

SOME IMPORTANT POINTS OF CHAPTER NO: 2

The following points will enable you to solve the problems of chapter No.2

- (i) A rectangular array of numbers written in the form of rows and columns and enclosed in square brackets is called a matrix.
- (ii) Order of a matrix = no. of rows $\times$ number of columns = $m \times n$
- (iii) There is associated a unique real no. with every square matrix which is called its determinant.
- (iv) $m = n \Rightarrow$ matrix is square, $m \neq n \Rightarrow$ matrix is rectangular.
- (v) Formula for cofactors, $A_{ij} = (-1)^{i+j} M_{ij}$, M_{ij} is called minor of a_{ij} .
- (vi) $A^{-1} = \frac{1}{\det A} \text{adj} A$ (Inverse by cofactors)
- (vii) $|A| = 0 \Rightarrow A$ is singular, $|A| \neq 0 \Rightarrow A$ is non-singular.
- (viii) A matrix which is obtained by interchanging its rows & columns, is called transpose of the matrix denoted by A^t .
- (ix) The following three steps are called elementary row operations (ERO).
 - (i) $R_i \leftrightarrow R_j$ (ii) kR_i (iii) $R_i + kR_j$
- (x) $A \rightarrow B$ means B is obtained from A by applying ERO on A .
- (xi) The first non-zero element in a non-zero row is called key entry.
- (xii) A matrix is called in Echelon form if
 - (i) The number of zeroes to the L.H.S of key entry increases row by row.
 - (ii) If there is any zero row it comes at the bottom.
- (xiii) A matrix is called in reduced echelon form if

- (i) It is in echelon form.
 (ii) K-entry is 1 in each non-zero row.
 (iii) If K-entry lies in jth column then the rest of the elements in that column must be zero.

(xiv) A system of linear equations.

(xv) ① can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$A X = B$

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \right\} \text{①}$$

augmented matrix
 $\uparrow$

(xvi) A is called coefficient matrix. $[A|B]$ or $Ab = \begin{bmatrix} a_{11} & a_{12} & a_{13} & | & b_1 \\ a_{21} & a_{22} & a_{23} & | & b_2 \\ a_{31} & a_{32} & a_{33} & | & b_3 \end{bmatrix}$

(xvii) In Gaussian Elimination method, Ab is

changed into echelon form and in Gauss-Jordan method Ab is changed into reduced echelon form.

(xviii) If $B=0$, the system ① is called homogeneous system.

If $B \neq 0$, " " " " " Non-homogeneous system.

(xix) If the solution of system ① exists, then it is called consistent.

" " " " " does not exist, " " " inconsistent.

(xx) The system $AX=0$ has non-trivial solution if $|A|=0$.

(xxi) The number of non-zero rows in echelon form of a matrix is called its rank ($\text{Rank}(A)$)

(xxii) Cramer's rule

$x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}$, $x_3 = \frac{D_3}{D}$ where $D = |A|$, For D_1 replace c_1 of D by B , For D_2 replace c_2 of D by B and for D_3 replace c_3 of D by B .

(xxiii) To find A^{-1} by row operations, change $[A|I]$ into $[I|A^{-1}]$ by ERO.

(xxiv) Properties of determinants:

- (i) If any row or ^{column} of a determinant is zero, the value of determinant is zero.
 - (ii) Interchanging two rows or two columns of a determinant the value of the determinant changes by a minus sign.
 - (iii) Transpose does not change the value of the determinant i.e. $|A| = |A^T|$
 - (iv) $\begin{vmatrix} a+k & b \\ c+l & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} k & b \\ l & d \end{vmatrix}$ etc
 - (v) $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ etc
 - (vi) $R_i + kR_j$ or $C_i + kC_j$ does not change the value of the determinant.
- (xxv) If M is any square matrix

Matrix Property	Name
(i) $M^T = M$	$\longleftrightarrow$ Symmetric
(ii) $M^T = -M$	$\longleftrightarrow$ Skew-symmetric
(iii) $M^2 = M$	$\longleftrightarrow$ Idempotent
(iv) $M^2 = I$	$\longleftrightarrow$ Involutory
(v) $M^P = O$ (smallest such P is called index)	$\longleftrightarrow$ Nilpotent of index P
(vi) $M^{k+1} = M$	$\longleftrightarrow$ Periodic of period k (where k is smallest)
(vii) $(\bar{M})^T = M$	$\longleftrightarrow$ Hermitian
(viii) $(\bar{M})^T = -M$	$\longleftrightarrow$ Skew-Hermitian
(ix) $M^T = M^{-1}$	$\longleftrightarrow$ orthogonal
(x) $(\bar{M})^T = M^{-1}$	$\longleftrightarrow$ Unitary.

EXERCISE 2.1

(Q.1) Express the following as a single matrix.

$$(i) [1 \ 2 \ 4] \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$\begin{aligned} \text{Sol: } [1 \ 2 \ 4] \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} &= [1 \ 2 \ 4] \begin{bmatrix} 2+0+12 \\ 4+0+6 \\ 0+4+12 \end{bmatrix} = [1 \ 2 \ 4] \begin{bmatrix} 14 \\ 10 \\ 16 \end{bmatrix} \\ &= [14+20+64] = [98] \text{ Ans} \end{aligned}$$

$$(ii) [1 \ -2 \ 3] \begin{bmatrix} 2 & -1 & 5 \\ 0 & 2 & 4 \\ -7 & 5 & 0 \end{bmatrix} - [2 \ -5 \ 7]$$

$$\begin{aligned} \text{Sol: } [1 \ -2 \ 3] \begin{bmatrix} 2 & -1 & 5 \\ 0 & 2 & 4 \\ -7 & 5 & 0 \end{bmatrix} - [2 \ -5 \ 7] \\ &= [24+0-21 \quad -1-4+15 \quad 5-8+0] - [2 \ -5 \ 7] \\ &= [-19 \quad 10 \quad -3] - [2 \ -5 \ 7] = [-19-2 \quad 10+5 \quad -3-7] \\ &= [-21 \quad 15 \quad -10] \text{ Ans} \end{aligned}$$

$$(iii) \begin{bmatrix} 7 & 1 & 2 \\ 9 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \text{Sol: } \begin{bmatrix} 7 & 1 & 2 \\ 9 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} &= \begin{bmatrix} 21+4+10 \\ 27+8+5 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 35 \\ 40 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 35+8 \\ 40+4 \end{bmatrix} = \begin{bmatrix} 43 \\ 44 \end{bmatrix} \text{ Ans} \end{aligned}$$

$$(iv) \left\{ \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \right\} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{aligned} \text{Sol: } \left\{ \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \right\} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} &= \begin{bmatrix} 1+3 & 3-2 \\ -1-1 & -4+1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 4 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 4+2 & 12+4 & 20+6 \\ -2-6 & -6-12 & -10-18 \end{bmatrix} = \begin{bmatrix} 6 & 16 & 26 \\ -8 & -18 & -28 \end{bmatrix} \text{Ans}$$

Full Q.2:- Let $A = \begin{bmatrix} 2 & -5 & 1 \\ 3 & 0 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & -2 \\ 0 & -1 & -1 \end{bmatrix}$.
Find $2A + 3B - 4C$.

Sol:- $2A + 3B - 4C = 2 \begin{bmatrix} 2 & -5 & 1 \\ 3 & 0 & -4 \end{bmatrix} + 3 \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & 5 \end{bmatrix} - 4 \begin{bmatrix} 0 & 1 & -2 \\ 0 & -1 & -1 \end{bmatrix}$
 $= \begin{bmatrix} 4 & -10 & 2 \\ 6 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 3 & -6 & -9 \\ 0 & -3 & 15 \end{bmatrix} + \begin{bmatrix} 0 & -4 & 8 \\ 0 & 4 & 4 \end{bmatrix}$
 $= \begin{bmatrix} 4+3+0 & -10-6-4 & 2-9+8 \\ 6+0+0 & 0-3+4 & -8+15+4 \end{bmatrix} = \begin{bmatrix} 7 & -20 & 1 \\ 6 & 1 & 11 \end{bmatrix} \text{Ans}$

Q.3: (i) if $A = \begin{bmatrix} x & y & z \end{bmatrix}$, $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and $C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, verify that
 $(AB)C = A(BC)$

Sol:- $(AB)C = \left(\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
 $= \begin{bmatrix} ax+hy+gz & hx+by+fz & gx+fy+cz \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
 $= [(ax+hy+gz)x + (hx+by+fz)y + (gx+fy+cz)z]$
 $= [ax^2 + hxy + gxz + hxy + by^2 + fyz + gxz + fyz + cz^2]$
 $= [ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz] \rightarrow (i)$

$A(BC) = \begin{bmatrix} x & y & z \end{bmatrix} \left(\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$
 $= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} ax+hy+gz \\ hx+by+fz \\ gx+fy+cz \end{bmatrix}$
 $= x(ax+hy+gz) + y(hx+by+fz) + z(gx+fy+cz)$
 $= ax^2 + hxy + xgz + hxy + by^2 + fyz + gxz + fyz + cz^2$

$$= ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz \rightarrow (ij)$$

From (i) and (ii).

$$(AB)C = A(BC) \text{ verified}$$

(ii) If $A = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 4 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & 0 \end{bmatrix}$,

verify that:

a) $A(B+C) = AB+AC$

Sol:- $A(B+C) = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 & -3 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & 0 \end{bmatrix} \right)$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2+3 & 1-1 & -3+5 \\ 0+2 & 4+1 & 2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 & 2 \\ 2 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5+6 & 0+15 & 2+6 \\ -5+8 & 0+20 & -2+8 \end{bmatrix} = \begin{bmatrix} 11 & 15 & 8 \\ 3 & 20 & 6 \end{bmatrix} \rightarrow (i)$$

$$AB+AC = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 1+12 & -3+6 \\ -2+0 & -1+16 & 3+8 \end{bmatrix} + \begin{bmatrix} 3+6 & -1+3 & 5+0 \\ -3+8 & 1+4 & -5+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 13 & 3 \\ -2 & 15 & 11 \end{bmatrix} + \begin{bmatrix} 9 & 2 & 5 \\ 5 & 5 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+9 & 13+2 & 3+5 \\ -2+5 & 15+5 & 11-5 \end{bmatrix} = \begin{bmatrix} 11 & 15 & 8 \\ 3 & 20 & 6 \end{bmatrix} \rightarrow (ii)$$

From (i) and (ii), we have

$$A(B+C) = AB+AC \text{ verified}$$

(b) $A(B-C) = AB-AC$

Sol:- $A(B-C) = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 & -3 \\ 0 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & 0 \end{bmatrix} \right)$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2-3 & 1+1 & -3-5 \\ 0-2 & 4-1 & 2-0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 & -8 \\ -2 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1-6 & 2+9 & -8+6 \\ -2+8 & -2+12 & 8+8 \end{bmatrix} = \begin{bmatrix} -7 & 11 & -2 \\ -7 & 10 & 16 \end{bmatrix} \rightarrow (i)$$

$$AB-AC = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 0 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 1+12 & -3+6 \\ -2+0 & -1+16 & 3+8 \end{bmatrix} - \begin{bmatrix} 3+6 & -1+3 & 5+0 \\ -3+8 & 1+4 & -5+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 13 & 3 \\ -2 & 15 & 11 \end{bmatrix} - \begin{bmatrix} 9 & 2 & 5 \\ 5 & 5 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 2-9 & 13-2 & 3-5 \\ -2-5 & 15-5 & 11+5 \end{bmatrix} = \begin{bmatrix} -7 & 11 & -2 \\ -7 & 10 & 16 \end{bmatrix} \rightarrow (ii)$$

From (i) and (ii), we have

$$A(B-C) = AB-AC \text{ verified}$$

Full Q.4 Let $A = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix}$, show that $\frac{1}{3}A^2 - 2A - 9I = 0$

Sol:-

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1+16+16 & 4+4+16 & 4+16+4 \\ 4+4+16 & 16+1+16 & 16+4+4 \\ 4+16+4 & 16+4+4 & 16+16+1 \end{bmatrix}$$

$$= \begin{bmatrix} 33 & 24 & 24 \\ 24 & 33 & 24 \\ 24 & 24 & 33 \end{bmatrix}$$

$$\text{L.H.S} = \frac{1}{3}A^2 - 2A - 9I = \frac{1}{3} \begin{bmatrix} 33 & 24 & 24 \\ 24 & 33 & 24 \\ 24 & 24 & 33 \end{bmatrix} - 2 \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix} - 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 8 & 8 \\ 8 & 11 & 8 \\ 8 & 8 & 11 \end{bmatrix} + \begin{bmatrix} -2 & -8 & -8 \\ -8 & -2 & -8 \\ -8 & -8 & -2 \end{bmatrix} + \begin{bmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 11-2-9 & 8-8+0 & 8-8+0 \\ 8-8+0 & 11-2-9 & 8-8+0 \\ 8-8+0 & 8-8+0 & 11-2-9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O = \text{R.H.S.}$$

Proved.

