrow \frac{a+b}{2}=8 \Rightarrow a+b=16 \longrightarrow (1)$$

$$H=6 \Rightarrow \frac{2ab}{a+b}=6 \Rightarrow \frac{2ab}{16}=6 \text{ (Using (1))}$$

$$\Rightarrow ab=48 \longrightarrow (2)$$

From (1), $a=16-b \longrightarrow (3)$

$$(2) \Rightarrow (16-b)b=48$$

$$16b-b^2=48$$

$$b^2-16b+48=0$$

$$b^2-12b-4b+48=0$$

$$b(b-12)-4(b-12)=0$$

$$(b-12)(b-4)=0$$

$b-12=0$	$b-4=0$
$\Rightarrow b=12$	$\Rightarrow b=4$
$(3) \Rightarrow a=16-12=4$	$(3) \Rightarrow a=16-4=12$

Hence the numbers are $\boxed{4, 12 \text{ or } 12, 4}$ Ans

Q.10: The harmonic mean of two numbers is $4\frac{4}{5}$ and the geometric mean is 6. What are the numbers?

Sol: Let the two numbers be a and b.

$$H=4\frac{4}{5}=\frac{24}{5}, \quad G=6$$

$G = 6 \Rightarrow \sqrt{ab} = 6$

$\Rightarrow ab = 36 \longrightarrow (1)$

$H = \frac{24}{5} \Rightarrow \frac{2ab}{a+b} = \frac{24}{5}$

$\Rightarrow \frac{ab}{a+b} = \frac{12}{5}$

$\Rightarrow \frac{36}{a+b} = \frac{12}{5}$ (By (1))

$\Rightarrow \frac{3}{a+b} = \frac{1}{5}$

$\Rightarrow a+b = 15 \longrightarrow (2)$

From (2), $a = 15 - b \longrightarrow (3)$

(1) $\Rightarrow$ $(15-b)b = 36 \Rightarrow b^2 - 15b + 36 = 0$

$\Rightarrow b^2 - 12b - 3b + 36 = 0$

$\Rightarrow b(b-12) - 3(b-12) = 0$

$\Rightarrow (b-12)(b-3) = 0$

$b - 12 = 0$	$b - 3 = 0$
$\Rightarrow b = 12$	$\Rightarrow b = 3$
(3) $\Rightarrow a = 15 - 12$	(3) $\Rightarrow a = 15 - 3$
$= 3$	$= 12$

Hence the numbers are $\boxed{3, 12 \text{ or } 12, 3}$. Ans

REVIEW EXERCISE 4

Q.1: Choose the correct option.

(i) The sum to 200 terms of the series $1+4+6+5+11+6+\cdots$ is

(a) 20,300 (b) 29,800 (c) 30,200 (d) None of these

Sol: Method I

$1+4+6+5+11+6+\cdots$ up to 200 terms (It is the sum of two A.P$_s$)

$=(1+6+11+\cdots$ up to 100 terms$)+(4+5+6+\cdots$ up to 100 terms$)$

$$=\frac{100}{2}\left[2(1)+(100-1)5\right]+\frac{100}{2}\left[2(4)+(100-1)1\right]$$

$$=\frac{100}{2}\left[497+107\right]=50(604)=\boxed{30,200}$$

Method II

$1+4+6+5+11+6+\cdots$ up to 200 terms

$(1+4)+(6+5)+(11+6)+\cdots$ up to 100 terms

$=5+11+17+\cdots$ up to 100 terms

$$=\frac{100}{2}\left[2(5)+(100-1)6\right]=50(604)=\boxed{30,200}$$

Thus the correct option is (c). Ans

(ii) If the sum of the series $2+5+8+11+\cdots$ is 60100, then the number of the terms is

(a) 100 (b) 200 (c) 150 (d) 250

Sol: $S_n=\frac{n}{2}\left[2a+(n-1)d\right] \Rightarrow 60100=\frac{n}{2}\left[2(2)+(n-1)3\right]$

