Exercise # 12.1 CH=12

Find the domain of each function

① $y = \sin 2x$

Sol In $\sin A$ where A is angle, any real value can be given to the angle. Hence A = Real

Then for $y = \sin 2x$

Also Dom $= R$ Ans

② $y = 4\cos x$

Dom $= R$ (Any real value can be given in $\cos x$)

③ $y = 3\sin 3x$

Dom $= R$

④ $y = \sec 2x$

Note $\sec ?$

$? \neq (2n+1)\frac{\pi}{2}$

where ? shows any real angle

Sol Here angle is $2x$

And angle $\neq (2n+1)\frac{\pi}{2}$

$\Rightarrow 2x \neq (2n+1)\frac{\pi}{2}$

$\Rightarrow x \neq (2n+1)\frac{\pi}{4}$

Hence Dom $= R - \{(2n+1)\frac{\pi}{4}\}$

⑤ $y = \tan(\frac{x}{2})$

Note In $y = \tan A$

$A \neq (2n+1)\frac{\pi}{2}$

Sol Here angle is $\frac{x}{2}$

Angle $\neq (2n+1)\frac{\pi}{2}$

$\frac{x}{2} \neq (2n+1)\frac{\pi}{2}$

$\Rightarrow x \neq (2n+1)\pi$

Hence Dom $= R - \{(2n+1)\pi\}$

⑥ $y = \operatorname{cosec} 2x$

Note $y = \operatorname{cosec} ?$

$? \neq n\pi$

Sol Here angle $= 2x$

and angle $\neq n\pi$ in $\operatorname{cosec} x$

So $2x \neq n\pi$

$\Rightarrow x \neq n\frac{\pi}{2}$

Hence Dom $= R - \{n\frac{\pi}{2}\}$

⑦ $y = 3\cos 2x$

Ans Dom $= R$

⑧ $y = 6\sec\frac{x}{2}$

Sol Here angle $= \frac{x}{2}$

and angle $\neq (2n+1)\frac{\pi}{2}$

$\Rightarrow \frac{x}{2} \neq (2n+1)\frac{\pi}{2}$

$\Rightarrow x \neq (2n+1)\pi$

Dom $= R - \{(2n+1)\pi\}$

⑨ $y = 5\tan 3x$

Sol Here angle $= 3x$

and angle $\neq (2n+1)\frac{\pi}{2}$

$\Rightarrow 3x \neq (2n+1)\frac{\pi}{2}$

$\Rightarrow x \neq (2n+1)\frac{\pi}{6}$

Hence Dom $= R - \{(2n+1)\frac{\pi}{6}\}$

⑩ $y = 5\sin 5x$

Sol $Dom = R$

Q:- Find the range of the following functions.

⑪ $y = \sin 2x$

Sol Range $= [-1 \quad 1]$

Note: The value of $\sin ?$ is $[-1 \quad 1]$ when $? \in R$

⑫ $y = \cos 4x$

Sol Range $= [-1 \quad 1]$

Note: The value of $\cos ?$ is $[-1 \quad 1]$

⑬ $y = 2\sin 3x$

$\sin ? = [-1 \quad 1]$

Sol Range $= 2[-1 \quad 1]$

$= [-2 \quad 2]$ Ans

⑭ $y = 5\cos x$

Sol Range $= 5[-1 \quad 1]$

$= [-5 \quad 5]$

⑮ $y = 3\cot x$

Note The range of $\cot ?$ is Real

Sol Range $= 3R$

$= R$

⑯ $y = 2\sec 2x$

Note The range of $\sec ?$ is $-1 \geq y \geq 1$

Sol for $y = \sec 2x$

$-1 \geq y \geq 1$

and for $y = 2\sec 2x$

Range $-2 \geq y \geq 2$

⑰ $y = \operatorname{cosec} 2x$

$y = \operatorname{cosec} ?$

$-1 \geq y \geq 1$

Sol Range $-1 \geq y \geq 1$

⑱ $y = \sin \pi x$

Range $y = [-1 \quad 1]$

⑲ $y = \tan \frac{\pi}{4} x$

Range $= R$

⑳ $y = \sec(2\pi x + 3)$

$y = \sec ?$

$-1 \geq y \geq 1$

Range $-1 \geq y \geq 1$

Golden words by Albert Einstein

If A is success in life, then A equals x plus y plus z where.

x is work

y is play

z is keeping your mouth shut.