Full

Q.5

Matrix $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ is given to be symmetric,

find values of a and b .

Sol:- $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{bmatrix}$

$\because A$ is symmetric

$$A^t = A$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$$

Comparing corresponding elements

$$3a = -2 \Rightarrow a = -\frac{2}{3}$$

$$2b = 3 \Rightarrow b = \frac{3}{2}$$

Hence $\boxed{a = -\frac{2}{3}, b = \frac{3}{2}}$ Ans

Q.6

Solve the following matrix equations for X .

(i) $X - 3A = 2B$, if $A = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 4 \end{bmatrix}$

Sol:-

$$X - 3A = 2B$$

$$\Rightarrow X = 3A + 2B$$

$$\begin{aligned}
 X &= 3 \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & 1 \end{bmatrix} + 2 \begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 3 & 0 & 9 \\ -6 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 2 \\ 6 & -2 & 8 \end{bmatrix} \\
 &= \begin{bmatrix} 3+4 & 0+2 & 9+2 \\ -6+6 & 6-2 & 3+8 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 11 \\ 0 & 4 & 11 \end{bmatrix} \text{ Ans}
 \end{aligned}$$

✓ (ii) $2(X-A) = B$, if $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & -4 & 2 \end{bmatrix}$

Sol:- $2(X-A) = B \Rightarrow X-A = \frac{1}{2}B$

$$\Rightarrow X = A + \frac{1}{2}B$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 3 & -1 & 2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 4 & 6 & 2 \\ 0 & -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 3 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & 2+3 & 2+1 \\ 3+0 & -1-2 & 2+1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ 3 & -3 & 3 \end{bmatrix} \text{ Ans}$$

Full Q.7. If $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$,

then show that $(A+B)^t = A^t + B^t$

Sol: $A+B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1+2 & 0-1 & -1+3 & 2+1 \\ 3+1 & 1+3 & 2-1 & 5+4 \\ 0+3 & -2+1 & 1+2 & 6-1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 & 3 \\ 4 & 4 & 1 & 9 \\ 3 & -1 & 3 & 5 \end{bmatrix}$$

$$\Rightarrow (A+B)^t = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \rightarrow (i)$$

$$A^t + B^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & 3+1 & 0+3 \\ 0-1 & 1+3 & -2+1 \\ -1+3 & 2-1 & 1+2 \\ 2+1 & 5+4 & 6-1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \rightarrow (ii)$$

From (i) and (ii), we have

$$(A+B)^t = A^t + B^t$$

Q. 8: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$. Show that

✓ (i) $(A^t)^t = A$

Sol:- $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix}$

✓ (ii) $AA^t \neq A^tA$ $\Rightarrow (A^t)^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix} = A$ Proved

Sol:- $AA^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+0 & 3-2+0 \\ 3-2+0 & 9+1+16 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 26 \end{bmatrix} \rightarrow (i)$$

$$A^tA = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+9 & 2-3 & 0+12 \\ 2-3 & 4+1 & 0-4 \\ 0+12 & 0-4 & 0+16 \end{bmatrix} = \begin{bmatrix} 10 & -1 & 12 \\ -1 & 5 & -4 \\ 12 & -4 & 16 \end{bmatrix} \rightarrow (ii)$$

From (i) and (ii), we have

$$AA^t \neq A^tA \text{ Proved}$$

Q. 9: Verify that $(AB)^t = B^tA^t$ if

✓ (i) $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 0 \end{bmatrix}$

Sol:- $AB = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2-2+9 & 4-2+0 \\ 1+0+3 & 2+0+0 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 4 & 2 \end{bmatrix}$

$$\Rightarrow (AB)^t = \begin{bmatrix} 9 & 4 \\ 2 & 2 \end{bmatrix} \longrightarrow (i)$$

$$B^t A^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2-2+9 & 1+0+3 \\ 4-2+0 & 2+0+0 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 2 & 2 \end{bmatrix} \longrightarrow (ii)$$

From (i) and (ii), we have

$$(AB)^t = B^t A^t \text{ Proved}$$

$$(ii) \quad A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & -2 \end{bmatrix}$$

$$\underline{\text{Sol:-}} \quad AB = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1+4+0 & 1+6+0 \\ -1+2+4 & -1+3-8 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & -6 \end{bmatrix}$$

$$\Rightarrow (AB)^t = \begin{bmatrix} 5 & 5 \\ 7 & -6 \end{bmatrix} \longrightarrow (iii)$$

$$B^t A^t = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1+4+0 & -1+2+4 \\ 1+6+0 & -1+3-8 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 7 & -6 \end{bmatrix} \longrightarrow (iv)$$

From (iii) and (iv), we have

$$(AB)^t = B^t A^t \text{ Proved.}$$

$$\underline{\text{Q10:}} \quad \text{Let } A = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 6 & 7 \\ 6 & -8 & 3 \\ 7 & 3 & 1 \end{bmatrix}$$

Verify that A and B are symmetric. Also verify that $A+B$ is symmetric.

$$\underline{\text{Sol:-}} \quad A = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix} = A \Rightarrow A \text{ is symmetric.} \quad \underline{\text{verified}}$$

$$B = \begin{bmatrix} 5 & 6 & 7 \\ 6 & -8 & 3 \\ 7 & 3 & 1 \end{bmatrix} \Rightarrow B^t = \begin{bmatrix} 5 & 6 & 7 \\ 6 & -8 & 3 \\ 7 & 3 & 1 \end{bmatrix} = B \Rightarrow B \text{ is symmetric.} \quad \underline{\text{verified}}$$

$$A+B = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 6 & 7 \\ 6 & -8 & 3 \\ 7 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1+5 & -3+6 & 4+7 \\ -3+6 & 2-8 & -5+3 \\ 4+7 & -5+3 & 0+1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 11 \\ 3 & -6 & -2 \\ 11 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow (A+B)^t = \begin{bmatrix} 6 & 3 & 11 \\ 3 & -6 & -2 \\ 11 & -2 & 1 \end{bmatrix} = A+B \Rightarrow A+B \text{ is symmetric. verified}$$

Q.11: Let $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -6 & 11 \\ 6 & 0 & -7 \\ -11 & 7 & 0 \end{bmatrix}$

verify that $A+B$ is skew-symmetric.

Sol:

$$A+B = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -6 & 11 \\ 6 & 0 & -7 \\ -11 & 7 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 1-6 & -2+11 \\ -1+6 & 0+0 & 3-7 \\ 2-11 & -3+7 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & -5 & 9 \\ 5 & 0 & -4 \\ -9 & 4 & 0 \end{bmatrix}$$

$$(A+B)^t = \begin{bmatrix} 0 & 5 & -9 \\ -5 & 0 & 4 \\ 9 & -4 & 0 \end{bmatrix}$$

$$= - \begin{bmatrix} 0 & -5 & 9 \\ 5 & 0 & -4 \\ -9 & 4 & 0 \end{bmatrix} = -(A+B)$$

$\Rightarrow A+B$ is skew-symmetric. verified

Full

Q.12. If $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ -2 & 3 & 4 \end{bmatrix}$, then verify that

(i) $A+A^t$ is symmetric

Sol:- $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ -2 & 3 & 4 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 5 & 3 \\ 1 & 6 & 4 \end{bmatrix}$

$$A+A^t = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ -2 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -2 \\ 2 & 5 & 3 \\ 1 & 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3+3 & 2+4 & 1-2 \\ 4+2 & 5+5 & 6+3 \\ -2+1 & 3+6 & 4+4 \end{bmatrix} = \begin{bmatrix} 6 & 6 & -1 \\ 6 & 10 & 9 \\ -1 & 9 & 8 \end{bmatrix}$$

$$(A + A^t)^t = \begin{bmatrix} 6 & 6 & -1 \\ 6 & 10 & 9 \\ -1 & 9 & 8 \end{bmatrix} = A + A^t$$

$\Rightarrow A + A^t$ is symmetric. verified

(ii) $A - A^t$ is skew-symmetric. verified

Sol:- $A - A^t = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ -2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 4 & -2 \\ 2 & 5 & 3 \\ 1 & 6 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 3-3 & 2-4 & 1+2 \\ 4-2 & 5-5 & 6-3 \\ -2-1 & 3-6 & 4-4 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 3 \\ -3 & -3 & 0 \end{bmatrix}$$

$$(A - A^t)^t = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -3 \\ 3 & 3 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 3 \\ -3 & -3 & 0 \end{bmatrix} = -(A - A^t)$$

$\Rightarrow A - A^t$ is skew-symmetric. verified

Q.13 If A is a square matrix of order 3, then show that:

(i) $A + A^t$ is symmetric.

Sol:- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$$A + A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & a_{22}+a_{22} & a_{23}+a_{32} \\ a_{31}+a_{13} & a_{32}+a_{23} & a_{33}+a_{33} \end{bmatrix}$$

$$(A + A^t)^t = \begin{bmatrix} a_{11}+a_{11} & a_{21}+a_{12} & a_{31}+a_{13} \\ a_{12}+a_{21} & a_{22}+a_{22} & a_{32}+a_{23} \\ a_{13}+a_{31} & a_{23}+a_{32} & a_{33}+a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & a_{22}+a_{22} & a_{23}+a_{32} \\ a_{31}+a_{13} & a_{32}+a_{23} & a_{33}+a_{33} \end{bmatrix} = A+A^t$$

$\Rightarrow A+A^t$ is symmetric. Proved

(ii) $A-A^t$ is skew-symmetric.

Sol:- $A-A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}-a_{11} & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & a_{22}-a_{22} & a_{23}-a_{32} \\ a_{31}-a_{13} & a_{32}-a_{23} & a_{33}-a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & 0 & a_{23}-a_{32} \\ a_{31}-a_{13} & a_{32}-a_{23} & 0 \end{bmatrix}$$

$$(A-A^t)^t = \begin{bmatrix} 0 & a_{21}-a_{12} & a_{31}-a_{13} \\ a_{12}-a_{21} & 0 & a_{32}-a_{23} \\ a_{13}-a_{31} & a_{32}-a_{23} & 0 \end{bmatrix}$$

$$= - \begin{bmatrix} 0 & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & 0 & a_{23}-a_{32} \\ a_{31}-a_{13} & a_{32}-a_{23} & 0 \end{bmatrix}$$

$$= -(A-A^t)$$

$\Rightarrow A-A^t$ is skew-symmetric.
Proved

EXERCISE 2.2

Full Q.1: If $A = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$, then find $A_{11}, A_{21}, A_{23}, A_{31}, A_{32}, A_{33}$.