$601\times100=\frac{n}{2}(3n+1) \Rightarrow 200\times601=n(3n+1)$

$\Rightarrow 200\times(3\times200+1)=n(3n+1) \Rightarrow \boxed{n=200}$

Thus the correct option is (b). Ans

(iii) If a, b, c are in G.P, then

(a) a^2, b^2, c^2 are in G.P (b) $a^2(b+c), c^2(a+b), b^2(a+c)$ are in G.P

(c) $\frac{a}{a+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in G.P (d) None of these

Sol: $\because a, b, c$ are in G.P $\Rightarrow b^2 = ac \Rightarrow b^4 = a^2c^2$

$\Rightarrow a^2, b^2, c^2$ are in G.P. Thus the correct option is (a). Ans

(iv) If the nth term of an A.P is $4n+1$ then the common difference is

(a) 3 (b) 4 (c) 5 (d) 6

Sol: $\because a_n = 4n+1 \Rightarrow a_1 = 4(1)+1 = 5$ and $a_2 = 4(2)+1 = 9$

$\therefore d = a_2 - a_1 = 9 - 5 = 4$. Thus the correct option is (b) Ans

(v) Which of the following is not a G.P?

(a) 2, 4, 8, 16, ... (b) 5, 25, 125, 625, ...

(c) 1.5, 3.0, 6.0, 12.0, ... (d) 8, 16, 24, 32, ...

Sol: $\because \frac{16}{8} \neq \frac{24}{16}$ $(\because 2 \neq \frac{3}{2})$

$\therefore$ 8, 16, 24, 32 is not a G.P. Thus the correct option is (d) Ans

(vi) There are four arithmetic means between 2 and -18. The means are:

(a) $-4, -7, -10, -13$ (b) $1, -4, -7, -10$

(c) $-2, -5, -9, -13$ (d) $-2, -6, -10, -14$

Sol: options (b) and (c) are not A.Ps so rejected. While $2, -4, -7, -10, -13, -18$ is not A.P. So (a) is not correct as well. Thus the correct option must be (d). Ans

(Note: $2, -2, -6, -10, -14, -18$ is an A.P) $\Rightarrow$ (d) is correct.

(vii) If A, G and H are A.M, G.M and H.M of any two positive numbers, then find the relation between A, G and H.

(a) $A^2=GH$ (b) $G^2=AH$ (c) $H^2=AG$ (d) $G^2=A^2H$

Sol: This is well-known that $G^2=AH$. Thus the correct option is (b) Ans

(viii) Find the number of terms to be added in the series 27, 9, 3, ... so that the sum is $1093/27$.

(a) 6 (b) 7 (c) 8 (d) 9

Sol:

$$\frac{1093}{27} = 27+9+3+\cdots+n\text{ terms}$$

$$\frac{1093}{27} = 27\left[\frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}\right]$$

$$\frac{1093}{3^3} = 3^3\left[\frac{1-\frac{1}{3^n}}{\frac{2}{3}}\right] \Rightarrow \frac{1093}{3^3} = \frac{3^4}{2}\left(\frac{3^n-1}{3^n}\right)$$

$$\Rightarrow \frac{2186}{37} = \frac{3^n-1}{3^n} \Rightarrow \boxed{n=7}$$. Thus the correct option is (b) Ans

(ix) Find the value of p $(p>0)$ if $\frac{15}{4}+p$, $\frac{5}{2}+2p$ and $2+p$ are the three consecutive terms of a geometric progression.

