SOME IMPORTANT POINTS OF CHAPTER NO: 5

The following points will enable you to solve the problems of this chapter.

(Exercise 5.1)

(i) $S_n = T_1 + T_2 + T_3 + \cdots + T_n$

$= \sum T_n$

(ii) $\sum 1 = n$, $\sum n = \frac{n(n+1)}{2}$, $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum n^3 = \left[\frac{n(n+1)}{2}\right]^2$

(iii) For S_{2n}, first find S_n and then replace n by $2n$.

(Exercise 5.2)

(iv) Sum of Arithmetico-Geometric series (A-G series)

Procedure: $S_n = a + (a+d)r + (a+2d)r^2 + \cdots + (a+(n-1)d)\, r^{n-1}$ (A-G series) $\rightarrow$ ①

$rS_n = ar + (a+d)r^2 + \cdots + [a+(n-2)d]\, r^{n-1} + [a+(n-1)d]\, r^n$ $\rightarrow$ ②

Then subtract & simplify.

(Exercise 5.3)

(v) The method of differences.

This method is used for a sequence whose successive terms differences form A.P or G.P.

Procedure: $a_1 + a_2 + a_3 + \cdots$

$a_2 - a_1 = \square_1$ 1st

$a_3 - a_2 = \square_2$ 2nd

$a_4 - a_3 = \square_3$ 3rd

$\cdots$

$\square_{n-1}$ $(n-1)$st

$a_n - a_{n-1} = (n-1)$th term of the sequence $a_1, a_2, a_3, \ldots$

Adding column-wise we get

$$a_n - a_1 = \boxed{\square_1 + \square_2 + \square_3 + \cdots + \square_{n-1}}$$

If A.P then use $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ then simplify using Point (ii)

If G.P then use $S_n = a_1\frac{(r^n - 1)}{r-1}$ or $S_n = a_1\frac{(1-r^n)}{1-r}$ accordingly

(Exercise 5.4)

Procedure: Take T_n. Use Partial fractions & then use $\sum T_n$.

EXERCISE 5.1

Q.1: Sum the following series up to n terms:

(i) $1^2+3^2+5^2+7^2+\cdots$

Sol: $1^2+3^2+5^2+7^2+\cdots$

Here $T_n = [1+(n-1)2]^2 = [1+2n-2]^2 = (2n-1)^2 = 4n^2-4n+1$

$$S_n = \sum T_n$$

$$= \sum(4n^2-4n+1)$$

$$= 4\sum n^2 - 4\sum n + \sum 1$$

$$= \cancel{4}^{2}\left[\frac{n(n+1)(2n+1)}{\cancel{6}_{3}}\right] - \cancel{4}^{2}\left[\frac{n(n+1)}{\cancel{2}}\right] + n$$

$$= n\left[\frac{2(n+1)(2n+1)}{3} - 2(n+1) + 1\right]$$

$$= n\left[\frac{2(n+1)(2n+1) - 6(n+1) + 3}{3}\right]$$

$$= \frac{n}{3}\left[2(2n^2+n+2n+1) - 6n - 6 + 3\right]$$

$$= \frac{n}{3}\left[2(2n^2+3n+1) - 6n - 6 + 3\right]$$

$$= \frac{n}{3}\left[4n^2 + \cancel{6n} + 2 - \cancel{6n} - 6 + 3\right]$$

$$= \boxed{\frac{n}{3}(4n^2-1)} \quad \text{Ans}$$

(ii) $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\cdots$

Sol: $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\cdots$

Here $T_n=1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}=\frac{n(2n^2+n+2n+1)}{6}$

$=\frac{n(2n^2+3n+1)}{6}=\frac{2n^3+3n^2+n}{6}=\frac{1}{6}[2n^3+3n^2+n]$

$$S_n=\sum T_n$$

$$=\sum\frac{1}{6}[2n^3+3n^2+n]$$

$$=\frac{1}{6}\sum[2n^3+3n^2+n]$$

$$=\frac{1}{6}\left[2\sum n^3+3\sum n^2+\sum n\right]$$

$$=\frac{1}{6}\left[\cancel{2}.\frac{n^2(n+1)^2}{\cancel{4}_2}+\cancel{3}.\frac{n(n+1)(2n+1)}{\cancel{6}_2}+\frac{n(n+1)}{2}\right]$$

$$=\frac{n(n+1)}{6\times2}\left[n(n+1)+2n+1+1\right]$$

$$=\frac{n(n+1)}{12}\left[n^2+n+2n+2\right]$$

$$=\frac{n(n+1)}{12}(n^2+3n+2)$$

$$=\frac{n(n+1)}{12}(n+1)(n+2)$$

$$=\boxed{\frac{n(n+1)^2(n+2)}{12}}\ \text{Ans}$$

(iii) $2^2+4^2+6^2+\cdots$

Sol: $2^2+4^2+6^2+\cdots$

Here $T_n=[2+(n-1)2]^2=[2+2n-2]^2=(2n)^2=4n^2$

$$S_n = \sum T_n$$
$$= \sum 4n^2$$
$$= 4\sum n^2$$
$$= \cancel{4}^{2} \frac{n(n+1)(2n+1)}{\cancel{6}_{3}}$$
$$= \boxed{\frac{2n(n+1)(2n+1)}{3}} \quad \text{Ans}$$

(iv) $1^3+3^3+5^3+\cdots$

Sol: $1^3+3^3+5^3+\cdots$

Here $T_n = [1+(n-1)2]^3 = (1+2n-2)^3 = (2n-1)^3$

$$= (2n)^3 - 3(2n)^2 + 3(2n) - 1$$
$$= 8n^3 - 3(4n^2) + 6n - 1$$
$$= 8n^3 - 12n^2 + 6n - 1$$

$$S_n = \sum T_n$$
$$= \sum (8n^3 - 12n^2 + 6n - 1)$$
$$= 8\sum n^3 - 12\sum n^2 + 6\sum n - \sum 1$$
$$= \cancel{8}^{2}\frac{n^2(n+1)^2}{\cancel{4}} - \cancel{12}^{2}\frac{n(n+1)(2n+1)}{\cancel{6}} + \cancel{6}^{3}\frac{n(n+1)}{\cancel{2}} - n$$
$$= n[2n(n+1)^2 - 2(n+1)(2n+1) + 3(n+1) - 1]$$
$$= n[2n(n^2+2n+1) - 2(2n^2+n+2n+1) + 3n+3-1]$$
$$= n[2n^3 + \cancel{4n^2} + 2n - \cancel{4n^2} - 6n - \cancel{2} + 3n + \cancel{2}]$$
$$= \boxed{n[2n^3 - n]} \quad \text{Ans}$$

(v) $1^3+5^3+9^3+\cdots$

Sol: $1^3+5^3+9^3+\cdots$

Here $T_n=[1+(n-1)4]^3=(1+4n-4)^3=(4n-3)^3$

$=(4n)^3-3(4n)^2(3)+3(4n)(3)^2-(3)^3$

$=64n^3-9(16n^2)+12n(9)-27$

$=64n^3-144n^2+108n-27$

$S_n=\Sigma T_n$

$=\Sigma(64n^3-144n^2+108n-27)$

$=64\Sigma n^3-144\Sigma n^2+108\Sigma n-27\Sigma 1$

$=\overset{16}{\cancel{64}}\frac{n^2(n+1)^2}{\cancel{4}}-\overset{24}{\cancel{144}}\frac{n(n+1)(2n+1)}{\cancel{6}}+\overset{54}{\cancel{108}}\frac{n(n+1)}{\cancel{2}}-27n$

$=n[16n(n+1)^2-24(n+1)(2n+1)+54(n+1)-27]$

$=n[16n(n^2+2n+1)-24(2n^2+n+2n+1)+54n+54-27]$

$=n[16n^3+32n^2+16n-48n^2-72n-24+54n+54-27]$

$=\boxed{n(16n^3-16n^2-2n+3)}$ Ans.