Also find $|A|$.

Sol:-

For Cofactors,

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^{1+1} M_{11} = + \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 - 0 = -4$$

$$A_{21} = (-1)^{2+1} M_{21} = - \begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix} = -(-6 - 0) = 6$$

$$A_{23} = (-1)^{2+3} M_{23} = - \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -(0 - 6) = 6$$

$$A_{31} = (-1)^{3+1} M_{31} = + \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} = 0 - 2 = -2$$

$$A_{32} = (-1)^{3+2} M_{32} = - \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -(0 + 1) = -1$$

$$A_{33} = (-1)^{3+3} M_{33} = + \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = 2 + 3 = 5$$

$$\text{Now } |A| = a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$$

$$= (1)(-4) + (-1)(6) + (2)(-2)$$

$$= -4 - 6 - 4$$

$$= \boxed{-14} \text{ Ans}$$

Q.2:- Without evaluating state the reasons for the following equalities.

$$(i) \begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 0 \end{vmatrix} = 0 \quad \text{Sol:- } \because C_3 = 0$$

$$(ii) \begin{vmatrix} 1 & 2 & 3 \\ -8 & 4 & -12 \\ 2 & -1 & 3 \end{vmatrix} = 0$$

$$\underline{\text{Sol:}} \begin{vmatrix} 1 & 2 & 3 \\ -8 & 4 & -12 \\ 2 & -1 & 3 \end{vmatrix} \xrightarrow{R} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & -1 & 3 \end{vmatrix} R_2 + 4R_3 = 0 (\because R_2 = 0)$$

$$(iii) \begin{vmatrix} 1 & 3 & -2 \\ 3 & -1 & 1 \\ 2 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ -2 & 1 & 4 \end{vmatrix}$$

$$\underline{\text{Sol:}} \text{ Let } \Delta = \begin{vmatrix} 1 & 3 & -2 \\ 3 & -1 & 1 \\ 2 & 1 & 4 \end{vmatrix} \Rightarrow \Delta^t = \begin{vmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ -2 & 1 & 4 \end{vmatrix}$$

$$(iv) \begin{vmatrix} 3 & 2 & 0 \\ 1 & 1 & -3 \\ 2 & 4 & -6 \end{vmatrix} = -3 \begin{vmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{vmatrix}$$

$$\underline{\text{Sol:}} \text{ Let } \Delta = \begin{vmatrix} 3 & 2 & 0 \\ 1 & 1 & -3 \\ 2 & 4 & -6 \end{vmatrix}$$

$$\text{Taking } (-3) \text{ common from } C_3 \\ \Delta = -3 \begin{vmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{vmatrix}$$

$$(v) \begin{vmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 1 & -1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\underline{\text{Sol:}} \text{ Let } \Delta = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$\xrightarrow{R} \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix} R_{23}$$

$$(vi) \begin{vmatrix} 2 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 5 & 5 & 6 \\ 1 & 2 & 2 \end{vmatrix}$$

$$\underline{\text{Sol:}} \text{ Let } \Delta = \begin{vmatrix} 2 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix} \xrightarrow{R} \begin{vmatrix} 2 & 0 & 1 \\ 5 & 5 & 6 \\ 1 & 2 & 2 \end{vmatrix} R_2 + 2R_3$$

Full

Q.3: Let A be a square matrix of order 3, then verify $|A^t| = |A|$

Sol:- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} (a_{22}a_{33} - a_{32}a_{23}) - a_{12} (a_{21}a_{33} - a_{31}a_{23}) + a_{13} (a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \quad \rightarrow (i)$$

$$|A^t| = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix}$$

$$= a_{11} (a_{22}a_{33} - a_{23}a_{32}) - a_{21} (a_{12}a_{33} - a_{13}a_{32}) + a_{31} (a_{12}a_{23} - a_{13}a_{22})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \quad \rightarrow (ii)$$

From (i) and (ii), we have

$$|A^t| = |A| \quad \text{Proved}$$

Q.4:

Evaluate the following determinants.

Q.4. Evaluate the following determinants.

(i) $\begin{vmatrix} 0 & 1 & 3 \\ -1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix}$

Sol: $\begin{vmatrix} 0 & 1 & 3 \\ -1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}$
 $= 0(2-1) - 1(-1-2) + 3(-1-4)$
 $= 0 - 1(-3) + 3(-5)$
 $= 3 - 15 = -12$ Ans

(ii) $\begin{vmatrix} 3 & 4 & -2 \\ 2 & 4 & -6 \\ -4 & 2 & 0 \end{vmatrix}$

Sol: $\begin{vmatrix} 3 & 4 & -2 \\ 2 & 4 & -6 \\ -4 & 2 & 0 \end{vmatrix} = 3 \begin{vmatrix} 4 & -6 \\ 2 & 0 \end{vmatrix} - 4 \begin{vmatrix} 2 & -6 \\ -4 & 0 \end{vmatrix} - 2 \begin{vmatrix} 2 & 4 \\ -4 & 2 \end{vmatrix}$
 $= 3(0+12) - 4(0-24) - 2(4+16)$
 $= 3(12) - 4(-24) - 2(20)$
 $= 36 + 96 - 40 = 92$ Ans

(iii) $\begin{vmatrix} 3 & 1 & 2 \\ 6 & -5 & 4 \\ -9 & 8 & -7 \end{vmatrix}$

Sol: $\begin{vmatrix} 3 & 1 & 2 \\ 6 & -5 & 4 \\ -9 & 8 & -7 \end{vmatrix} = 3 \begin{vmatrix} -5 & 4 \\ 8 & -7 \end{vmatrix} - 1 \begin{vmatrix} 6 & 4 \\ -9 & -7 \end{vmatrix} + 2 \begin{vmatrix} 6 & -5 \\ -9 & 8 \end{vmatrix}$
 $= 3(35-32) - 1(-42+36) + 2(48-45)$
 $= 3(3) - 1(-6) + 2(3)$
 $= 9 + 6 + 6 = 21$ Ans

(iv) $\begin{vmatrix} 2 & 1 & -3 \\ 1 & 1 & 0 \\ -2 & 3 & 4 \end{vmatrix}$

Sol: $\begin{vmatrix} 2 & 1 & -3 \\ 1 & 1 & 0 \\ -2 & 3 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 \\ -2 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ -2 & 4 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix}$

$$= 2(4-0) - 1(4+0) - 3(3+2)$$

$$= 2(4) - 1(4) - 3(5)$$

$$= 8 - 4 - 15 = -11 \text{ Ans}$$

Q.5 Show that

✓ (i) $\begin{vmatrix} a & b & c \\ l & m & n \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & l & x \\ b & m & y \\ c & n & z \end{vmatrix}$

Sol: Let $\Delta = \begin{vmatrix} a & b & c \\ l & m & n \\ x & y & z \end{vmatrix}$

Taking transpose

$$\Delta = \begin{vmatrix} a & l & x \\ b & m & y \\ c & n & z \end{vmatrix} \text{ Proved}$$

✓ (ii) $\begin{vmatrix} a & b & c \\ 1-3a & 2-3b & 3-3c \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$

Sol: Let $\Delta = \begin{vmatrix} a & b & c \\ 1-3a & 2-3b & 3-3c \\ 4 & 5 & 6 \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ 1-3a+3a & 2-3b+3b & 3-3c+3c \\ 4 & 5 & 6 \end{vmatrix} \quad R_2 + 3R_1$$

$$= \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

Proved

✓ (iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} = 0$

Sol: Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{vmatrix} \quad R_3 \rightarrow R_3 + R_2 + R_1$$

Taking $(a+b+c)$ common from R_3

$$\Delta = (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (a+b+c)(0) \quad (\because R_1 = R_3)$$

$$= 0 \text{ Proved}$$

$$\checkmark (iv) \begin{vmatrix} bc & ca & ab \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Sol:

$$\text{Let } \Delta = \begin{vmatrix} bc & ca & ab \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$\times$ and $\div C_1$ by a , C_2 by b , C_3 by c .

$$\Delta = \frac{1}{abc} \begin{vmatrix} abc & abc & abc \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Taking (abc) common from R_1

$$\Delta = \frac{abc}{abc} \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Proved

Q.6

Prove that

$$\checkmark (i) \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-a \end{vmatrix} = 0$$

Sol:-

$$\text{Let } \Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-a \end{vmatrix}$$

$$= \begin{vmatrix} a-b+b-c+c-a & b-c+c-a+a-b & c-a+a-b+b-a \\ b-c & c-a & a-b \\ c-a & a-b & b-a \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-a \end{vmatrix}$$

$$= 0 \quad (\because R_1 = 0)$$

$$* (ii) \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Sol:- Let $\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Expanding by C_1

$$\Delta = 1 \begin{vmatrix} b-a & b^3-a^3 \\ c-a & c^3-a^3 \end{vmatrix} - 0 + 0$$

$$= \begin{vmatrix} b-a & (b-a)(b^2+ab+a^2) \\ c-a & (c-a)(c^2+ac+a^2) \end{vmatrix}$$

Taking common $(b-a)$ from R_1 , $(c-a)$ from R_2

$$\Delta = (b-a)(c-a) \begin{vmatrix} 1 & b^2+ab+a^2 \\ 1 & c^2+ac+a^2 \end{vmatrix}$$

$$= (b-a)(c-a) [(c^2+ac+a^2) - (b^2+ab+a^2)]$$

$$\begin{aligned}
 \Delta &= (b-a)(c-a)[c^2 + ac + a^2 - b^2 - ab - a^2] \\
 &= (b-a)(c-a)[c^2 - b^2 + ac - ab] \\
 &= (b-a)(c-a)[(c-b)(c+b) + a(c-b)] \\
 &= (b-a)(c-a)[(c-b)(c+b+a)] \\
 &= (b-a)(c-a)(c-b)(a+b+c) \\
 &= (b-a)(c-b)(c-a)(a+b+c) \\
 &= \{-(a-b)\} \{-(b-c)\} (c-a)(a+b+c) \\
 \Delta &= (a-b)(b-c)(c-a)(a+b+c) \quad \text{Proved}
 \end{aligned}$$

$$\checkmark \text{(iii)} \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Sol:- Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Expanding by C_1

$$\Delta = 1 \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b-a) & (b-a)(b+a) \\ (c-a) & (c-a)(c+a) \end{vmatrix}$$

Taking Common $(b-a)$ from R_1 , $(c-a)$ from R_2

$$\begin{aligned}
 \Delta &= (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} \\
 &= (b-a)(c-a)[(c+a)-(b+a)] \\
 &= (b-a)(c-a)[c+a-b-a] \\
 &= (b-a)(c-a)(c-b) \\
 &= (b-a)(c-b)(c-a) \\
 &= \{-(a-b)\} \{-(b-c)\} (c-a) \\
 \Delta &= (a-b)(b-c)(c-a) \quad \text{Proved}
 \end{aligned}$$

$$(iv) \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Sol:-

$$\text{Let } \Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

Taking common a from R_1 , b from R_2 , c from R_3

$$\Delta = abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Taking common a from C_1 , b from C_2 , c from C_3

$$\Delta = a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} \begin{matrix} R_2+R_1 \\ R_3+R_1 \end{matrix}$$