(a) $\frac{3}{4}$ (b) $\frac{1}{4}$ (c) $\frac{5}{3}$ (d) $\frac{1}{2}$

Sol: $\frac{15}{4}+p$, $\frac{5}{2}+2p$, $2+p$ is a G.P.

$$\Rightarrow \left(\frac{5}{2}+2p\right)^2 = \left(\frac{15}{4}+p\right)(2+p)$$

$$\Rightarrow \frac{(5+4p)^2}{\cancel{4}} = \frac{(15+4p)(2+p)}{\cancel{4}} \Rightarrow 25+16p^2+40p = 30+15p+8p+4p^2$$

$$\Rightarrow 12p^2+17p-5=0$$

$$p = \frac{-17 \pm \sqrt{(17)^2 - 4(12)(-5)}}{2\cdot 12} = \frac{-17 \pm \sqrt{529}}{24} = \frac{-17 \pm 23}{24}$$

$\because p > 0 \quad \therefore p = \frac{-17+23}{24} = \frac{6}{24} = \frac{1}{4}$. Thus the correct option is (b) Ans

(x) The 10^{th} term of harmonic progression $\frac{1}{5}, \frac{4}{19}, \frac{2}{9}, \frac{4}{17}, \ldots$ is

(a) $11/4$ (b) $13/4$ (c) $4/13$ (d) $4/11$

Sol: The corresponding A.P is $5, \frac{19}{4}, \frac{9}{2}, \frac{17}{4}, \ldots$

Here $a_1 = 5$, $d = \frac{19}{4} - 5 = -\frac{1}{4}$

$a_{10} = 5 + 9(-\frac{1}{4}) = \frac{20-9}{4} = \frac{11}{4}$. In A.P, $a_{10} = \frac{11}{4}$.

In H.P, $a_{10} = \frac{4}{11}$. Thus the correct option is (d). Ans

(xi) Find the sum of 3 geometric means between $\frac{1}{3}$ and $\frac{1}{48}$ $(r>0)$.

(a) $\frac{1}{4}$ (b) $\frac{5}{24}$ (c) $\frac{7}{24}$ (d) $\frac{1}{3}$

Sol: Let G_1, G_2, G_3 be the 3 G.Ms b/w $\frac{1}{3}$ and $\frac{1}{48}$.

$\Rightarrow \frac{1}{3}, G_1, G_2, G_3, \frac{1}{48}$ is a G.P.

Here $a_1 = \frac{1}{3}$, $n = 5$, $a_5 = \frac{1}{48}$

We first find r.

$a_n = a_1 r^{n-1} \Rightarrow a_5 = a_1 r^4 \Rightarrow \frac{1}{\cancel{48}_{16}} = \frac{1}{\cancel{3}} r^4 \Rightarrow r^4 = (\frac{1}{2})^4 \Rightarrow r = \frac{1}{2} < 1$

Now $G_n = a_1 r^n$

$G_1 = a_1 r = (\frac{1}{3})(\frac{1}{2}) = \frac{1}{6}$

$G_2 = a_1 r^2 = (\frac{1}{3})(\frac{1}{2})^2 = (\frac{1}{3})(\frac{1}{4}) = \frac{1}{12}$

$G_3 = a_1 r^3 = \frac{1}{3}(\frac{1}{2})^3 = (\frac{1}{3})(\frac{1}{8}) = \frac{1}{24}$

$G_1 + G_2 + G_3 = \frac{1}{6} + \frac{1}{12} + \frac{1}{24} = \frac{4+2+1}{24} = \frac{7}{24}$

Thus the correct option is (c). Ans

Q.2: If the first term and common difference in an A.P are 8 and -1 respectively, then find:

(i) General term

Sol: Given $a_1 = 8$, $d = -1$

$\because a_n = a_1 + (n-1)d = 8 + (n-1)(-1) = 8 - n + 1 = 9 - n$

i.e $\boxed{a_n = 9 - n}$ Ans

(ii) The progression

$\because a_n = 9 - n$. Put $n = 1, 2, 3, \ldots$

$n=1$: $a_1 = 9 - 1 = 8$

$n=2$: $a_2 = 9 - 2 = 7$

$n=3$: $a_3 = 9 - 3 = 6$

......

......

Hence the progression is $\boxed{8, 7, 6, 5, \ldots}$ Ans

(iii) The 10th term and

$\because a_n = 9 - n$

$\therefore a_{10} = 9 - 10 = \boxed{-1}$ Ans

(iv) The expression for sum to n terms and hence sum to 10 terms.