Engr. Majid Amin

Exercise # 12.2

Q:- Find the value of each function

① $\sin(-\pi)$

Sol $\sin(-\pi) = -\sin\pi$
$= -(0)$
$= 0$ Ans

Note (i) $\sin(-\theta) = -\sin\theta$
(ii) $\sin\pi = 0$

② $\cos\left(-\frac{\pi}{4}\right)$

Sol $\cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4}$
$= \frac{1}{\sqrt{2}}$ Ans

Note $\cos(-\theta) = \cos\theta$
$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

③ $y = \tan\left(-\frac{\pi}{4}\right)$
$= -\tan\frac{\pi}{4}$ $\quad\left(\tan\frac{\pi}{4} = 1\right)$
$= -(1)$
$= -1$ Ans

Note $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)}$
$= -\frac{\sin\theta}{\cos\theta}$
$= -\tan\theta$

④ $\cot\left(-3\frac{\pi}{2}\right)$
$= \frac{\cos\left(-3\frac{\pi}{2}\right)}{\sin\left(-3\frac{\pi}{2}\right)}$
$= \frac{\cos 3\frac{\pi}{2}}{-\sin\frac{3\pi}{2}} = \frac{0}{-1} = 0$ Ans

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

⑤ $\operatorname{cosec}\left(-\frac{\pi}{4}\right)$
$= \frac{1}{\sin\left(-\frac{\pi}{4}\right)} = \frac{1}{-\sin\frac{\pi}{4}} = \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$ Ans

⑥ $y = \sec(-\pi)$
$= \frac{1}{\cos(-\pi)}$
$= \frac{1}{\cos\pi} = \frac{1}{-1} = -1$ Ans

Q:- Find the period of each function

⑦ $2\sin x$

Ftn	Period
$2\sin x$	$\to 2\pi$

Sol As period of $\sin x = 2\pi$
By theorem period of $k f(x) =$ Period of $f(x)$
$\Rightarrow$ Period of $2\sin x =$ Period of $\sin x$
$\Rightarrow$ Period of $2\sin x = 2\pi$

⑧ $3\tan x$

Ftn	Period
$3\tan x$	π

Sol As period of $\tan x = \pi$
By theorem period of $k.f(x) =$ Period of $f(x)$
$\Rightarrow$ Period of $3\tan x =$ Period of $\tan x$
$\Rightarrow$ Period of $3\tan x = \pi$

⑨ $5\cos 3x$

Ftn	Period
$5\cos 3x$	$\to \frac{2\pi}{3}$

Sol As Period of $\cos x = 2\pi$
By theorems
(i) Period of $k f(x) =$ Period of $f(x)$
(ii) Period of $f(kx) = \frac{\text{Period of } f(x)}{k}$
we have
Period of $5\cos 3x = \frac{\text{Period of } \cos x}{3}$
$= \frac{2\pi}{3}$ Ans

⑩ $\frac{1}{2}\sec x$

Sol Period $= 2\pi$

$\frac{1}{2}\sec x$

No role in period

⑪ $y = -2\operatorname{cosec}\pi x$

Sol As period of $\operatorname{cosec} x = 2\pi$

By theorems Period of $k f(x) =$ Period of $f(x)$

& Period of $f(kx) = \frac{\text{Period of } f(x)}{k}$

we have

Period of $-2\operatorname{cosec}\pi x = \frac{\text{Period of } \operatorname{cosec} x}{\pi}$

$= \frac{2\pi}{\pi}$

$= 2$ Ans

⑫ $\frac{7}{9}\cot\frac{2\pi}{3}x$

Sol As Period of $\tan x = \pi$

By theorems (i) Period of $k f(x) =$ Period of $f(x)$

& (ii) Period of $f(kx) = \frac{\text{Period of } f(x)}{k}$

we have

Period of $\frac{7}{2}\cot\frac{2\pi}{3}x = \frac{\text{Period of } \cot x}{\frac{2\pi}{3}}$

$= \frac{\pi}{\frac{2\pi}{3}}$

$= 3/2$ Ans

⑬ $3\operatorname{cosec}\frac{\pi}{2}x$

Sol Period $= \frac{2\pi}{\pi/2} = 4$

⑭ $-1\cot\frac{1}{2\pi}x$

Sol Period $= \frac{2\pi}{\frac{1}{2\pi}} = 4\pi^2$.

⑮ $-\frac{2}{5}\sec\frac{3}{\pi}x$.

Sol Period $= \frac{2\pi}{\frac{3}{\pi}} = \frac{2\pi^2}{3}$

⑯ $\frac{7}{9}\sec\frac{2}{\theta}x$

Sol As period of $\sec x = 2\pi$

By theorems (i) Period of $k f(x) =$ Period of $f(x)$

(ii) Period of $f(kx) = \frac{\text{Period of } f(x)}{k}$

we have

Period of $\frac{7}{9}\sec\frac{2}{\theta}x = \frac{\text{Period of } \sec x}{\frac{2}{\theta}}$.