Q.2 Find the sum: $1\cdot2+2\cdot3+3\cdot4+\cdots+99\cdot100$

Sol: $1\cdot2+2\cdot3+3\cdot4+\cdots+99\cdot100$

Here $T_n=[1+(n-1)1][2+(n-1)1]=(1+n-1)(2+n-1)$

$=n(n+1)=n^2+n$

$S_n=\Sigma(n^2+n)=\Sigma n^2+\Sigma n$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right] = \frac{n(n+1)}{2}\left[\frac{2n+1+3}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+4}{3}\right] = \frac{n(n+1)}{\cancel{2}}\cdot\cancel{2}\left(\frac{n+2}{3}\right)$$

$$S_n = \frac{n(n+1)(n+2)}{3}$$

$$S_{99} = \frac{\overset{33}{\cancel{99}}(99+1)(99+2)}{\cancel{3}} = 33(100)(101) = \boxed{333300}$$

Ans

Q.3: Find the sum: $1^2+3^2+5^2+\dots+99^2$

Sol: $1^2+3^2+5^2+\dots+99^2$ (50 terms)

Here $T_n = [1+(n-1)2]^2 = (1+2n-2)^2 = (2n-1)^2 = 4n^2-4n+1$

$S_n = \Sigma T_n$

$= \Sigma(4n^2-4n+1)$

$= 4\Sigma n^2 - 4\Sigma n + \Sigma 1$

$$= \overset{2}{\cancel{4}}\frac{n(n+1)(2n+1)}{\underset{3}{\cancel{6}}} - \overset{2}{\cancel{4}}\frac{n(n+1)}{\cancel{2}} + n$$

$$= n\left[\frac{2(n+1)(2n+1)}{3} - 2(n+1) + 1\right]$$

$$= n\left[\frac{2(n+1)(2n+1) - 6(n+1) + 3}{3}\right]$$

$$= \frac{n}{3}\left[2(2n^2+n+2n+1) - 6n - 6 + 3\right]$$

$$= \frac{n}{3}\left[4n^2 + \cancel{6n} + 2 - \cancel{6n} - 3\right]$$

$$S_n = \frac{n}{3}(4n^2-1) \Rightarrow S_{50} = \frac{50}{3}[4(50)^2-1]$$

$= \frac{50}{3}[4(2500)-1] = \frac{50}{3}[10000-1] = \frac{50}{3}[9999]$

$= \boxed{50 \times 3333}$ Ans

Q.4: Find the sum: $2+(2+5)+(2+5+8)+\ldots$ to n terms

Sol: $2+(2+5)+(2+5+8)+\ldots\ldots$ to n terms

Here $T_n = 2+5+8+\ldots+n$

$= \frac{n}{2}[2a+(n-1)d]$

$= \frac{n}{2}[2(2)+(n-1)3]$

$= \frac{n}{2}[4+3n-3]$

$= \frac{n}{2}(3n+1)$

$= \frac{3}{2}n^2+\frac{n}{2}$

$S_n = \sum T_n$

$= \sum\left(\frac{3}{2}n^2+\frac{n}{2}\right)$

$= \frac{3}{2}\sum n^2+\frac{1}{2}\sum n$

$= \frac{\cancel{3}}{2}\frac{n(n+1)(2n+1)}{\cancel{6}_2}+\frac{1}{2}\frac{n(n+1)}{2}$

$= \frac{n(n+1)}{2}\left[\frac{2n+1}{2}+\frac{1}{2}\right]$

$= \frac{n(n+1)}{2}\left[\frac{2n+2}{2}\right] = \frac{n(n+1)}{2}\left[\frac{\cancel{2}(n+1)}{\cancel{2}}\right]$

$= \boxed{\frac{n(n+1)^2}{2}}$ Ans

Q.5: Sum: $2+5+10+17+\cdots$ to n terms

Sol: $2+5+10+17+\cdots$ to n terms

$$= 1^2+1+(2^2+1)+(3^2+1)+(4^2+1)+\cdots+(n^2+1)$$

$$= (1^2+2^2+3^2+4^2+\cdots+n^2)+(1+1+1+\cdots+1)$$

$$= \frac{n(n+1)(2n+1)}{6}+n = n\left[\frac{(n+1)(2n+1)}{6}+1\right]$$

$$= n\left[\frac{2n^2+3n+1+6}{6}\right] = \boxed{\frac{n}{6}(2n^2+3n+7)} \text{ Ans.}$$

Q.6: Sum to n terms $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots$

Sol: $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots$

Here $T_n = [1+(n-1)1][2+(n-1)1][3+(n-1)1]$

$$= [1+n-1][2+n-1][3+n-1]$$

$$= n(n+1)(n+2) = n[n^2+3n+2] = n^3+3n^2+2n$$

$$S_n = \sum T_n$$

$$= \sum[n^3+3n^2+2n]$$

$$= \sum n^3+3\sum n^2+2\sum n$$

$$= \frac{n^2(n+1)^2}{4}+3\frac{n(n+1)(2n+1)}{6}+2\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{3(2n+1)}{3}+2\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n(n+1)+6(2n+1)+12}{6}\right]$$

$$= \frac{3n(n+1)}{12}\left[n(n+1)+2(2n+1)+4\right]$$

$$= \frac{n(n+1)}{4}\left[n^2+n+4n+2+4\right]$$

$$= \frac{n(n+1)}{4}(n^2+5n+6)$$

$$= \frac{n(n+1)}{4}\left[n^2+3n+2n+6\right]$$

$$= \frac{n(n+1)}{4}\left[n(n+3)+2(n+3)\right]$$

$$= \frac{n(n+1)}{4}\left[(n+2)(n+3)\right]$$

$$= \boxed{\frac{n(n+1)(n+2)(n+3)}{4}} \quad \text{Ans}$$

Q.7: Sum to n terms $1\cdot5\cdot9+2\cdot6\cdot10+3\cdot7\cdot11+\cdots$

Sol: $1\cdot5\cdot9+2\cdot6\cdot10+3\cdot7\cdot11+\cdots$

Here $T_n = [1+(n-1)1][5+(n-1)1][9+(n-1)1]$

$$= (1+n-1)(5+n-1)(9+n-1)$$

$$= n(n+4)(n+8)$$

$$= n(n^2+8n+4n+32)$$

$$= n(n^2+12n+32)$$

$$= n^3+12n^2+32n$$

$$S_n = \sum T_n$$

$$= \sum(n^3+12n^2+32n)$$

$$= \sum n^3 + 12\sum n^2 + 32\sum n$$

$$= \frac{n^2(n+1)^2}{4} + 12\frac{n(n+1)(2n+1)}{6} + 32\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{12(2n+1)}{3} + 32\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + 4(2n+1) + 32\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1) + 8(2n+1) + 64}{2}\right]$$

$$= \frac{n(n+1)}{4}\left[n^2 + n + 16n + 8 + 64\right]$$

$$= \frac{n(n+1)}{4}\left[n^2 + 17n + 72\right]$$

$$= \frac{n(n+1)}{4}\left[n^2 + 9n + 8n + 72\right]$$

$$= \frac{n(n+1)}{4}\left[n(n+9) + 8(n+9)\right]$$

$$= \frac{n(n+1)}{4}\left[(n+8)(n+9)\right]$$

$$= \boxed{\frac{n(n+1)(n+8)(n+9)}{4}} \text{ Ans}$$

Q.8. Find the sum to 2n terms of the series whose nth term is $4n^2+5n+1$.