Sol: $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$= \frac{n}{2}[2(8) + (n-1)(-1)] = \frac{n}{2}[16 - (n-1)] = \frac{n}{2}[16 - n + 1] = \boxed{\frac{n}{2}(17-n)}$ Ans

$S_{10} = \frac{10}{2}(17 - 10) = 5(-3) = \boxed{-15}$ Ans

Q.3: If the sum of the n terms of the series $54, 51, 48, \ldots$ is 513, then find the value of n.

Sol: $54, 51, 48, \ldots$ (This is A.P with $a_1 = 54, d = 51-54 = -3$)

Given $S_n = 513$. n?

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

$$513 = \frac{n}{2}[2(54) + (n-1)(-3)]$$

$$513 = \frac{n}{2}[108 - 3n + 3]$$

$$1026 = n(111 - 3n) \Rightarrow 1026 = 111n - 3n^2$$

$$\Rightarrow 3n^2 - 111n + 1026 = 0$$

$$\Rightarrow n^2 - 37n + 342 = 0 \quad (\div \text{by } 3)$$

$$\Rightarrow n^2 - 18n - 19n + 342 = 0$$

$$n(n-18) - 19(n-18) = 0$$

$$(n-18)(n-19) = 0$$

$n - 18 = 0$	$n - 19 = 0$
$\Rightarrow n = 18$	$\Rightarrow n = 19$

Hence $\boxed{n = 18 \text{ or } 19}$ Ans.

Q.4: If the sum of the n terms of an A.P is $2n + 3n^2$, generate the progression and find the nth term.

Sol: Given $S_n = 2n + 3n^2$

$n=1$: $S_1 = 2(1) + 3(1)^2 = 2 + 3 = 5$

$n=2$: $S_2 = 2(2) + 3(2)^2 = 4 + 12 = 16$

$n=3: \quad S_3 = 2(3)+3(3)^2 = 6+27 = 33$

Now $a_1 = S_1 = 5$

$a_2 = S_2 - S_1 = 16-5 = 11$

$a_3 = S_3 - S_2 = 33-16 = 17$

....

....

Thus the sequence is $\boxed{5, 11, 17, \ldots \quad (A.P)}$

Here $a_1 = 5, d = 11-5 = 6$

$\therefore a_n = a_1 + (n-1)d = 5+(n-1)6 = 5+6n-6 = 6n-1$

i.e. $\boxed{a_n = 6n-1}$

Q.5: Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3.

Sol: All natural numbers b/w 250 and 1000 which are exactly divisible by 3 are

$252, 255, 258, \ldots, 999$ (This is an A.P)

Here $a_1 = 252, d = 3, a_n = 999$

We first find n.

$a_n = a_1 + (n-1)d$

$999 = 252 + (n-1)3 \Rightarrow 3(n-1) = 999-252$

$\Rightarrow 3(n-1) = 747 \Rightarrow n-1 = \frac{747}{3} = 249$

$\Rightarrow \boxed{n = 250}$

Now $S_n = 252+255+258+\cdots+999$

$= \frac{n}{2}(a_1+a_n)$

$S_{250} = \frac{250}{2}(252+999) = 125(1251) = \boxed{156375}$ Ans

Q.6: Find the sum of the series $1, \frac{2}{5}, \frac{4}{25}, \frac{8}{125}, \cdots \infty$

Sol: Let $S_\infty = 1+\frac{2}{5}+\frac{4}{25}+\frac{8}{125}+\cdots\infty$ (This is an infinite G.Series with $a_1=1$, $r=\frac{\frac{2}{5}}{1}=\frac{2}{5}<1$)

$= \frac{a_1}{1-r}$

$= \frac{1}{1-\frac{2}{5}} = \frac{1}{\frac{5-2}{5}} = \boxed{\frac{5}{3}}$ Ans

Q.7: If a, b, c are in A.P, and x, y, z are in G.P, show that $x^b y^c z^a = x^c y^a z^b$