$= \frac{2\pi}{\frac{2}{\theta}}$

$= \pi\theta$.

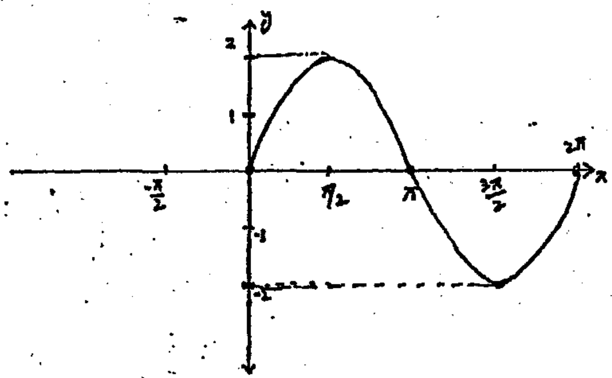
Exercise # 12.3

Draw the graph of the following functions in the indicated interval?

① $y = 2\sin x$ $\quad 0 \le x \le 2\pi$

Sol Table

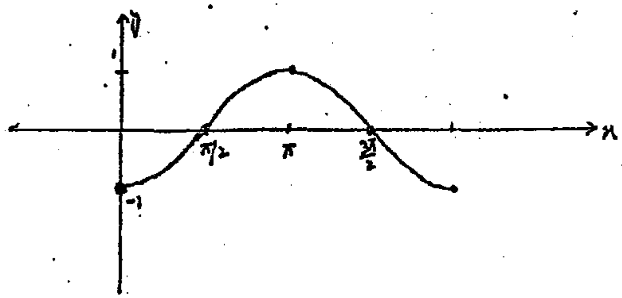
x	0	$\pi/2$	π	$3\pi/2$	2π
y	0	2	0	-2	0



② $y = -\cos x$ $\quad 0 \le x \le 2\pi$

Table

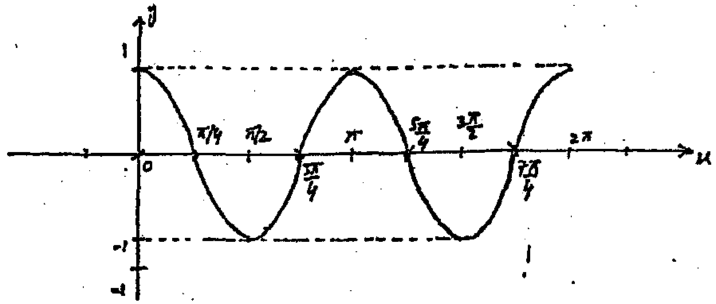
x	0	$\pi/2$	π	$3\pi/2$	2π
y	-1	0	1	0	-1



③ $y = \cos 2x$ $\quad 0 \le x \le 2\pi$

Sol Table

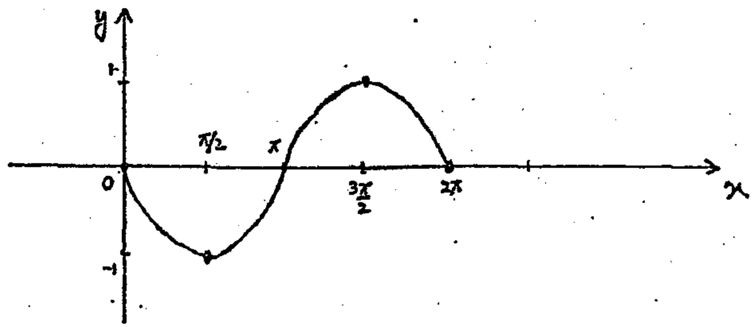
x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$\frac{3\pi}{2}$	$7\pi/4$	2π
y	1	0	-1	0	1	0	-1	0	1



④ $y = \sin(-x)$ $\quad 0 \le x \le 2\pi$

Table

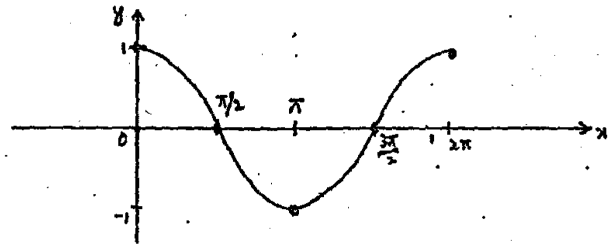
x	0	$\pi/2$	π	$3\frac{\pi}{2}$	2π
y	0	-1	0	1	0