Sol: $T_n = 4n^2 + 5n + 1$

$S_n = \Sigma T_n$

$= \Sigma(4n^2+5n+1)$

$= 4\Sigma n^2 + 5\Sigma n + \Sigma 1$

$= \overset{2}{\cancel{4}}\dfrac{n(n+1)(2n+1)}{\underset{3}{\cancel{6}}} + 5\dfrac{n(n+1)}{2} + n$

$= n\left[\dfrac{2(n+1)(2n+1)}{3} + 5\dfrac{(n+1)}{2} + 1\right]$

$= n\left[\dfrac{4(n+1)(2n+1)+15(n+1)+6}{6}\right]$

$= \dfrac{n}{6}\left[4(2n^2+n+2n+1)+15n+15+6\right]$

$= \dfrac{n}{6}\left[8n^2+12n+4+15n+21\right]$

$S_n = \dfrac{n}{6}\left[8n^2+27n+25\right]$

Put $n = 2n$

$S_{2n} = \dfrac{2n}{6}\left[8(2n)^2+27(2n)+25\right]$

$= \boxed{\dfrac{n}{3}\left[32n^2+54n+25\right]}$ Ans

Q.4: Find the sum of n terms of the series whose nth term is:

(i) $n^2(2n+3)$

Sol: $T_n = n^2(2n+3) = 2n^3+3n^2$

$S_n = \Sigma T_n$

$= \Sigma(2n^3+3n^2)$

$$= 2\sum n^3 + 3\sum n^2$$

$$= \cancel{2}\frac{n^2(n+1)^2}{\cancel{4}_2} + \cancel{3}\frac{n(n+1)(2n+1)}{\cancel{6}_2}$$

$$= \frac{n(n+1)}{2}\left[n(n+1) + 2n+1\right]$$

$$= \frac{n(n+1)}{2}(n^2+n+2n+1)$$

$$= \boxed{\frac{n(n+1)(n^2+3n+1)}{2}} \text{ Ans}$$

(ii) $3(4^n+2n^2)-4n^3$

<u>Sol:</u> $T_n = 3(4^n+2n^2)-4n^3 = 3\cdot 4^n + 6n^2 - 4n^3$

$$S_n = \sum T_n$$

$$= \sum(3\cdot 4^n + 6n^2 - 4n^3)$$

$$= 3\sum 4^n + 6\sum n^2 - 4\sum n^3$$

$$= 3\left[\frac{a_1(r^n-1)}{r-1}\right] + \cancel{6}\frac{n(n+1)(2n+1)}{\cancel{6}} - \cancel{4}\frac{n^2(n+1)^2}{\cancel{4}}$$

$$= 3\left[\frac{4(4^n-1)}{4-1}\right] + n(n+1)\left[2n+1 - n(n+1)\right]$$

$$= \cancel{3}\left[\frac{4(4^n-1)}{\cancel{3}}\right] + n(n+1)\left[2n+1-n^2-n\right]$$

$$= 4^{n+1} - 4 + n(n+1)(-n^2+n+1)$$

$$= \boxed{4^{n+1} - 4 - n(n+1)(n^2-n-1)} \text{ Ans}$$

EXERCISE 5.2

Q.1 Sum to n terms the following series.

(i) $1\cdot2+2\cdot2^2+3\cdot2^3+4\cdot2^4+\ldots$ to n terms.

Sol: Let $S_n=1\cdot2+2\cdot2^2+3\cdot2^3+4\cdot2^4+\ldots+n2^n$

$$\frac{1}{2}S_n=1+2\cdot2+3\cdot2^2+4\cdot2^3+\ldots+n2^{n-1} \longrightarrow (i) \text{ (A-G series)}$$

$$\pm S_n=\pm1\cdot2\pm2\cdot2^2\pm3\cdot2^3\pm4\cdot2^4+\ldots\pm(n-1)2^{n-1}\pm n2^n \longrightarrow (ii)$$

(i) − (ii) $\Rightarrow$ $-\frac{1}{2}S_n=1+1\cdot2+1\cdot2^2+1\cdot2^3+1\cdot2^4+\ldots+1\cdot2^{n-1}-n2^n$

$$-\frac{1}{2}S_n=1+2+2^2+2^3+2^4+\ldots+2^{n-1}-n2^n$$

$$=\frac{1(2^n-1)}{2-1}-n2^n$$

$$=2^n-1-n2^n$$

$$-\frac{1}{2}S_n=(1-n)2^n-1$$

$$S_n=\boxed{(n-1)2^{n+1}+2} \text{ Ans}$$

(ii) $1+4x+7x^2+10x^3+\ldots$ to n terms

Sol: Let $S_n=1+4x+7x^2+10x^3+\ldots+[1+(n-1)3]x^{n-1}$

$$S_n=1+4x+7x^2+10x^3+\ldots+(3n-2)x^{n-1} \longrightarrow (i) \text{ (A-G series)}$$

$$\pm xS_n=\pm x\pm4x^2\pm7x^3\pm10x^4\pm\ldots\pm(3n-5)x^{n-1}\pm(3n-2)x^n \longrightarrow (ii)$$

(i) − (ii) $\Rightarrow$ $(1-x)S_n=1+3x+3x^2+3x^3+\ldots+3x^{n-1}-(3n-2)x^n$

$$=1+3(x+x^2+x^3+\ldots+x^{n-1})-(3n-2)x^n$$

$$(1-x)S_n=1+3\left[\frac{x(1-x^{n-1})}{1-x}\right]-(3n-2)x^n$$

$$S_n=\boxed{\frac{1-(3n-2)x^n}{1-x}+\frac{3x(1-x^{n-1})}{(1-x)^2}} \text{ Ans}$$

(iii) $1+2x+3x^2+4x^3+\dots$ to n terms

Sol: Let $S_n = 1+2x+3x^2+4x^3+\dots+[1+(n-1)1]x^{n-1}$

$S_n = 1+2x+3x^2+4x^3+\dots+nx^{n-1} \longrightarrow$ (i) (A-G series)

$\pm x S_n = \pm x \pm 2x^2 \pm 3x^3 \pm 4x^4 \pm \dots \pm (n-1)x^{n-1} \pm nx^n \longrightarrow$ (ii)