Expanding by C_1

$$\Delta = a^2 b^2 c^2 \begin{bmatrix} -1 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= a^2 b^2 c^2 [-1(0-4)]$$

$$= a^2 b^2 c^2 (4)$$

$$= 4 a^2 b^2 c^2 \text{ Proved}$$

✓ (v) $\begin{vmatrix} bc & a^3 & \frac{1}{a} \\ ca & b^3 & \frac{1}{b} \\ ab & c^3 & \frac{1}{c} \end{vmatrix} = 0, \quad a \neq 0, b \neq 0, c \neq 0$

Sol:-

$$\text{Let } \Delta = \begin{vmatrix} bc & a^3 & \frac{1}{a} \\ ca & b^3 & \frac{1}{b} \\ ab & c^3 & \frac{1}{c} \end{vmatrix}$$

× and ÷ C_3 by (abc)

$$\Delta = \frac{1}{abc} \begin{vmatrix} bc & a^3 & \frac{abc}{a} \\ ca & b^3 & \frac{abc}{b} \\ ab & c^3 & \frac{abc}{c} \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} bc & a^3 & bc \\ ca & b^3 & ca \\ ab & c^3 & ab \end{vmatrix}$$

$$= \frac{1}{abc} (0) \quad (\because C_1 = C_3)$$

$$\Delta = 0 \text{ Proved}$$

Q.7: Evaluate

(i) $\begin{vmatrix} 3860 & 3861 \\ 3862 & 3863 \end{vmatrix}$

Sol:- Let $\Delta = \begin{vmatrix} 3860 & 3861 \\ 3862 & 3863 \end{vmatrix}$
 $C_2 - C_1$

$$\Delta = \begin{vmatrix} 3860 & 1 \\ 3862 & 1 \end{vmatrix}$$

$$= 3860 - 3862$$

$$= -2$$

✓ (ii) $\begin{vmatrix} 81 & 82 & 83 \\ 84 & 85 & 86 \\ 87 & 88 & 89 \end{vmatrix}$

Sol:- Let $\Delta = \begin{vmatrix} 81 & 82 & 83 \\ 84 & 85 & 86 \\ 87 & 88 & 89 \end{vmatrix}$

$$= \begin{vmatrix} 81 & 82 & 83 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= 0 \quad (\because R_2 = R_3)$$

Proved

Full

Q.8 Prove that $\begin{vmatrix} 1+x & y & z \\ x & 1+y & z \\ x & y & 1+z \end{vmatrix} = 1+x+y+z$

Sol.:- Let $\Delta = \begin{vmatrix} 1+x & y & z \\ x & 1+y & z \\ x & y & 1+z \end{vmatrix}$

$$C_1 + C_2 + C_3$$

$$\Delta = \begin{vmatrix} 1+x+y+z & y & z \\ 1+x+y+z & 1+y & z \\ 1+x+y+z & y & 1+z \end{vmatrix}$$

Taking common $(1+x+y+z)$ from C_1

$$\Delta = (1+x+y+z) \begin{vmatrix} 1 & y & z \\ 1 & 1+y & z \\ 1 & y & 1+z \end{vmatrix}$$

$$= (1+x+y+z) \begin{vmatrix} 1 & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= (1+x+y+z) (1)$$

$$\Delta = 1+x+y+z \text{ Proved.}$$

Q.9 Prove that $\begin{vmatrix} x & p & q \\ p & x & q \\ p & q & x \end{vmatrix} = (x-p)(x-q)(x+p+q)$

Sol:- Let $\Delta = \begin{vmatrix} x & p & q \\ p & x & q \\ p & q & x \end{vmatrix}$

$$= \begin{vmatrix} x+p+q & p & q \\ x+p+q & x & q \\ x+p+q & q & x \end{vmatrix} \quad \begin{matrix} C_1 + C_2 + C_3 \end{matrix}$$

Taking common $(x+p+q)$ from C_1

$$\Delta = (x+p+q) \begin{vmatrix} 1 & p & q \\ 1 & x & q \\ 1 & q & x \end{vmatrix}$$

$$= (x+p+q) \begin{vmatrix} 1 & p & q \\ 0 & x-p & 0 \\ 0 & q-p & x-q \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Expanding by C_1

$$\Delta = (x+p+q) \begin{bmatrix} 1 & x-p & 0 \\ 0 & q-p & x-q \end{bmatrix}$$

$$= (x+p+q) [(x-p)(x-q) - 0]$$

$$= (x+p+q)(x-p)(x-q)$$

$$\Delta = (x-p)(x-q)(x+p+q) \text{ Proved}$$

Full

Q.10: Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c})$

Sol:- Let $\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$
 $\div$ and $\times$ C_1 by a , C_2 by b , C_3 by c

$$\Delta = (abc) \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b}+1 & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$$

$C_1 + C_2 + C_3$

$$\Delta = (abc) \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b}+1 & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$$

Taking $(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c})$ common from C_1 ,

$$\Delta = (abc) \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 1 & \frac{1}{b}+1 & \frac{1}{c} \\ 1 & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$$

$$= (abc) \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} R_2-R_1 \\ R_3-R_1 \end{matrix}$$

$$= (abc) \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) (1)$$

$$\Delta = (abc) \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \quad \text{Proved}$$

!! Identify singular and non-singular matrices

(i) $\begin{bmatrix} 7 & 1 & 3 \\ 6 & 2 & -2 \\ 5 & 1 & 1 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 7 & 1 & 3 \\ 6 & 2 & -2 \\ 5 & 1 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 7 & 1 & 3 \\ 6 & 2 & -2 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= 7 \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 6 & -2 \\ 5 & 1 \end{vmatrix} + 3 \begin{vmatrix} 6 & 2 \\ 5 & 1 \end{vmatrix}$$

$$= 7(2+2) - 1(6+10) + 3(6-10)$$

$$= 7(4) - 1(16) + 3(-4)$$

$$= 28 - 16 - 12 = 0 \Rightarrow A \text{ is singular. } \text{Ans}$$

(ii) $\begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 1 \\ -2 & -3 & 2 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 1 \\ -2 & -3 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 3 & -2 & 1 \\ -2 & -3 & 2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & -2 \\ -2 & -3 \end{vmatrix}$$

$$= 1(-4+3) + 1(6+2) + 1(-9-4)$$

$$= 1(-1) + 1(8) + 1(-13)$$

$$= -1 + 8 - 13 \neq 0 \Rightarrow A \text{ is non-singular. } \text{Ans}$$

(iii) $\begin{bmatrix} 3 & 2 & -3 \\ 3 & 6 & -3 \\ -1 & 0 & 1 \end{bmatrix}$

Sol:- Let $A = \begin{bmatrix} 3 & 2 & -3 \\ 3 & 6 & -3 \\ -1 & 0 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 2 & -3 \\ 3 & 6 & -3 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 6 & -3 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & -3 \\ -1 & 1 \end{vmatrix} - 3 \begin{vmatrix} 3 & 6 \\ -1 & 0 \end{vmatrix}$$

$$= 3(6+0) - 2(3-3) - 3(0+6)$$

$$= 3(6) - 2(0) - 3(6)$$

$$= 18 - 0 - 18 = 0 \Rightarrow A \text{ is singular. Ans}$$

Full

Q.12 Find the value of λ if A is singular where $A = \begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix}$

Sol:- $|A| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & -\lambda \end{vmatrix} + 0 \begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 1) - 1(-\lambda - 0) = -\lambda^3 + \lambda + \lambda = 2\lambda - \lambda^3$$

Since A is singular

$$\Rightarrow |A| = 0 \Rightarrow 2\lambda - \lambda^3 = 0 \Rightarrow \lambda(2 - \lambda^2) = 0$$

$$\Rightarrow \lambda = 0 \quad \left| \begin{array}{l} 2 - \lambda^2 = 0 \\ \lambda^2 = 2 \\ \lambda = \pm\sqrt{2} \end{array} \right.$$

Thus $\lambda = 0, \pm\sqrt{2}$ Ans

✓ Q.13 Solve for x .

$$(i) \begin{vmatrix} x & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 4 & 5 \end{vmatrix} = 9$$

Sol:- $\begin{vmatrix} x & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 4 & 5 \end{vmatrix} = 9$

$$\Rightarrow x \begin{vmatrix} -1 & 1 \\ 4 & 5 \end{vmatrix} = 9 \Rightarrow x(-5-4) = 9 \Rightarrow -9x = 9 \Rightarrow \boxed{x = -1}$$

Ans

$$(ii) \begin{vmatrix} -1 & 0 & 1 \\ x^2 & 1 & x \\ 2 & 3 & 4 \end{vmatrix} = -6$$

Sol:- $\begin{vmatrix} -1 & 0 & 1 \\ x^2 & 1 & x \\ 2 & 3 & 4 \end{vmatrix} = -6$

$$-1 \begin{vmatrix} 1 & x \\ 3 & 4 \end{vmatrix} + 0 + 1 \begin{vmatrix} x^2 & 1 \\ 2 & 3 \end{vmatrix} = -6$$

$$-1(4-3x) + 0 + 1(3x^2-2) = -6$$

$$-4 + 3x + 3x^2 - 2 = -6$$

$$3x^2 + 3x = 0 \Rightarrow x^2 + x = 0$$

$$\Rightarrow x(x+1) = 0$$

$$\Rightarrow x = 0 \quad \left| \begin{array}{l} x+1 = 0 \\ \Rightarrow x = -1 \end{array} \right.$$

Hence $\boxed{x = 0, -1}$ Ans

Q.14: Show that if inverse of a square matrix exists, then it is unique.

Sol: Let A be a square matrix.

Let B and C are inverses of A

Then by definition,

$$AB = BA = I \longrightarrow (1)$$

$$AC = CA = I \longrightarrow (2)$$

Now $B = IB$ (by definition of identity matrix)

$$= CAB \quad (\text{from (2)})$$

$$= CI \quad (\text{from (1)})$$

$$B = C \quad (\text{by definition of identity matrix})$$

Thus the inverse is unique. (Proved)

Q.15:

Let $A = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 3 & 2 \\ 1 & 0 & 5 \end{bmatrix}$. Find A^{-1}

Sol:- $A = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 3 & 2 \\ 1 & 0 & 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 0 & 2 & 2 \\ -1 & 3 & 2 \\ 1 & 0 & 5 \end{vmatrix}$$

$$= 0 \begin{vmatrix} 3 & 2 \\ 0 & 5 \end{vmatrix} - 2 \begin{vmatrix} -1 & 2 \\ 1 & 5 \end{vmatrix} + 2 \begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix}$$

$$= 0 - 2(-5-2) + 2(-3)$$

$$= -2(-7) + 2(-3) = 14 - 6 = 8 \neq 0 \Rightarrow A^{-1} \text{ exists}$$

For $\text{adj } A$ $A_{ij} = (-1)^{i+j} M_{ji}$

$$A_{11} = (-1)^{1+1} M_{11} = + \begin{vmatrix} 3 & 2 \\ 0 & 5 \end{vmatrix} = 15 - 0 = 15$$

$$\therefore A_{12} = (-1)^{1+2} M_{12} = - \begin{vmatrix} -1 & 2 \\ 1 & 5 \end{vmatrix} = -(-5-2) = -(-7) = 7$$

$$A_{13} = (-1)^{1+3} M_{13} = + \begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} = +(0-3) = -3$$

$$A_{21} = (-1)^{2+1} M_{21} = - \begin{vmatrix} 2 & 2 \\ 0 & 5 \end{vmatrix} = -(10-0) = -10$$

$$A_{22} = (-1)^{2+2} M_{22} = + \begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 0-2 = -2$$

$$A_{23} = (-1)^{2+3} M_{23} = - \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_{31} = (-1)^{3+1} M_{31} = + \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = 4-6 = -2$$

$$A_{32} = (-1)^{3+2} M_{32} = - \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} = -(0+2) = -2$$

$$A_{33} = (-1)^{3+3} M_{33} = + \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} = 0+2 = 2$$

$$\text{adj } A = \begin{bmatrix} 15 & -10 & -2 \\ 7 & -2 & -2 \\ -3 & 2 & 2 \end{bmatrix}$$

$$\bar{A} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{8} \begin{bmatrix} 15 & -10 & -2 \\ 7 & -2 & -2 \\ -3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 15/8 & -5/4 & -1/4 \\ 7/8 & -1/4 & -1/4 \\ -3/8 & 1/4 & 1/4 \end{bmatrix} \text{ Ans}$$

Full

Q.16

Let $A = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$. Show that $|\bar{A}| = \frac{1}{|A|}$

Sol:- $A = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} = 6 + 4 = 10 \neq 0 \Rightarrow A^{-1} \text{ exists}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{10} \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} \\ -\frac{2}{5} & \frac{3}{10} \end{bmatrix}$$

$$|A^{-1}| = \begin{vmatrix} \frac{1}{5} & \frac{1}{10} \\ -\frac{2}{5} & \frac{3}{10} \end{vmatrix} = \left(\frac{1}{5}\right)\left(\frac{3}{10}\right) - \left(-\frac{2}{5}\right)\left(\frac{1}{10}\right) = \frac{3}{50} + \frac{2}{50}$$

$$= \frac{3+2}{50} = \frac{5}{50} = \frac{1}{10} = \frac{1}{|A|}$$

Thus $|A^{-1}| = \frac{1}{|A|}$ Proved

Full

Q.17.