Sol: Since a,b,c are in A.P $\Rightarrow 2b = a+c \longrightarrow (i)$

x,y,z are in G.P $\Rightarrow y^2 = xz \longrightarrow (ii)$

$\Rightarrow \frac{y}{x} = \frac{z}{y}$

Now $\frac{x^b y^c z^a}{x^c y^a z^b} = \left(\frac{x}{z}\right)^b\left(\frac{y}{x}\right)^c\left(\frac{z}{y}\right)^a = \left(\frac{x}{z}\right)^b\left(\frac{y}{x}\right)^c\left(\frac{y}{x}\right)^a = \left(\frac{x}{z}\right)^b\left(\frac{y}{x}\right)^{c+a}$

$= \frac{x^b}{z^b}\left(\frac{y}{x}\right)^{2b}$ ($\because c+a=2b$)

$= \frac{x^b}{z^b}\frac{(y^2)^b}{x^{2b}} = \frac{1}{z^b}\cdot\frac{(zx)^b}{x^b}$ ($\because y^2 = zx$)

$= \frac{1}{z^b}\cdot\frac{z^b x^b}{x^b} = 1$. Hence $x^b y^c z^a = x^c y^a z^b$ Ans

Q.8: Find the arithmetic mean between $10\frac{1}{2}$ and $25\frac{1}{2}$.

Sol: Let $a = 10\frac{1}{2} = \frac{21}{2}$, $b = 25\frac{1}{2} = \frac{51}{2}$

$$A = \frac{a+b}{2} = \frac{\frac{21}{2}+\frac{51}{2}}{2} = \frac{72}{4} = \boxed{18} \text{ Ans.}$$

Q.9: Find three numbers of a G.P whose sum is 26 and product is 216.

Sol: Let the three numbers be $\frac{a}{r}, a, ar$.

By Ist condition: $\frac{a}{r} + a + ar = 26 \Rightarrow a\left(\frac{1}{r} + 1 + r\right) = 26 \longrightarrow (1)$

By 2nd condition: $\left(\frac{a}{r}\right)(a)(ar) = 216 \Rightarrow a^3 = (6)^3 \Rightarrow \boxed{a = 6}$

$(1) \Rightarrow 6\left(\frac{1}{r} + 1 + r\right) = 26 \Rightarrow 6 + 6r + 6r^2 = 26r$

$\Rightarrow 6r^2 - 20r + 6 = 0 \Rightarrow 6r^2 - 18r - 2r + 6 = 0$

$\Rightarrow 6r(r-3) - 2(r-3) = 0 \Rightarrow (r-3)(6r-2) = 0 \Rightarrow \boxed{r = 3, \frac{1}{3}}$

Hence the numbers are:

when $r=3$	when $r=\frac{1}{3}$
$\frac{a}{r} = \frac{6}{3} = 2$	$\frac{a}{r} = \frac{6}{1/3} = 18$
$a = 6$	$a = 6$
$ar = (6)(3) = 18$	$ar = 6\left(\frac{1}{3}\right) = 2$

$\boxed{2, 6, 18 \text{ or } 18, 6, 2}$ Ans

Q.10: How many odd integers beginning with 15 must be taken for their sum to be equal to 975.

Sol: The odd integers beginning with 15 are 15, 17, 19, ... (This is A.P)

Here $a = 15$, $d = 17 - 15 = 2$, $S_n = 975$

$S_n = 15 + 17 + 19 + \dots$ to n terms

$\because S_n = \frac{n}{2}[2a_1 + (n-1)d] \Rightarrow 975 = \frac{n}{2}[2(15) + (n-1)2] \Rightarrow 975 = n(n+14)$

$\Rightarrow n^2 + 14n - 975 = 0 \Rightarrow n^2 + 39n - 25n - 975 = 0$

$\Rightarrow n(n+39) - 25(n+39) = 0 \Rightarrow (n-25)(n+39) = 0 \Rightarrow n = 25$ or $n = -39$

But $n = -39$ is not possible since number of terms cannot be negative.

Hence $\boxed{n = 25}$ Ans