(i) − (ii) $\Rightarrow (1-x)S_n = 1+x+x^2+x^3+\dots+x^{n-1}-nx^n$

$(1-x)S_n = \dfrac{1[1-x^n]}{1-x} - nx^n$

$S_n = \boxed{\dfrac{(1-x^n)}{(1-x)^2} - \dfrac{nx^n}{1-x}}$ Ans

(iv) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\dots$ to n terms

Sol: Let $S_n = 1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\dots+[1+(n-1)2]\frac{1}{2^{n-1}}$

$S_n = 1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\dots+\frac{(2n-1)}{2^{n-1}} \longrightarrow$ (i) (A-G series)

$\pm\frac{1}{2}S_n = \pm\frac{1}{2}\pm\frac{3}{2^2}\pm\frac{5}{2^3}\pm\frac{7}{2^4}\pm\dots\pm\frac{(2n-3)}{2^{n-1}}\pm\frac{(2n-1)}{2^n} \longrightarrow$ (ii)

(i) − (ii) $\Rightarrow \frac{1}{2}S_n = 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots+\frac{1}{2^{n-2}} - \frac{(2n-1)}{2^n}$

$= 1+\dfrac{1\left(1-\frac{1}{2^{n-1}}\right)}{1-\frac{1}{2}} - \dfrac{(2n-1)}{2^n}$

$\frac{1}{2}S_n = 1+\dfrac{1-\frac{1}{2^{n-1}}}{\frac{1}{2}} - \dfrac{(2n-1)}{2^n}$

$\frac{1}{2}S_n = 1+\dfrac{1-\frac{1}{2^{n-1}}}{\frac{1}{2}} - \dfrac{(2n-1)}{2^n} \Rightarrow S_n = \boxed{2+4\left(1-\frac{1}{2^{n-1}}\right)-\frac{2n-1}{2^{n-1}}}$

Ans

(v) $1-7x+13x^2-19x^3+\cdots$

Sol: $1-7x+13x^2-19x^3+\cdots$ to n terms

Let $S_n = 1-7x+13x^2-19x^3+\cdots$ to n terms

$= 1+7(-x)+13(-x)^2+19(-x)^3+\cdots+$ to n terms

$S_n = 1+7(-x)+13(-x)^2+19(-x)^3+\cdots+[1+(n-1)6](-x)^{n-1}$

$S_n = 1+7(-x)+13(-x)^2+19(-x)^3+\cdots+(6n-5)(-x)^{n-1} \longrightarrow$ (i) (A.G series)

$\mp x S_n = \pm(-x) \pm 7(-x)^2 \pm 13(-x)^3 \pm 19(-x)^4 \pm\cdots\pm (6n-11)(-x)^{n-1} \pm (6n-5)(-x)^n \longrightarrow$ (ii)

(i)−(ii) $\Rightarrow (1+x)S_n = 1+6(-x)+6(-x)^2+6(-x)^3+\cdots+6(-x)^{n-1}-(6n-5)(-x)^n$

$= 1+6[(-x)+(-x)^2+(-x)^3+\cdots+(-x)^{n-1}]-(6n-5)(-x)^n$

$(1+x)S_n = 1+6\left[(-x)\left\{\dfrac{1-(-x)^{n-1}}{1-(-x)}\right\}\right]-(6n-5)(-x)^n$

$= 1-6x\dfrac{[1-(-x)^{n-1}]}{1+x}-(6n-5)(-x)^n$

$S_n = \boxed{\dfrac{1-(6n-5)(-x)^n}{1+x} - \dfrac{6x[1-(-x)^{n-1}]}{(1+x)^2}}$ Ans

Q.2 Find the sum to infinity of the following series:

(i) $1^2+3^2x+5^2x+7^2x^3+\cdots,\ x<1$

Sol: Let $S_\infty = 1^2+3^2x+5^2x^2+7^2x^3+\cdots$ (Not Arithmetic/Geometric series) $\longrightarrow$ (i)

$\pm x S_\infty = \pm 1^2x \pm 3^2x^2 \pm 5^2x^3 \pm 7^2x^4+\cdots \longrightarrow$ (ii)

(i)−(ii) $\Rightarrow (1-x)S_\infty = 1^2+(3^2-1^2)x+(5^2-3^2)x^2+(7^2-5^2)x^3+\cdots$

$= 1+(3-1)(3+1)x+(5-3)(5+3)x^2+(7-5)(7+5)x^3+\cdots$

$$(1-x)S_\infty = 1+(2)(4)x+(2)(8)x^2+(2)(12)x^3+\cdots$$
$$= 1+8x+16x^2+24x^3+\cdots$$
$$= 1+8x(1+2x+3x^2+\cdots)$$
$$(1-x)S_\infty = 1+8xS \longrightarrow \text{(iii)}$$
$$S = 1+2x+3x^2+\cdots \quad \text{(Arithmetico-Geometric series)} \longrightarrow \text{(iv)}$$
$$\pm xS = \pm x \pm 2x^2 \pm 3x^3 \pm \cdots \longrightarrow \text{(v)}$$

$$\text{(iv)}-\text{(v)} \Rightarrow (1-x)S = 1+(x+x^2+x^3+\cdots) \longrightarrow \text{infinite Geometric series with } a_1 = x,\ r = x$$
$$= 1+\frac{x}{1-x}$$
$$S = \frac{1}{1-x}+\frac{x}{(1-x)^2}$$

$$\text{(iii)} \Rightarrow (1-x)S_\infty = 1+8x\left[\frac{1}{1-x}+\frac{x}{(1-x)^2}\right]$$
$$= 1+\frac{8x}{1-x}+\frac{8x^2}{(1-x)^2}$$
$$= \frac{1-x+8x}{1-x}+\frac{8x^2}{(1-x)^2}$$
$$(1-x)S_\infty = \frac{1+7x}{1-x}+\frac{8x^2}{(1-x)^2}$$
$$S_\infty = \boxed{\frac{1+7x}{(1-x)^2}+\frac{8x^2}{(1-x)^3}} \text{ Ans}$$

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(ii) $1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\dots$

Sol: Let $S_\infty = 1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\dots$ (Not A-G series) $\longrightarrow$ (i)

$\pm\frac{1}{3}S_\infty = \frac{1}{-3} \pm\frac{4}{3^2}\pm\frac{9}{3^3}\pm\frac{16}{3^4}\pm\frac{25}{3^5}+\dots \longrightarrow$ (ii)

(i) − (ii) $\Rightarrow (1-\frac{1}{3})S_\infty = 1+\frac{3}{3}+\frac{5}{3^2}+\frac{7}{3^3}+\dots$

$\frac{2}{3}S_\infty = 1+\frac{3}{3}+\frac{5}{3^2}+\frac{7}{3^3}+\dots$ (Arithmeticogeometric series)

$\frac{2}{3}S_\infty = S \longrightarrow$ (iii)

$S = 1+\frac{3}{3}+\frac{5}{3^2}+\frac{7}{3^3}+\dots \longrightarrow$ (iv)

$\pm\frac{1}{3}S = \frac{1}{-3}\pm\frac{3}{3^2}\pm\frac{5}{3^3}\pm\frac{7}{3^4}+\dots \longrightarrow$ (v)

(iv) − (v) $\Rightarrow (1-\frac{1}{3})S = 1+\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\dots$