Verify that $(AB)^{-1} = B^{-1}A^{-1}$ if $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$

Sol:- $AB = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2+6 & 2+9 \\ -1+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ -1 & 1 \end{bmatrix}$

$$|AB| = \begin{vmatrix} 4 & 11 \\ -1 & 1 \end{vmatrix} = 4 + 11 = 15 \neq 0 \Rightarrow (AB)^{-1} \text{ exists}$$

$$(AB)^{-1} = \frac{1}{|AB|} \begin{bmatrix} 1 & -11 \\ 1 & 4 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 1 & -11 \\ 1 & 4 \end{bmatrix} \rightarrow (1)$$

$$B^{-1}A^{-1} = \frac{1}{|B|} \begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix} \times \frac{1}{|A|} \begin{bmatrix} 0 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{-5} \begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix} \times \frac{1}{-3} \begin{bmatrix} 0 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{15} \begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 0+1 & -9-2 \\ 0+1 & 6-2 \end{bmatrix}$$

$$= \frac{1}{15} \begin{bmatrix} 1 & -11 \\ 1 & 4 \end{bmatrix} \rightarrow (2)$$

From (1) and (2), we get $(AB)^{-1} = B^{-1}A^{-1}$ Proved

Q.18 If A and B are non-singular matrices, then show that

(i) $(\bar{A}')^T = A$

Sol:- Since A is non-singular, so $\bar{A}'$ exists such that

$$A\bar{A}' = \bar{A}'A = I$$

$\Rightarrow A$ is the inverse of $\bar{A}'$.

But the inverse of $\bar{A}'$ is $(\bar{A}')^{-1}$

$$\therefore (\bar{A}')^{-1} = A \quad (\text{by the uniqueness of inverses})$$

Proved.

(ii) $(AB)^T = \bar{B}'\bar{A}'$

Sol:- Since A and B are non-singular, so $\bar{A}'$ and $\bar{B}'$ exist such that

$$A\bar{A}' = \bar{A}'A = I \longrightarrow (1)$$

$$B\bar{B}' = \bar{B}'B = I \longrightarrow (2)$$

Consider $(AB)(\bar{B}'\bar{A}') = A(B\bar{B}')\bar{A}'$ (by associative law)

$$= A I \bar{A}' \quad (\text{by (2)})$$

$$= A\bar{A}' \quad (\text{by the definition of identity matrix})$$

$$= I \longrightarrow (3) \quad (\text{by (1)})$$

Again consider $(\bar{B}'\bar{A}')(AB) = \bar{B}'(\bar{A}'A)B$ (by associative law)

$$= \bar{B}' I B \quad (\text{by (1)})$$

$$= \bar{B}' B \quad (\text{by the definition of identity matrix})$$

$$= I \longrightarrow (4) \quad (\text{by (2)})$$

From (3) and (4), we have

$$(AB)(\bar{B}^1\bar{A}^1) = (\bar{B}^1\bar{A}^1)(AB) = I$$

$\Rightarrow \bar{B}^1\bar{A}^1$ is the inverse of AB .

But the inverse of AB is $(AB)^{-1}$

Hence $(AB)^{-1} = \bar{B}^1\bar{A}^1$ (by the uniqueness of inverses)
Proved

Q.19

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$. Verify $(\bar{A}^1)^t = (A^t)^{-1}$

Sol:- $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

$$\begin{aligned} \bar{A}^1 &= \frac{1}{|A|} \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} = \frac{1}{2+3} \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \end{aligned}$$

$$(\bar{A}^1)^t = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} \longrightarrow (1)$$

$$A^t = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

$$\begin{aligned} (A^t)^{-1} &= \frac{1}{|A^t|} \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} = \frac{1}{2+3} \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} \longrightarrow (2) \end{aligned}$$

From (1) and (2), we have

$$(\bar{A}^1)^t = (A^t)^{-1} \quad \text{Proved}$$

EXERCISE 2.3

Q.1: Reduce each of the following matrices to the indicated form.

(i) $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 3 & 4 & -5 \end{bmatrix}$ Echelon form

Sol:- $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 3 & 4 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 6 \\ 0 & -5 & -2 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & -8 \end{bmatrix} R_3 - R_2$

(ii) $\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ Reduced Echelon form

Sol:- $\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix} \begin{matrix} \\ R_{12} \\ \end{matrix}$

$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 4 & -3 & 11 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$

$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ R_3 - 4R_2 \\ R_1 - R_3 \\ R_2 + R_3 \end{matrix}$

(iii) $\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 2 \\ 4 & 1 & 7 \end{bmatrix}$ Reduced echelon form

Sol:- $\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 2 \\ 4 & 1 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & 1 \\ 4 & 1 & 7 \end{bmatrix} R_{12}$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -3 \\ 0 & -3 & -1 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -3 & -1 \end{bmatrix} R_2 - 2R_3$

$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -4 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ R_3 + 3R_2 \end{matrix}$

$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} -\frac{1}{4} R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 3R_3 \\ R_2 + R_3 \end{matrix}$

(iv) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$ Echelon form.

Sol:- $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 2 & 9 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} R_3 - 2R_2$

which is the required echelon form.

Q.2 Find the inverses of the following matrices by using elementary row operations.

(i)

$$\begin{bmatrix} 4 & -2 & 5 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

Sol:

Let $A = \begin{bmatrix} 4 & -2 & 5 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 4 & -2 & 5 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{R} \left[\begin{array}{ccc|ccc} -1 & 2 & 3 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 1 & 0 & 0 \end{array} \right] R_{13}$$

$$\xrightarrow{R} \left[\begin{array}{ccc|ccc} -1 & 2 & 3 & 0 & 0 & 1 \\ 0 & 5 & 6 & 0 & 1 & 2 \\ 0 & 6 & 17 & 1 & 0 & 4 \end{array} \right] \begin{array}{l} R_2 + 2R_1 \\ R_3 + 4R_1 \end{array}$$

$$\xrightarrow{R} \left[\begin{array}{ccc|ccc} -1 & 2 & 3 & 0 & 0 & 1 \\ 0 & -1 & -11 & -1 & 1 & -2 \\ 0 & 6 & 17 & 1 & 0 & 4 \end{array} \right] R_2 - R_1$$

$$\xrightarrow{R} \left[\begin{array}{ccc|ccc} -1 & 0 & -19 & -2 & 2 & -3 \\ 0 & -1 & -11 & -1 & 1 & -2 \\ 0 & 0 & -49 & -5 & 6 & -8 \end{array} \right] \begin{array}{l} R_1 + 2R_2 \\ R_3 + 6R_2 \end{array}$$

$$\xrightarrow{R} \left[\begin{array}{ccc|ccc} -1 & 0 & -19 & -2 & 2 & -3 \\ 0 & -1 & -11 & -1 & 1 & -2 \\ 0 & 1 & \frac{5}{49} & \frac{-6}{49} & \frac{8}{49} & -\frac{1}{49} \end{array} \right] -\frac{1}{49}R_3$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -3/49 & -16/49 & 5/49 \\ 0 & -1 & 0 & 6/49 & -17/49 & -10/49 \\ 0 & 0 & 1 & 5/49 & -6/49 & 8/49 \end{array} \right] \begin{array}{l} R_1 + 19R_3 \\ R_2 + 11R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/49 & 16/49 & -5/49 \\ 0 & 1 & 0 & -6/49 & 17/49 & 10/49 \\ 0 & 0 & 1 & 5/49 & -6/49 & 8/49 \end{array} \right] \begin{array}{l} (-1)R_1 \\ (-1)R_2 \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} 3/49 & 16/49 & -5/49 \\ -6/49 & 17/49 & 10/49 \\ 5/49 & -6/49 & 8/49 \end{bmatrix} \text{ Ans}$$

(ii) $\begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & 4 \\ -1 & 5 & 1 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & 4 \\ -1 & 5 & 1 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & -1 & 6 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ -1 & 5 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 1 & 0 \\ 3 & -1 & 6 & 1 & 0 & 0 \\ -1 & 5 & 1 & 0 & 0 & 1 \end{array} \right] R_{12}$$

$$R \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & -10 & -6 & 1 & 0 & 0 \\ 0 & 8 & 5 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ R_2 - 3R_1 \\ R_3 + R_1 \end{array}$$

$$R \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 3/5 & 1/10 & 3/10 & 1 \\ 0 & 8 & 5 & 0 & 1 & 0 \end{array} \right] \begin{array}{l} \\ \\ -\frac{1}{10}R_2 \end{array}$$

$$R \left[\begin{array}{ccc|ccc} 1 & 0 & 11/5 & 3/10 & 3/10 & 1 \\ 0 & 1 & 3/5 & -1/10 & 1/10 & 0 \\ 0 & 0 & 1/5 & 4/5 & -7/5 & 0 \end{array} \right] \begin{array}{l} \\ \\ R_1 - 3R_2 \\ R_3 - 8R_2 \end{array}$$

$$R \left[\begin{array}{ccc|ccc} 1 & 0 & 11/5 & 3/10 & 1/10 & 0 \\ 0 & 1 & 3/5 & -1/10 & 3/10 & 0 \\ 0 & 0 & 1 & 4 & -7 & 5 \end{array} \right] \begin{array}{l} \\ \\ 5R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17/2 & 3/2 & -11 \\ 0 & 1 & 0 & -5/2 & 9/2 & -3 \\ 0 & 0 & 1 & 4 & -7 & 5 \end{array} \right] \begin{array}{l} \\ \\ R_1 - 11/5R_3 \\ R_2 - 3/5R_3 \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} -17/2 & 3/2 & -11 \\ -5/2 & 9/2 & -3 \\ 4 & -7 & 5 \end{bmatrix} \quad \underline{\underline{\text{Ans}}}$$

✓ (ii)

$$\left[\begin{array}{ccc} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{array} \right] I$$

Sol: Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -2 & -2 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -4 & 2 & 0 & 1 \end{array} \right] R_3 + 2R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 2 & 1 & 1 \end{array} \right] \begin{array}{l} R_1 + R_2 \\ R_3 + R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array} \right] \begin{array}{l} -\frac{1}{2}R_2 \\ -\frac{1}{4}R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array} \right] R_1 + 3R_2$$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

(iv) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 0 & -2 & 3 & -1 & 0 & 1 \end{array} \right] R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 0 & 0 & -3 & -1 & -2 & 1 \end{array} \right] \begin{array}{l} R_1 + 2R_2 \\ R_3 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array} \right] \begin{array}{l} (-1)R_2 \\ -\frac{1}{3}R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array} \right] \begin{array}{l} R_1 - 5R_3 \\ R_2 + 3R_3 \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Q.3: Find the rank of each of the following matrices.