$\frac{2}{3}S = 1+2\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots\right)$

infinite geometric series with $a_1=\frac{1}{3}, r=\frac{1}{3}$

$= 1+2\left(\frac{\frac{1}{3}}{1-\frac{1}{3}}\right)$

$= 1+2\left(\frac{\frac{1}{3}}{\frac{2}{3}}\right)$

$= 1+2\left(\frac{1}{2}\right)$

$\left(\frac{2}{3}\right)S = 2$

$\Rightarrow S = 3$

(iii) $\Rightarrow \frac{2}{3}S_\infty = 3$

$\boxed{S_\infty = \frac{9}{2}}$ Ans

Q.3 Find the nth term of the following arithmetico-geometric series.

$$\frac{0}{1}+\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots$$

Sol: $\frac{0}{1}+\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots$ (Arithmetico-geometric series)

Here $T_n=$ nth term of A.P × nth term of G.P

$=$ nth term of $0,1,2,3,4,5,\cdots$ × nth term of $1,2,4,8,16,32,\cdots$

$=[0+(n-1)1]\times[1\cdot 2^{n-1}]$

$=\boxed{n\,2^{n-1}}$ Ans

Q.4: Find the sum of the following Arithmetico-geometric series:

$$5+\frac{7}{3}+1+\frac{11}{27}+\cdots$$

Sol: Let $S_\infty=5+\frac{7}{3}+\frac{9}{9}+\frac{11}{27}+\cdots$ (A-G series)

$S_\infty=5+\frac{7}{3}+\frac{9}{3^2}+\frac{11}{3^3}+\cdots \longrightarrow$ (i)

$\pm\frac{1}{3}S_\infty=\pm\frac{5}{3}\pm\frac{7}{3^2}\pm\frac{9}{3^3}\pm\frac{11}{3^4}+\cdots \longrightarrow$ (ii)

(i) − (ii) $\Rightarrow$ $\left(1-\frac{1}{3}\right)S_\infty=5+\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\cdots$

$\frac{2}{3}S_\infty=5+2\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)$ $\longrightarrow$ infinite Geometric series with $a_1=\frac{1}{3}, r=\frac{1}{3}$

$=5+2\left[\frac{\frac{1}{3}}{1-\frac{1}{3}}\right]$

$$\frac{2}{3}S_\infty = 5+2\left[\frac{\frac{1}{3}}{\frac{2}{3}}\right]$$
$$= 5+2\left(\frac{1}{2}\right)$$
$$\frac{2}{3}S_\infty = 6 \Rightarrow S_\infty = \cancel{6}^{3}\times\frac{3}{\cancel{2}} \Rightarrow \boxed{S_\infty = 9}$$ Ans

Q.5 If the sum to infinity of the series $3+5r+7r^2+\dots\infty$ is $\frac{44}{9}$. Find the value of r.

Sol: Let $S_\infty = 3+5r+7r^2+\dots\infty$ (Arithmetico-geometric series) $\longrightarrow$ (i)

$$\underline{-\,rS_\infty = \pm 3r \pm 5r^2 \pm 7r^3 \pm \dots\infty} \longrightarrow \text{(ii)}$$

$$\text{(i)}-\text{(ii)} \Rightarrow (1-r)S_\infty = 3+2r+2r^2+2r^3+\dots\infty$$
$$= 3+2(r+r^2+r^3+\dots\infty)$$
$$(1-r)S_\infty = 3+2\left(\frac{r}{1-r}\right)$$
$$\Rightarrow S_\infty = \frac{3}{1-r}+\frac{2r}{(1-r)^2}$$
$$= \frac{3(1-r)+2r}{(1-r)^2}$$
$$= \frac{3-3r+2r}{1-2r+r^2}$$
$$S_\infty = \frac{3-r}{1-2r+r^2}$$

But $S_\infty = \frac{44}{9}$ (given)

$$\therefore \frac{44}{9} = \frac{3-r}{1-2r+r^2} \Rightarrow 44-88r+44r^2 = 27-9r$$
$$\Rightarrow 44r^2-88r+9r+44-27=0 \Rightarrow 44r^2-79r+17=0$$
$$r = \frac{-(-79)\pm\sqrt{(-79)^2-4(44)(17)}}{2\times 44} = \frac{79\pm\sqrt{6241-2992}}{88}$$
$$= \frac{79\pm\sqrt{3249}}{88} = \frac{79\pm 57}{88} = \frac{79+57}{88}, \frac{79-57}{88} = \frac{17}{11}, \frac{1}{4}$$

$\frac{17}{11}$ is not possible. Hence $\boxed{r=\frac{1}{4}}$ Ans

EXERCISE 5.3

Find the nth term and the sum to n terms of each of the following series.

Q.1: $4+13+28+49+76+\cdots$

Sol: $\underset{a_1}{4}+\underset{a_2}{13}+\underset{a_3}{28}+\underset{a_4}{49}+\underset{a_5}{76}+\cdots$

$a_2-a_1=9$

$\cancel{a_3}-\cancel{a_2}=15$

$a_4-\cancel{a_3}=21$

$a_n-\cancel{a_{n-1}}=(n-1)$th term of the sequence $9,16,21,\ldots$

Adding column-wise, we get

$$a_n-a_1=9+15+21+\ldots\text{ to }(n-1)\text{ terms}\quad (A.P)$$

$$=\frac{n-1}{2}[2\cdot 9+(n-2)6]\qquad\left(\text{Using } S_n=\frac{n}{2}[2a_1+(n-1)d]\right)$$

$$=(n-1)[9+(n-2)3]$$

$$=(n-1)[9+3n-6]$$

$$a_n=a_1+(n-1)(3n+3)$$

$$=4+3(n-1)(n+1)\quad[\because a_1=4]$$

$$a_n = 4+3(n^2-1)$$
$$= 4+3n^2-3$$
$$a_n = \boxed{3n^2+1} \text{ Ans}$$
$$\sum_1^n a_n = 3\sum_1^n n^2 + \sum_1^n 1$$
$$= 3\frac{n(n+1)(2n+1)}{6} + n = \frac{n(n+1)(2n+1)}{2} + n$$
$$= n\left[\frac{(n+1)(2n+1)}{2} + 1\right] = n\left[\frac{(n+1)(2n+1)+2}{2}\right]$$
$$= \frac{n}{2}\left[2n^2+n+2n+1+2\right] = \boxed{\frac{n}{2}\left[2n^2+3n+3\right]} \text{ Ans}$$

Q.2: $4+14+30+52+80+114+\cdots$

Sol: $4+14+30+52+80+114+\cdots$

$$\underset{a_1}{4} + \underset{a_2}{14} + \underset{a_3}{30} + \underset{a_4}{52} + \underset{a_5}{80} + \underset{a_6}{114}$$

$$\cancel{a_2} - a_1 = 10$$
$$\cancel{a_3} - \cancel{a_2} = 16$$
$$a_4 - \cancel{a_3} = 22$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$a_n - \cancel{a_{n-1}} = (n-1)\text{th term of the sequence } 10, 16, 22, \ldots$$

Adding column-wise, we get

$$a_n - a_1 = 10+16+22+\ldots \text{ to } (n-1) \text{ terms (A.P)}$$
$$= \frac{n-1}{2}[2\cdot 10+(n-2)6] \quad \left(\text{Using } S_n = \frac{n}{2}[2a_1+(n-1)d]\right)$$

$$a_n - a_1 = (n-1)\left[10 + (n-2)3\right]$$

$$= (n-1)(10+3n-6)$$

$$a_n = a_1 + (n-1)(3n+4)$$

$$a_n = 4 + 3n^2 + 4n - 3n - 4$$

$$a_n = \boxed{3n^2 + n} \text{ Ans}$$

$$\sum_1^n a_n = 3\sum n^2 + \sum n$$

$$= \cancel{3}\frac{n(n+1)(2n+1)}{\cancel{6}_{2}} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[2n+1+1\right]$$

$$= \frac{n(n+1)}{2}(2n+2)$$

$$= n(n+1)(n+1)$$

$$= \boxed{n(n+1)^2} \text{ Ans}$$

Q.3: $4+10+18+28+40+\cdots$

Sol: $4+10+18+28+40+\cdots$

$\downarrow a_1 \quad \downarrow a_2 \quad \downarrow a_3 \quad \downarrow a_4 \quad \downarrow a_5$

$$\cancel{a_2} - a_1 = 6$$

$$\cancel{a_3} - \cancel{a_2} = 8$$

$$a_4 - \cancel{a_3} = 10$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$a_n - \cancel{a_{n-1}} = (n-1)\text{th term of the sequence } 6, 8, 10, \cdots$$

Adding column-wise, we get

$$a_n - a_1 = 6+8+10+\cdots \text{ to } n \text{ terms (A.P)}$$

$$a_n - a_1 = \frac{n-1}{2}[2\cdot 6 + (n-2)2] \quad \left\{ \text{Using } S_n = \frac{n}{2}[2a + (n-1)d] \right\}$$

$$= (n-1)(6+n-2)$$

$$a_n = a_1 + (n-1)(n+4)$$

$$= 4 + (n^2 + 4n - n - 4) \quad [\because a_1 = 4]$$

$$= \cancel{4} + n^2 + 3n - \cancel{4}$$

$$a_n = \boxed{n^2 + 3n} \text{ Ans}$$

$$\Sigma a_n = \Sigma n^2 + 3\Sigma n$$

$$= \frac{n(n+1)(2n+1)}{6} + 3\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+1}{3} + 3\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+1+9}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+10}{3}\right] = \boxed{\frac{n(n+1)(n+5)}{3}} \text{ Ans}$$

Q.4: $3 + 5 + 11 + 29 + 83 + 245 + \cdots$

Sol: $3 + 5 + 11 + 29 + 83 + 245 + \cdots$

$$\downarrow a_1 \quad \downarrow a_2 \quad \downarrow a_3 \quad \downarrow a_4 \quad \downarrow a_5 \quad \downarrow a_6$$

$$\cancel{a_2} - a_1 = 2$$
$$\cancel{a_3} - \cancel{a_2} = 6$$
$$a_4 - \cancel{a_3} = 18$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$a_n - \cancel{a_{n-1}} = (n-1)\text{th term of the sequence } 2, 6, 18, \ldots$$

Adding column-wise we get

$a_n - a_1 = 2+6+18+\cdots$ to $(n-1)$ terms (G.P)

$= \frac{2(3^{n-1}-1)}{3-1} = \frac{\cancel{2}(3^{n-1}-1)}{\cancel{2}}$

$= 3^{n-1}-1$

$a_n = a_1 + 3^{n-1}$

$a_n = 3 + 3^{n-1} - 1$

$a_n = \boxed{3^{n-1}+2}$ Ans

$\sum_1^n a_n = \sum_1^n 3^{n-1} + \sum_1^n 2$

$= [1+3+3^2+\cdots+3^{n-1}]+2n$

$= \frac{1[3^n-1]}{3-1} + 2n$

$= \boxed{\frac{3^n-1}{2}+2n}$ Ans

Q.5: $3+9+21+45+93+189+\cdots$

Sol: $3+9+21+45+93+189+\cdots$

$\downarrow a_1 \quad \downarrow a_2 \quad \downarrow a_3 \quad \downarrow a_4 \quad \downarrow a_5 \quad \downarrow a_6$

$a_2 - a_1 = 6$

$a_3 - a_2 = 12$

$a_4 - a_3 = 24$

$\cdots\cdots$

$a_n - a_{n-1} = (n-1)$th term of the sequence $6, 12, 24, \ldots$

Adding column-wise, we get

$a_n - a_1 = 6+12+24+\cdots$ to $(n-1)$ terms (G.P)

$$a_n - a_1 = 6\frac{[2^{n-1} - 1]}{2-1} \quad \left(\because S_n = a_1\frac{[r^n - 1]}{r-1}\right)$$

$$a_n = a_1 + 6(2^{n-1} - 1)$$

$$a_n = 3 + 6(2^{n-1} - 1) = 3 + 6\cdot 2^{n-1} - 6$$

$$a_n = 6\cdot 2^{n-1} - 3 = 3\cdot 2\cdot 2^{n-1} - 3 = 3\cdot 2^n - 3$$

$$a_n = \boxed{3(2^n - 1)} \text{ Ans}$$

$$a_n = 3\cdot 2^n - 3$$

$$\sum_1^n a_n = \sum_1^n 3\cdot 2^n - \sum_1^n 3$$

$$= 3\sum_1^n 2^n - 3\sum_1^n 1$$

$$= 3[2 + 2^2 + 2^3 + \cdots + 2^n] - 3n$$

$$= 3\left[2\frac{[2^n - 1]}{2-1}\right] - 3n$$

$$= 6(2^n - 1) - 3n$$

$$= 3[2(2^n - 1) - n]$$

$$= 3(2^{n+1} - 2 - n)$$

$$= \boxed{3(2^{n+1} - n - 2)} \text{ Ans}$$

Q.6: $28 + 32 + 52 + 152 + 652 + \cdots$

Sol: $28 + 32 + 52 + 152 + 652 + \cdots$

$\downarrow a_1 \quad \downarrow a_2 \quad \downarrow a_3 \quad \downarrow a_4 \quad \downarrow a_5$

$a_2 - a_1 = 4$

$a_3 - a_2 = 20$

$a_4 - a_3 = 100$

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$a_n - a_{n-1} = (n-1)$th term of the sequence $4, 20, 100, \ldots$

Adding column-wise, we get

$$a_n - a_1 = 4 + 20 + 100 + \ldots \text{ to } (n-1) \text{ terms} \quad (G.P)$$

$$= \frac{4(5^{n-1} - 1)}{5-1} \quad \left(\text{Using } S_n = \frac{a_1(r^n - 1)}{r-1}\right)$$

$$= \frac{\cancel{4}(5^{n-1} - 1)}{\cancel{4}}$$

$$a_n = a_1 + 5^{n-1} - 1$$

$$= 28 + 5^{n-1} - 1 \quad [\because a_1 = 28]$$

$$a_n = \boxed{5^{n-1} + 27}$$

$$\sum_1^n a_n = \sum_1^n 5^{n-1} + 27 \sum_1^n 1$$

$$= [1 + 5 + 5^2 + \cdots + 5^{n-1}] + 27n$$

$$= \frac{1(5^n - 1)}{5-1} + 27n$$

$$\sum_1^n a_n = \boxed{\frac{5^n - 1}{4} + 27n} \text{ Ans}$$

EXERCISE 5.4

Q1) Find the sum of the following:

(i) $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\dots$ to n terms

Sol: Here $T_n = \frac{1}{[1+(n-1)1][2+(n-1)1]}$

$$\Rightarrow T_n = \frac{1}{n(n+1)}$$

By Partial fractions,

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \longrightarrow (i)$$

X by $n(n+1)$,

$$1 = A(n+1) + Bn \longrightarrow (ii)$$

Put $n=0$ in (ii)

$$1 = A(0+1) + B(0) \Rightarrow \boxed{A=1}$$

Put $n=-1$ in (ii)

$$1 = A(-1+1) + B(-1) \Rightarrow 1 = 0 - B \Rightarrow \boxed{B=-1}$$

$$(i) \Rightarrow \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\Rightarrow T_n = \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{n=1}^{n} T_n = \sum_{n=1}^{n}\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= \left[\left(1-\cancel{\tfrac{1}{2}}\right)+\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}\right)+\dots+\left(\cancel{\tfrac{1}{n}}-\tfrac{1}{n+1}\right)\right]$$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n+\cancel{1}-\cancel{1}}{n+1}$$

$$= \boxed{\frac{n}{n+1}} \text{ Ans}$$

(ii) $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots$ to n terms

Sol: Here $T_n=\frac{1}{[1+(n-1)2][3+(n-1)2]}$

$$=\frac{1}{[1+2n-2][3+2n-2]}$$

$$=\frac{1}{(2n-1)(2n+1)}$$

By Partial fractions,

$$\frac{1}{(2n-1)(2n+1)}=\frac{A}{2n-1}+\frac{B}{2n+1} \longrightarrow (i')$$

$\times$ by $(2n-1)(2n+1)$

$$1=A(2n+1)+B(2n-1) \longrightarrow (ii)$$

Put $n=\frac{1}{2}$ in (ii)

$$1=A(2\cdot\tfrac{1}{2}+1)+B(2\cdot\tfrac{1}{2}-1) \Rightarrow 1=A(2)+B(0) \Rightarrow \boxed{A=\frac{1}{2}}$$

Put $n=-\frac{1}{2}$ in (ii)

$$1=A[2(-\tfrac{1}{2})+1]+B[2(-\tfrac{1}{2})-1] \Rightarrow 1=A(0)+B(-2)$$

$$\Rightarrow \boxed{B=-\frac{1}{2}}$$

$$(i') \Rightarrow \frac{1}{(2n-1)(2n+1)}=\frac{1}{2(2n-1)}-\frac{1}{2(2n+1)}$$

$$\Rightarrow T_n=\frac{1}{2(2n-1)}-\frac{1}{2(2n+1)}$$

$$\sum_{n=1}^{n} T_n=\sum_{n=1}^{n}\left[\frac{1}{2(2n-1)}-\frac{1}{2(2n+1)}\right]$$

$$\sum T_n=\left(\frac{1}{2\cdot 1}-\cancel{\frac{1}{2\cdot 3}}\right)+\left(\cancel{\frac{1}{2\cdot 3}}-\cancel{\frac{1}{2\cdot 5}}\right)+\left(\cancel{\frac{1}{2\cdot 5}}-\cancel{\frac{1}{2\cdot 7}}\right)+\cdots+\left(\cancel{\frac{1}{2(2n-1)}}-\frac{1}{2(2n+1)}\right)$$

$$=\frac{1}{2}-\frac{1}{2(2n+1)}$$

$$=\frac{2n+\cancel{1}-\cancel{1}}{2(2n+1)}=\boxed{\frac{n}{2n+1}} \text{ Ans}$$

(iii) $\frac{1}{2\cdot5}+\frac{1}{5\cdot8}+\frac{1}{8\cdot11}+\cdots$ to infinity

Sol: Here $T_n = \frac{1}{[2+(n-1)3][5+(n-1)3]}$

$$= \frac{1}{[2+3n-3][5+3n-3]}$$

$$= \frac{1}{(3n-1)(3n+2)}$$

By Partial fractions,

$$\frac{1}{(3n-1)(3n+2)} = \frac{A}{(3n-1)} + \frac{B}{(3n+2)} \longrightarrow (i)$$

$\times$ by $(3n-1)(3n+2)$

$$1 = A(3n+2)+B(3n-1) \longrightarrow (ii)$$

Put $n=\frac{1}{3}$ in (ii)

$$1 = A\left(3\cdot\frac{1}{3}+2\right)+B\left(3\cdot\frac{1}{3}-1\right)$$

$$1 = A(3)+B(0) \Rightarrow \boxed{A=\frac{1}{3}}$$

Put $n=-\frac{2}{3}$ in (ii)

$$1 = A\left[3\left(-\frac{2}{3}\right)+2\right]+B\left[3\left(-\frac{2}{3}\right)-1\right]$$

$$1 = A(0)+B(-3) \Rightarrow \boxed{B=-\frac{1}{3}}$$

$$(i) \Rightarrow \frac{1}{(3n-2)(3n+2)} = \frac{1}{3(3n-1)} - \frac{1}{3(3n+2)}$$

$$\Rightarrow T_n = \frac{1}{3(3n-1)} - \frac{1}{3(3n+2)}$$

$$\sum_{n=1}^{n} T_n = \sum_{n=1}^{n}\left[\frac{1}{3(3n-1)} - \frac{1}{3(3n+2)}\right]$$

$$= \left(\frac{1}{3\cdot2} - \cancel{\frac{1}{3\cdot5}}\right)+\left(\cancel{\frac{1}{3\cdot5}} - \cancel{\frac{1}{3\cdot8}}\right)+\left(\cancel{\frac{1}{3\cdot8}} - \cancel{\frac{1}{3\cdot11}}\right)+\cdots+\left(\cancel{\frac{1}{3(3n-1)}} - \frac{1}{3(3n+2)}\right)$$

$$= \frac{1}{3\cdot2} - \frac{1}{3(3n+2)} = \frac{1}{6} - \frac{1}{3(3n+2)}$$

$$\frac{1}{3(3n+2)} \to 0 \text{ as } n\to\infty$$

$$\therefore \sum_{n=1}^{\infty} T_n = \boxed{\frac{1}{6}} \text{ Ans}$$

(iv) $\frac{1}{4.13}+\frac{1}{13.22}+\frac{1}{22.31}+\dots$ to infinity.