(i) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$

$$R_1 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 5 \\ 0 & 2 & 1 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix}$$

$$R_1 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 5 \\ 0 & 0 & -4 \end{bmatrix} R_3 - R_2$$

$$\Rightarrow \boxed{\text{Rank } A = 3} \text{ Ans}$$

✓ (ii) $\begin{bmatrix} 3 & 1 & -4 \\ 0 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 0 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix}$

$$R_1 \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix} R_{13}$$

$$R_1 \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} R_3 - 3R_1$$

$$R_1 \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_3 - 2R_2$$

$$\Rightarrow \boxed{\text{Rank } A = 2} \text{ Ans}$$

Q.4: Find the rank of matrix $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

$$R_1 \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -2 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_2 \\ R_4 - 2R_3 \end{matrix}$$

$$R \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -2 \end{bmatrix} \begin{array}{l} R_1 - R_4 \\ R_3 - R_2 \end{array}$$

$$R \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -2 & -4 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & -9 \end{bmatrix} \begin{array}{l} R_2 - R_1 \\ R_4 - R_1 \end{array}$$

$$R \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{array}{l} (-\frac{1}{2}R_2) \\ (-\frac{1}{3})R_4 \end{array}$$

$$R \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 - R_2$$

$$\Rightarrow \boxed{\text{Rank } A = 2} \text{ Ans}$$

EXERCISE 2.4

Q.1: Solve the following systems of equations by matrix inversion method.

$$(i) \quad 4x - 3y + z = 11$$

$$2x + y - 4z = -1$$

$$x + 2y - 2z = 1$$

Sol: The matrix form is

$$\begin{bmatrix} 4 & -3 & 1 \\ 2 & 1 & -4 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \\ 1 \end{bmatrix}$$

A X B

i.e. $AX = B$

$$|A| = \begin{vmatrix} 4 & -3 & 1 \\ 2 & 1 & -4 \\ 1 & 2 & -2 \end{vmatrix} = 4 \begin{vmatrix} 1 & -4 \\ 2 & -2 \end{vmatrix} + 3 \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 4(-2+8) + 3(-4+4) + 1(4-1)$$

$$= 4(6) + 3(0) + 1(3)$$

$$= 24 + 0 + 3 = 27 \neq 0$$

$\therefore$ solution exists

$$\therefore X = A^{-1}B \rightarrow (1)$$

for adj A

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^{1+1} M_{11} = + \begin{vmatrix} 1 & -4 \\ 2 & -2 \end{vmatrix} = -2+8=6$$

$$A_{12} = (-1)^{1+2} M_{12} = - \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = -(-4+4) = 0$$

$$A_{13} = (-1)^{1+3} M_{13} = + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$A_{21} = (-1)^{2+1} M_{21} = - \begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} = -(6 - 2) = -4$$

$$A_{22} = (-1)^{2+2} M_{22} = + \begin{vmatrix} 4 & 1 \\ 1 & -2 \end{vmatrix} = -8 - 1 = -9$$

$$A_{23} = (-1)^{2+3} M_{23} = - \begin{vmatrix} 4 & -3 \\ 1 & 2 \end{vmatrix} = -(8 + 3) = -11$$

$$A_{31} = (-1)^{3+1} M_{31} = + \begin{vmatrix} -3 & 1 \\ 1 & -4 \end{vmatrix} = 12 - 1 = 11$$

$$A_{32} = (-1)^{3+2} M_{32} = - \begin{vmatrix} 4 & 1 \\ 2 & -4 \end{vmatrix} = -(-16 - 2) = 18$$

$$A_{33} = (-1)^{3+3} M_{33} = + \begin{vmatrix} 4 & -3 \\ 2 & 1 \end{vmatrix} = 4 + 6 = 10$$

$$\therefore \text{adj } A = \begin{bmatrix} 6 & -4 & 11 \\ 0 & -9 & 18 \\ 3 & -11 & 10 \end{bmatrix}$$

$$\bar{A} = \frac{1}{|A|} \text{adj } A = \frac{1}{27} \begin{bmatrix} 6 & -4 & 11 \\ 0 & -9 & 18 \\ 3 & -11 & 10 \end{bmatrix}$$

$$\textcircled{1} \Rightarrow X = \frac{1}{27} \begin{bmatrix} 6 & -4 & 11 \\ 0 & -9 & 18 \\ 3 & -11 & 10 \end{bmatrix} \begin{bmatrix} 11 \\ -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{27} \begin{bmatrix} 6+4+11 \\ 0+9+18 \\ 33+11+10 \end{bmatrix} = \frac{1}{27} \begin{bmatrix} 21 \\ 27 \\ 54 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \boxed{x=3, y=1, z=2} \text{ Ans}$$

$$\begin{aligned} \text{(ii)} \quad x + y + z &= 1 \\ x + y - 2z &= 3 \\ 2x + y + z &= 2 \end{aligned}$$

Sol:- The matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

A X B

i.e. $AX = B$

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ &= 1(1+2) - 1(1+4) + 1(1-2) \\ &= 1(3) - 1(5) + 1(-1) \\ &= 3 - 5 - 1 = -3 \neq 0 \\ &\therefore A^{-1} \text{ exists} \end{aligned}$$

$$\therefore X = A^{-1}B \longrightarrow (1)$$

For $\text{adj } A$

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^{1+1} M_{11} = + \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = 1+2=3$$

$$A_{12} = (-1)^{1+2} M_{12} = - \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = -(1+4) = -5$$

$$A_{13} = (-1)^{1+3} M_{13} = + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1-2 = -1$$

$$A_{21} = (-1)^{2+1} M_{21} = - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -(1-1) = 0$$

$$A_{22} = (-1)^{2+2} M_{22} = + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$$

$$A_{23} = (-1)^{2+3} M_{23} = - \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = - (1 - 2) = 1$$

$$A_{31} = (-1)^{3+1} M_{31} = + \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -2 - 1 = -3$$

$$A_{32} = (-1)^{3+2} M_{32} = - \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = - (-2 - 1) = 3$$

$$A_{33} = (-1)^{3+3} M_{33} = + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$$

$$\therefore \text{adj } A = \begin{bmatrix} 3 & 0 & -3 \\ -5 & -1 & 3 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\bar{A} = \frac{1}{|A|} \text{adj } A = \frac{1}{-3} \begin{bmatrix} 3 & 0 & -3 \\ -5 & -1 & 3 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\textcircled{1} \Rightarrow X = \frac{1}{-3} \begin{bmatrix} 3 & 0 & -3 \\ -5 & -1 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} 3+0-6 \\ -5-3+6 \\ -1+3+0 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$\text{i.e. } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2/3 \\ -2/3 \end{bmatrix} \Rightarrow \boxed{x=1, y=\frac{2}{3}, z=-\frac{2}{3}} \text{ Ans}$$

Q.2 Solve the following system of equations by the Gauss elimination method and Gauss-Jordan method.

✓ (i) $x - y + 4z = 4$
 $2x + 2y - z = 2$
 $3x - 2y + 3z = -3$

Sol:

Here $A_b = \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 2 & 2 & -1 & 2 \\ 3 & -2 & 3 & -3 \end{array} \right]$

Gauss Elimination Method

$$R \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 0 & 4 & -9 & -6 \\ 0 & 1 & -9 & -15 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$R \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 0 & 1 & -9 & -15 \\ 0 & 4 & -9 & -6 \end{array} \right] R_{23}$$

$$R \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 0 & 1 & -9 & -15 \\ 0 & 0 & 27 & 54 \end{array} \right] R_3 - 4R_2$$

$$(R_3) \Rightarrow 27z = 54$$

$$\boxed{z = 2}$$

$$(R_2) \Rightarrow y - 9z = -15$$

$$y - 9(2) = -15$$

$$y - 18 = -15$$

$$y = -15 + 18 \Rightarrow \boxed{y = 3}$$

$$(R_1) \Rightarrow x - y + 4z = 4$$

$$x - 3 + 4(2) = 4$$

$$x = 3 - 8 + 4 \Rightarrow \boxed{x = -1}$$

$\therefore$ the solution is $x = -1, y = 3, z = 2$

Gauss-Jordan Method

$$R \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 0 & 4 & -9 & -6 \\ 0 & 1 & -9 & -15 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$R \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 0 & 1 & -9 & -15 \\ 0 & 4 & -9 & -6 \end{array} \right] R_{23}$$

$$R \left[\begin{array}{ccc|c} 1 & 0 & -5 & -11 \\ 0 & 1 & -9 & -15 \\ 0 & 0 & 27 & 54 \end{array} \right] \begin{array}{l} R_1 + R_2 \\ R_3 - 4R_2 \end{array}$$

$$R \left[\begin{array}{ccc|c} 1 & 0 & -5 & -11 \\ 0 & 1 & -9 & -15 \\ 0 & 0 & 1 & 2 \end{array} \right] \frac{1}{27} R_3$$

$$R \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 + 5R_3 \\ R_2 + 9R_3 \end{array}$$

$$\therefore x = -1, y = 3, z = 2$$

which is the required solution.

$$\begin{aligned}
 (i) \quad & 2x + 4y - z = 0 \\
 & x - 2y - 2z = 2 \\
 & -5x - 8y + 3z = -2
 \end{aligned}$$

Sol:

Here

$$A_b = \left[\begin{array}{ccc|c} 2 & 4 & -1 & 0 \\ 1 & -2 & -2 & 2 \\ -5 & -8 & 3 & -2 \end{array} \right]$$

Gauss Elimination Method:

$$R_2 \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 2 & 4 & -1 & 0 \\ -5 & -8 & 3 & -2 \end{array} \right] R_{12}$$

$$R_2 \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 8 & 3 & -4 \\ 0 & -18 & -7 & 8 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 + 5R_1 \end{array}$$

$$R_2 \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 1 & 3/8 & -1/2 \\ 0 & -18 & -7 & 8 \end{array} \right] \frac{1}{8} R_2$$

$$R_2 \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 1 & 3/8 & -1/2 \\ 0 & 0 & -1/4 & -1 \end{array} \right] R_3 + 18R_2$$

$$(R_3) \Rightarrow -\frac{1}{4}z = -1 \Rightarrow \boxed{z=4}$$

$$\begin{aligned}
 (R_2) \Rightarrow y + \frac{3}{8}z &= -\frac{1}{2} \\
 y + \frac{3}{8}(4) &= -\frac{1}{2} \Rightarrow \boxed{y=-2}
 \end{aligned}$$

$$\begin{aligned}
 (R_1) \Rightarrow x - 2y - 2z &= 2 \\
 x - 2(-2) - 2(4) &= 2 \\
 \Rightarrow \boxed{x=6}
 \end{aligned}$$