Sol: Here $T_n=\frac{1}{[4+(n-1)9][13+(n-1)9]}$

$$=\frac{1}{(9n-5)(9n+4)}$$

By Partial fraction,

$$\frac{1}{(9n-5)(9n+4)}=\frac{A}{(9n-5)}+\frac{B}{(9n+4)}\longrightarrow(i)$$

$\times$ by $(9n-5)(9n+4)$

$$1=A(9n+4)+B(9n-5)\longrightarrow(ii)$$

Put $n=\frac{5}{9}$ in (ii)

$$1=A(9\cdot\tfrac{5}{9}+4)+B(9\cdot\tfrac{5}{9}-5)\Rightarrow 1=A(9)+B(0)\Rightarrow\boxed{A=\tfrac{1}{9}}$$

Put $n=-\frac{4}{9}$ in (ii)

$$1=A[9(-\tfrac{4}{9})+4]+B[9(-\tfrac{4}{9})-5]\Rightarrow 1=A(0)+B(-9)\Rightarrow\boxed{B=-\tfrac{1}{9}}$$

$$(iii)\Rightarrow\quad\frac{1}{(9n-5)(9n+4)}=\frac{1}{9(9n-5)}-\frac{1}{9(9n+4)}$$

$$\Rightarrow T_n=\frac{1}{9(9n-5)}-\frac{1}{9(9n+4)}$$

$$\sum_{n=1}^{n}T_n=\sum_{n=1}^{n}\left[\frac{1}{9(9n-5)}-\frac{1}{9(9n+4)}\right]$$

$$=\left(\frac{1}{9.4}-\cancel{\frac{1}{9.13}}\right)+\left(\cancel{\frac{1}{9.13}}-\cancel{\frac{1}{9.22}}\right)+\left(\cancel{\frac{1}{9.22}}-\cancel{\frac{1}{9.31}}\right)+\dots+\left(\cancel{\frac{1}{9(9n-5)}}-\frac{1}{9(9n+4)}\right)$$

$$=\frac{1}{36}-\frac{1}{9(9n+4)}$$

Now $\frac{1}{9(9n+4)}\to 0$ as $n\to\infty$

$$\sum_{n=1}^{\infty}T_n=\boxed{\frac{1}{36}}\text{ Ans}$$

REVIEW EXERCISES 5

Q.1: choose the correct option.

(i) If $t_n = 6n+5$, then $t_{n+1} =$

(a) $6n-1$ (b) $6n+11$ (c) $6n+6$ (d) $6n-5$

Sol: $t_{n+1} = 6(n+1)+5$

$= 6n+6+5 = 6n+11$

Thus the correct option is (b). Ans

(ii) The sum to infinity of the series $1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\cdots$

(a) -6 (b) 2 (c) 3 (d) 4

Sol: $S_\infty = 1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\cdots$ (A-G series) $\rightarrow$ (i)

$\frac{1}{3}S_\infty = \frac{1}{3}+\frac{2}{3^2}+\frac{6}{3^3}+\frac{10}{3^4}+\frac{14}{3^5}+\cdots \rightarrow$ (ii)

(i) - (ii) $\Rightarrow \left(1-\frac{1}{3}\right)S_\infty = 1+\frac{1}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\frac{4}{3^4}+\cdots$

$$\frac{2}{3}S_\infty = \frac{4}{3}+4\left(\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots\right)$$

$$= \frac{4}{3}+4\left(\frac{\frac{1}{3^2}}{1-\frac{1}{3}}\right)$$ ↳ infinite Geometric Series

$$= \frac{4}{3}+4\left(\frac{\frac{1}{3^2}}{\frac{2}{3}}\right)$$

$$= \frac{4}{3}+4\left(\frac{\frac{1}{3}}{2}\right)$$

$$\frac{2}{3}S_\infty = \frac{4}{3}+\frac{4}{6} = \frac{12}{6} = 2$$

$$S_\infty = 2\cdot\frac{3}{2} = 3.$$ Thus the correct option is (c)

(iii) Sum the series: $1+2\cdot2+3\cdot2^2+\cdots+100\cdot2^{99}$

(a) $99\cdot2^{100}$ (b) $100\cdot2^{100}$ (c) $99\cdot2^{100}+1$ (d) $1000\,2^{100}$

Sol: $S_{100} = 1+2\cdot2+3\cdot2^2+\cdots+100\cdot2^{99}$ (Arithmetico-geometric series) $\to$ (i)

$2S_{100} = 2+2\cdot2^2+3\cdot2^3+\cdots+99\cdot2^{99}+100\cdot2^{100} \to$ (ii)

(i)$-$(ii) $\Rightarrow -S_{100} = (1+2+2^2+2^3+\cdots+2^{99}) - 100\cdot2^{100}$ $\to$ Sum of finite geometric series with $a_1=1$, $r=2$ and $n=100$

$= \frac{1(2^{100}-1)}{2-1} - 100\cdot2^{100}$

$= 2^{100}-1-100\cdot2^{100}$

$-S_{100} = -99\cdot2^{100}-1$

$\Rightarrow S_{100} = 99\cdot2^{100}+1$. Thus the correct option is (c). Ans

(iv) The nth term of the series $1\cdot2+2\cdot3+3\cdot4+\cdots$ is

(a) (n^2-n) (b) (n^2+n) (c) n^2 (d) None of these

Sol: $1\cdot2+2\cdot3+3\cdot4+\cdots$

$T_n = [1+(n-1)1][2+(n-1)1] = n(n+1) = n^2+n$

Thus the correct option is (b), Ans

(v) The sum of n terms of the series whose nth term is $1+2^n$

(a) $n+2^{n-1}$ (b) $(n+1)+2^{n+1}$ (c) $n+2(2^n-1)$ (d) None of these

Sol: $\Sigma(1+2^n) = \Sigma1+\Sigma2^n = n+\frac{2(2^n-1)}{2-1} = n+2(2^n-1)$

Thus the correct option is (c). Ans.

(vi) Evaluate $\Sigma(3+2^r)$, where $r=1,2,3,\ldots,10$

(a) 2051 (b) 2049 (c) 2076 (d) 1052

Sol. $\sum_{r=1}^{10}(3+2^r) = \sum_{r=1}^{10}3 + \sum_{r=1}^{10}2^r = \underbrace{3+3+3+\cdots+3}_{10\text{ times}} + 2+2^2+\cdots+2^{10}$

$= 10\times 3 + \dfrac{2[2^{10}-1]}{2-1} = 30 + 2^{11} - 2 = 30+2048-2$

$= 2076$

Thus the correct option is (c). Ans

(vii) What is the nth term of the series $1+\dfrac{(1+2)}{2}+\dfrac{(1+2+3)}{3}+\cdots$?

(a) $\dfrac{n+1}{2}$ (b) $\dfrac{n(n+1)}{2}$ (c) $n^2-(n+1)$ (d) $\dfrac{(n+1)(2n+3)}{2}$

Sol: nth term $= \dfrac{1+2+3+\cdots+n}{n} = \dfrac{n(n+1)}{2}\times\dfrac{1}{n} = \dfrac{n+1}{2}$

Thus the correct option is (a). Ans

(viii) Sum of n terms of the series $1^3+3^3+5^3+7^3+\cdots$ is

(a) $n^2(2n^2-1)$ (b) $2n^3+3n^2$ (c) $n^3(n-1)$ (d) n^3+8n+4

Sol: $T_n = [1+(n-1)2]^3 = (2n-1)^3 = (2n)^3 - 3(2n)^2 + 3(2n) - 1$

$= 8n^3 - 12n^2 + 6n - 1$

$S_n = \sum T_n$

$= \sum(8n^3-12n^2+6n-1)$

$= 8\sum n^3 - 12\sum n^2 + 6\sum n - \sum 1$

$= 8\cdot\dfrac{n^2(n+1)^2}{4} - 12\dfrac{n(n+1)(2n+1)}{6} + 6\dfrac{n(n+1)}{2} - n$

$= n[2n(n^2+2n+1) - 2(2n^2+n+2n+1) + 3(n+1) - 1]$

$= n[2n^3 + \cancel{4n^2} + 2n - \cancel{4n^2} - 6n - \cancel{2} + 3n + \cancel{3} - \cancel{1}]$

$= n(2n^3 - n)$

$= n^2(2n^2-1)$

Thus the correct option is (a). Ans.