Ans

∴ The solution is $x=6, y=-2, z=4$ Gauss-Jordan Method:

$$R_2 \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 2 & 4 & -1 & 0 \\ -5 & -8 & 3 & -2 \end{array} \right] R_{12}$$

$$R_2 \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 8 & 3 & -4 \\ 0 & -18 & -7 & 8 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 + 5R_1 \end{array}$$

$$R_2 \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 1 & 3/8 & -1/2 \\ 0 & -18 & -7 & 8 \end{array} \right] \frac{1}{8} R_2$$

$$R_2 \left[\begin{array}{ccc|c} 1 & 0 & -5/4 & 1 \\ 0 & 1 & 3/8 & -1/2 \\ 0 & 0 & -1/4 & -1 \end{array} \right] \begin{array}{l} R_1 + 2R_2 \\ R_3 + 18R_2 \end{array}$$

$$R_2 \left[\begin{array}{ccc|c} 1 & 0 & -5/4 & 1 \\ 0 & 1 & 3/8 & -1/2 \\ 0 & 0 & 1 & 4 \end{array} \right] (-4)R_3$$

$$R_2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array} \right] \begin{array}{l} R_1 + 5/4 R_3 \\ R_2 - 3/8 R_3 \end{array}$$

$$\therefore \boxed{x=6, y=-2, z=4} \text{ Ans}$$

is the required solution

Q.3 Use Cramer's rule to solve the following system of equations.

$$\begin{aligned} \text{(i)} \quad & x - 2y = -4 \\ & 3x + y = -5 \\ & 2x + z = -1 \end{aligned}$$

Sol:

Here $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$$D = |A| = \begin{vmatrix} 1 & -2 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 1 + 6 = 7$$

$$D_1 = \begin{vmatrix} -4 & -2 & 0 \\ -5 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} -4 & -2 \\ -5 & 1 \end{vmatrix} = -4 - 10 = -14$$

$$D_2 = \begin{vmatrix} 1 & -4 & 0 \\ 3 & -5 & 0 \\ 2 & -1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -4 \\ 3 & -5 \end{vmatrix} = -5 + 12 = 7$$

$$\begin{aligned} D_3 &= \begin{vmatrix} 1 & -2 & -4 \\ 3 & 1 & -5 \\ 2 & 0 & -1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -5 \\ 0 & -1 \end{vmatrix} + 2 \begin{vmatrix} 3 & -5 \\ 2 & -1 \end{vmatrix} - 4 \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} \\ &= 1(-1+0) + 2(-3+10) - 4(0-2) \\ &= 1(-1) + 2(7) - 4(-2) \\ &= -1 + 14 + 8 = 21 \end{aligned}$$

$$\text{Now } x = \frac{D_1}{D} = \frac{-14}{7} = -2$$

$$y = \frac{D_2}{D} = \frac{7}{7} = 1$$

$$z = \frac{D_3}{D} = \frac{21}{7} = 3$$

Thus the solution is $\boxed{x = -2, y = 1, z = 3}$ Ans

$$\begin{aligned} \text{(ii)} \quad x - y + 2z &= 10 \\ 2x + y - 2z &= -4 \\ 3x + y + z &= 7 \end{aligned}$$

Sol: Here $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -2 \\ 3 & 1 & 1 \end{bmatrix}$

$$D = |A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -2 \\ 3 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= 1(1+2) + 1(2+6) + 2(2-3) = 3+8-2=9$$

$$D_1 = \begin{vmatrix} 10 & -1 & 2 \\ -4 & 1 & -2 \\ 7 & 1 & 1 \end{vmatrix} = 10 \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} -4 & -2 \\ 7 & 1 \end{vmatrix} + 2 \begin{vmatrix} -4 & 1 \\ 7 & 1 \end{vmatrix}$$

$$= 10(1+2) + 1(-4+14) + 2(-4-7)$$

$$= 10(3) + 1(10) + 2(-11)$$

$$= 30 + 10 - 22 = 18$$

$$D_2 = \begin{vmatrix} 1 & 10 & 2 \\ 2 & -4 & -2 \\ 3 & 7 & 1 \end{vmatrix} = 1 \begin{vmatrix} -4 & -2 \\ 7 & 1 \end{vmatrix} - 10 \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & -4 \\ 3 & 7 \end{vmatrix}$$

$$= 1(-4+14) - 10(2+6) + 2(14+12)$$

$$= 1(10) - 10(8) + 2(26)$$

$$= 10 - 80 + 52 = -18$$

$$D_3 = \begin{vmatrix} 1 & -1 & 10 \\ 2 & 1 & -4 \\ 3 & 1 & 7 \end{vmatrix} = 1 \begin{vmatrix} 1 & -4 \\ 1 & 7 \end{vmatrix} + 1 \begin{vmatrix} 2 & -4 \\ 3 & 7 \end{vmatrix} + 10 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= 1(7+4) + 1(14+12) + 10(2-3)$$

$$= 1(11) + 1(26) + 10(-1) = 11+26-10=27$$

$$\text{Now } x = \frac{D_1}{D} = \frac{18}{9} = 2$$

$$y = \frac{D_2}{D} = \frac{-18}{9} = -2$$

$$z = \frac{D_3}{D} = \frac{27}{9} = 3$$

Thus the solution is $x=2, y=-2, z=3$ Ans

Q.4: Solve the following system of homogeneous equations.

(i)
$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

Sol: Here $A_b = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 2 & 1 & 3 & 0 \end{array} \right]$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 3 & 1 & 0 \end{array} \right] \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{array} \right] R_3 - R_2$$

$(R_3) \Rightarrow 3x_3 = 0 \Rightarrow \boxed{x_3 = 0}$

$(R_2) \Rightarrow 3x_2 - 2x_3 = 0$
 $3x_2 - 2(0) = 0$

$3x_2 = 0$

$\Rightarrow \boxed{x_2 = 0}$

$(R_1) \Rightarrow x_1 - x_2 + x_3 = 0$

$x_1 - 0 + 0 = 0$

$\Rightarrow \boxed{x_1 = 0}$

$x_1 = 0, x_2 = 0, x_3 = 0$

Hence the given system has only trivial solution.

Ans

(ii)
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ -2x_1 + x_2 - x_3 &= 0 \\ -x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$$

Sol: Here $A_b = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -2 & 1 & -1 & 0 \\ -1 & 5 & 4 & 0 \end{array} \right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 6 & 6 & 0 \end{array} \right] \begin{array}{l} R_2 + 2R_1 \\ R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] R_3 - 2R_2$$

$(R_2) \Rightarrow 3x_2 + 3x_3 = 0$
 $\Rightarrow x_2 + x_3 = 0$

$\Rightarrow x_2 = -x_3$

Let $x_3 = t$

$\therefore x_2 = -t$

$(R_1) \Rightarrow x_1 + x_2 + 2x_3 = 0$

$x_1 - t + 2t = 0$

$x_1 + t = 0$

$\Rightarrow x_1 = -t$

Hence $\boxed{x_1 = -t, x_2 = -t, x_3 = t}$

is the solution.

Ans

Q.5. For what values of λ , the following system of homogeneous equations has a non-trivial solution. Solve the system.

$$x_1 + 5x_2 + 3x_3 = 0$$

$$5x_1 + x_2 - \lambda x_3 = 0$$

$$x_1 + 2x_2 + \lambda x_3 = 0$$

Sol Here $A = \begin{bmatrix} 1 & 5 & 3 \\ 5 & 1 & -\lambda \\ 1 & 2 & \lambda \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 5 & 3 \\ 5 & 1 & -\lambda \\ 1 & 2 & \lambda \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & -\lambda \\ 2 & \lambda \end{vmatrix} - 5 \begin{vmatrix} 5 & -\lambda \\ 1 & \lambda \end{vmatrix} + 3 \begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 1(\lambda + 2\lambda) - 5(5\lambda + \lambda) + 3(10 - 1)$$

$$= 1(3\lambda) - 5(6\lambda) + 3(9)$$

$$= 3\lambda - 30\lambda + 27$$

$$= 27 - 27\lambda$$

For non-trivial solution

$$|A| = 0 \Rightarrow 27 - 27\lambda = 0$$

$$\Rightarrow 27\lambda = 27$$

$$\Rightarrow \boxed{\lambda = 1}$$

The system becomes,

$$x_1 + 5x_2 + 3x_3 = 0$$

$$5x_1 + x_2 - x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

Here $A_b = \begin{bmatrix} 1 & 5 & 3 & | & 0 \\ 5 & 1 & -1 & | & 0 \\ 1 & 2 & 1 & | & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 5 & 3 & | & 0 \\ 0 & -24 & -16 & | & 0 \\ 0 & -3 & -2 & | & 0 \end{bmatrix} \begin{array}{l} R_2 - 5R_1 \\ R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 5 & 3 & | & 0 \\ 0 & -3 & -2 & | & 0 \\ 0 & -24 & -16 & | & 0 \end{bmatrix} R_{23}$$

$$\sim \begin{bmatrix} 1 & 5 & 3 & | & 0 \\ 0 & -3 & -2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} R_3 - 8R_2$$

$$(R_2) \Rightarrow -3x_2 - 2x_3 = 0$$

$$\Rightarrow 3x_2 = -2x_3$$

$$\text{let } x_3 = t$$

$$\therefore 3x_2 = -2t$$

$$x_2 = -\frac{2}{3}t$$

$$(R_1) \Rightarrow x_1 + 5x_2 + 3x_3 = 0$$

$$x_1 + 5\left(-\frac{2}{3}t\right) + 3t = 0$$

$$x_1 - \frac{10}{3}t + 3t = 0$$

$$x_1 - \frac{1}{3}t = 0 \Rightarrow x_1 = \frac{1}{3}t$$

$$\text{Hence } \boxed{x_1 = \frac{1}{3}t, x_2 = -\frac{2}{3}t, x_3 = t}$$

is the required non-trivial solution

REVIEW EXERCISE 2

Q.1 Choose the correct option.

(i) If $\begin{vmatrix} 7a-5b & 3c \\ -1 & 2 \end{vmatrix} = 0$, then which one of the following is correct.

(a) $14a + 3c = 5b$ (b) $14a - 3c = 5b$ (c) $14a + 3c = 10b$ (d) $14a + 10b = 3c$

Sol: $\begin{vmatrix} 7a-5b & 3c \\ -1 & 2 \end{vmatrix} = 0$

$$\Rightarrow 2(7a-5b) + 1(3c) = 0 \Rightarrow 14a - 10b + 3c = 0 \Rightarrow 14a + 3c = 10b$$

Thus the correct option is (c) Ans

(ii) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and A_{ij} is the cofactor of a_{ij} in A . Then the value of $|A|$ is given by

(a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$ (b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$ (d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Sol: The correct option is (d) Ans

(iii) If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$ then the value of α is

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 5

Sol: $\therefore |A^3| = |A|^3$

$$\therefore |A^3| = 125 \Rightarrow |A|^3 = 125 \Rightarrow |A|^3 = 5^3 \Rightarrow |A| = 5$$

$$\begin{vmatrix} \alpha & 2 \\ 2 & \alpha \end{vmatrix} = 5 \Rightarrow \alpha^2 - 4 = 5 \Rightarrow \alpha^2 = 9 \Rightarrow \alpha = \pm 3$$

Thus the correct option is (c). Ans

(iv) If $|A| = 47$, then find $|A^t|$

- (a) -47 (b) 47 (c) 0 (d) cannot be determined

Sol: $\because |A| = |A^t|$

$\therefore$ the correct option is (b) Ans

(v) If $\det(A) = 5$ then find $\det(15A)$ where A is of order 2×2 .

- (a) 225 (b) 75 (c) 375 (d) 1125

Sol: $\because \det(15A) = 15^2 \det(A) = 225 \det(A) = 225 \times 5 = 1125$

Thus the correct option is (d) Ans

(vi) If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then find A^n , (where $n \in \mathbb{N}$)

- (a) $\begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ (c) $3^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $I_{2 \times 2}$

Sol: $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow A^n = \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix} = 3^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thus the correct option is (c) Ans

Q.2 Compute the product $\begin{bmatrix} -5 & 1 \\ 6 & -1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 4 & -5 & -1 \\ 5 & 6 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$

Sol:

$$\begin{bmatrix} -5 & 1 \\ 6 & -1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 4 & -5 & -1 \\ 5 & 6 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 1 \\ 6 & -1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 20 + 20 - 1 \\ 25 - 24 + 1 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 6 & -1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 39 \\ 2 \end{bmatrix} = \begin{bmatrix} -195 + 2 \\ 234 - 2 \\ -78 + 0 \end{bmatrix} = \begin{bmatrix} -193 \\ 232 \\ -78 \end{bmatrix}$$

Ans

Q.3 Prove that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies $A^2 - 4A - 5I = 0$.

Sol: $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

Now $A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8+0 & 8-8+0 \\ 8-8+0 & 9-4-5 & 8-8+0 \\ 8-8+0 & 8-8+0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Thus $A^2 - 4A - 5I = 0$. Proved

Q.4 If $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 6 \\ 7 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & -1 \\ 2 & 0 & 3 \\ -1 & 2 & 4 \end{bmatrix}$. Find $|2A - B|$

Sol: $B^2 = BB$

$$= \begin{bmatrix} 4 & 0 & -1 \\ 2 & 0 & 3 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 0 & -1 \\ 2 & 0 & 3 \\ -1 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 16+0+1 & 0+0-2 & -4+0-4 \\ 8+0-3 & 0+0+6 & -2+0+12 \\ -4+4-4 & 0+0+8 & 1+6+16 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & -2 & -8 \\ 5 & 6 & 10 \\ -4 & 8 & 23 \end{bmatrix}$$

$$2A - B^2 = 2 \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 6 \\ 7 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 17 & -2 & -8 \\ 5 & 6 & 10 \\ -4 & 8 & 23 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 6 \\ 0 & 8 & 12 \\ 14 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 17 & -2 & -8 \\ 5 & 6 & 10 \\ -4 & 8 & 23 \end{bmatrix}$$

$$= \begin{bmatrix} 4-17 & 2+2 & 6+8 \\ 0-5 & 8-6 & 12-10 \\ 14-4 & 4-8 & 2-23 \end{bmatrix}$$

$$= \begin{bmatrix} -13 & 4 & 14 \\ -5 & 2 & 2 \\ 10 & -4 & -21 \end{bmatrix}$$

$$|2A - B^2| = \begin{vmatrix} -13 & 4 & 14 \\ -5 & 2 & 2 \\ 10 & -4 & -21 \end{vmatrix}$$

$$= -13 \begin{vmatrix} 2 & 2 \\ -4 & -21 \end{vmatrix} - 4 \begin{vmatrix} -5 & 2 \\ 18 & -21 \end{vmatrix} + 14 \begin{vmatrix} -5 & 2 \\ 18 & -4 \end{vmatrix}$$

$$= -13(-42+8) - 4(105-36) + 14(20-36)$$

$$= -13(-34) - 4(69) + 14(-16)$$

$$= 442 - 276 - 224 = -58 \text{ Ans}$$

Q.5 Using properties of determinants, prove that

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

Sol:

Let $\Delta = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$

$$= \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1-a^2 & 1-a & 0 \\ 2-2a & 1-a & 0 \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= \begin{vmatrix} 1-a^2 & 1-a \\ 2-2a & 1-a \end{vmatrix}$$

$$= \begin{vmatrix} (1-a)(1+a) & 1-a \\ 2(1-a) & 1-a \end{vmatrix}$$

$$= (1-a)^2 \begin{vmatrix} 1+a & 1 \\ 2 & 1 \end{vmatrix}$$

$$= (1-a)^2 (1+a-2)$$

$$= (1-a)^2 (a-1)$$

$$= [-(a-1)]^2 (a-1)$$

$$= (a-1)^2 (a-1)$$

$$= (a-1)^3$$

Proved

Q.6 If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, then show that AA^t and A^tA are both symmetric.

Sol: let $B = AA^t$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9+1+1 & 0+1-2 \\ 0+1-2 & 0+1+4 \end{bmatrix} = \begin{bmatrix} 11 & -1 \\ -1 & 5 \end{bmatrix}$$

$$B^t = \begin{bmatrix} 11 & -1 \\ -1 & 5 \end{bmatrix} = B$$

$\Rightarrow B$ is symmetric

let $C = A^tA$

$$= \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9+0 & 3+0 & -3+0 \\ 3+0 & 1+1 & -1+2 \\ -3+0 & -1+2 & 1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 & -3 \\ 3 & 2 & 1 \\ -3 & 1 & 5 \end{bmatrix}$$

$$C^t = \begin{bmatrix} 9 & 3 & -3 \\ 3 & 2 & 1 \\ -3 & 1 & 5 \end{bmatrix} = C$$

$\Rightarrow C$ is symmetric.
Proved

Q.7 If $A = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$, prove that $A^{-1} = A$.

Sol:

$$[A|I] = \left[\begin{array}{ccc|ccc} 4 & 3 & 3 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ -4 & -4 & -3 & 0 & 0 & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 0 & 1 & 0 \\ 4 & 3 & 3 & 1 & 0 & 0 \\ -4 & -4 & -3 & 0 & 0 & 1 \end{array} \right] R_{12}$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 4 & 0 \\ 0 & -4 & 1 & 0 & -4 & 1 \end{array} \right] \begin{array}{l} R_2 + 4R_1 \\ R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & -4 & 1 & 0 & -4 & 1 \end{array} \right] \begin{array}{l} R_2 + R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 & -4 & -3 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -4 & -3 & -3 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 & -4 & -3 \end{array} \right] \begin{array}{l} R_3 - 4R_2 \\ R_1 + R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 3 & 3 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -4 & -4 & -3 \end{array} \right]$$

Hence $A^{-1} = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix} = A$. Proved

Q.8 If $A = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$, then find $A + 10\bar{A}'$.

Sol: $\bar{A}' = \frac{1}{4+6} \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \Rightarrow 10\bar{A}' = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$

$$A + 10\bar{A}' = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4+1 & 3-3 \\ -2+2 & 1+4 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 5I \text{ Ans}$$

Q.9 Solve the system.

$$x + y + z = 4$$

$$2x - 3y + z = 2$$

$$-x + 2y - z = -1$$

by using the following methods:

(i) Matrix Inversion

The matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

$A \quad X \quad B$

ie $AX = B$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -3 & 1 \\ 2 & -1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix}$$

$$= 1(3-2) - 1(-2+1) + 1(4-3)$$

$$= 1(1) - 1(-1) + 1(1)$$

$$= 1 + 1 + 1$$

$$= 3 \neq 0 \Rightarrow \bar{A}' \text{ exists.}$$

$$\therefore X = \bar{A}' B \rightarrow (1)$$

For $\bar{A}'$

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^{1+1} M_{11} = + \begin{vmatrix} -3 & 1 \\ 2 & -1 \end{vmatrix} = 3-2=1$$

$$A_{12} = (-1)^{1+2} M_{12} = - \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} = -(-2+1)=1$$

$$A_{13} = (-1)^{1+3} M_{13} = + \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} = 4-3=1$$

$$A_{21} = (-1)^{2+1} M_{21} = - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -(-1-2)=3$$

$$A_{22} = (-1)^{2+2} M_{22} = + \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = -1+1=0$$

$$A_{23} = (-1)^{2+3} M_{23} = - \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -(2-1)=-1$$

$$A_{31} = (-1)^{3+1} M_{31} = + \begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} = 1-3=-2$$

$$A_{32} = (-1)^{3+2} M_{32} = - \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(1-2) = 1$$

$$A_{33} = (-1)^{3+3} M_{33} = + \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3-2 = -5$$

$$\text{adj } A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 0 & 1 \\ 1 & -3 & -5 \end{bmatrix}$$

$$\bar{A} = \frac{1}{|A|} \text{adj } A = \frac{1}{3} \begin{bmatrix} 1 & 3 & 4 \\ 1 & 0 & 1 \\ 1 & -3 & -5 \end{bmatrix}$$

$$\textcircled{1} \Rightarrow X = \frac{1}{3} \begin{bmatrix} 1 & -3 & 4 \\ 1 & 0 & 1 \\ 1 & -3 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 4+6-4 \\ 4+0-1 \\ 4-6+5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{i.e. } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{x=2, y=1, z=1}$$

is the solution. Ans

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -5 & -1 & -6 \\ 0 & 3 & 0 & 3 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 0 & 3 \end{bmatrix} \begin{array}{l} \\ R_2 + 2R_3 \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array}$$

$$(R_3) \Rightarrow 3z = 3 \Rightarrow \boxed{z=1}$$

$$(R_2) \Rightarrow y - z = 0$$

$$y - 1 = 0$$

$$\Rightarrow \boxed{y=1}$$

$$(R_1) \Rightarrow x + y + z = 4$$

$$x + 1 + 1 = 4$$

$$\Rightarrow \boxed{x=2}$$

$$\therefore \boxed{x=2, y=1, z=1}$$

is the solution.

(ii) Gauss Elimination Method

Sol: Here

$$A_b = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & -3 & 1 & 2 \\ -1 & 2 & -1 & -1 \end{bmatrix}$$

(iii) Gauss-Jordan Method

Sol:

$$\text{Here } A_b = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -3 & 1 & 2 \\ -1 & 2 & -1 & -1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -5 & -1 & -6 \\ 0 & 3 & 0 & 3 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -1 & 0 \\ 0 & 3 & 0 & 3 \end{array} \right] R_2 + 2R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 3 \end{array} \right] \begin{array}{l} R_1 - R_2 \\ R_3 - 3R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right] \frac{1}{3}R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right] \begin{array}{l} R_2 - 2R_3 \\ R_2 + R_3 \end{array}$$

$$\Rightarrow \boxed{x=1, y=1, z=1}$$

is the solution.

Ans

(iv) Cramer's Rule

Sol:

$$\text{Here } A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

$$D = |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -3 & 1 \\ 2 & -1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix}$$

$$= 1(3-2) - 1(-2+1) + 1(4-3)$$

$$= 1 + 1 + 1 = 3$$

$$D_1 = \begin{vmatrix} 4 & 1 & 1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= 4 \begin{vmatrix} -3 & 1 \\ 2 & -1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix}$$

$$= 4(3-2) - 1(-2+1) + 1(4-3)$$

$$= 4(1) - 1(-1) + 1(1)$$

$$= 4 + 1 + 1 = 6$$

$$D_2 = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 2 & 1 \\ -1 & -1 & -1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} - 4 \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix}$$

$$= 1(-2+1) - 4(-2+1) + 1(-2+2)$$

$$= -1 + 4 + 0 = 3$$

$$D_3 = \begin{vmatrix} 1 & 1 & 4 \\ 2 & -3 & 2 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -3 & 2 \\ 2 & -1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} + 4 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix}$$

$$= 1(3-4) - 1(-2+2) + 4(4-3)$$

$$= -1 + 0 + 4 = 3$$

$$\text{Now } x = \frac{D_1}{D} = \frac{6}{3} = 2$$

$$y = \frac{D_2}{D} = \frac{3}{3} = 1$$

$$z = \frac{D_3}{D} = \frac{3}{3} = 1$$

$$\therefore \boxed{x=2, y=1, z=1} \text{ is the solution.} \quad \text{Ans}$$