⑤ $y = \sin(x + \frac{\pi}{2})$ $0 \le x \le 2\pi$

Table

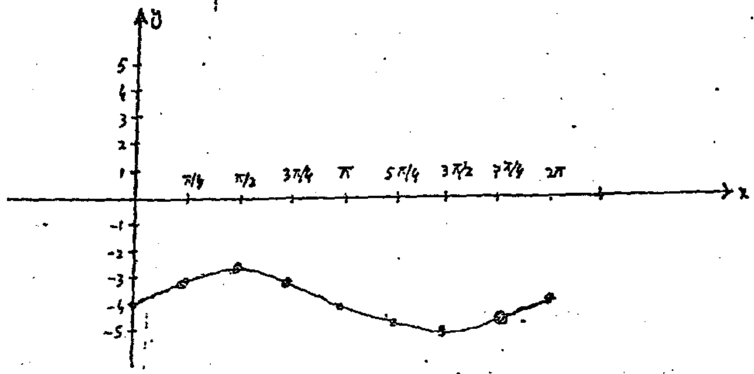
x	0	$\pi/2$	π	$3\pi/2$	2π
y	1	0	-1	0	1



⑥ $y = -4 + \sin x$ $0 \le x \le 2\pi$

Sol

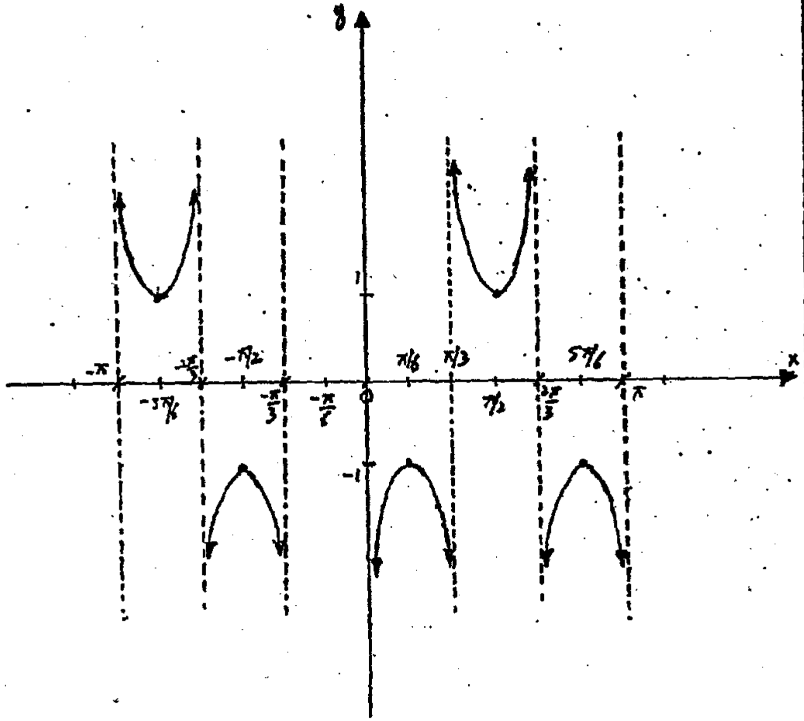
x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
y	-4	-3.3	-3	-3.3	-4	-4.7	-5	-4.7	-4



⑦ $y = \sec(3x + \frac{\pi}{2})$ $-\pi \le x \le \pi$

Sol

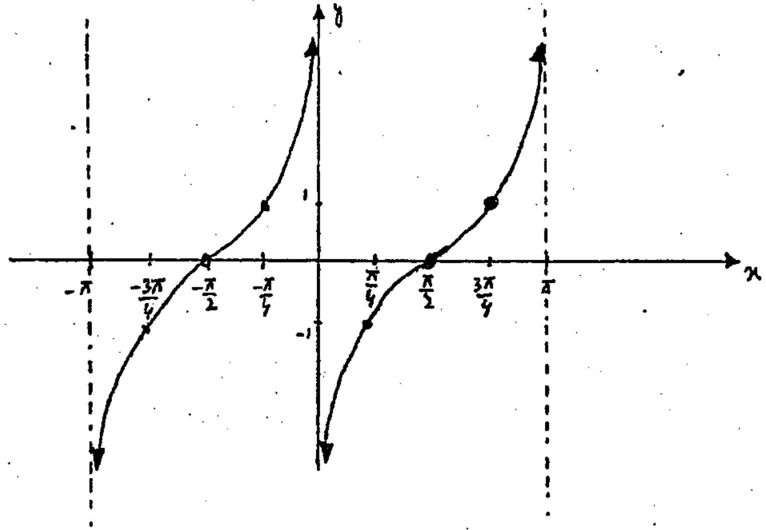
x		$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
y		∞	1	∞	-1	∞	1	∞	-1	∞	1	∞	-1	∞



Q:8 $y = -\cot x$ $\quad -\pi \le x \le \pi$

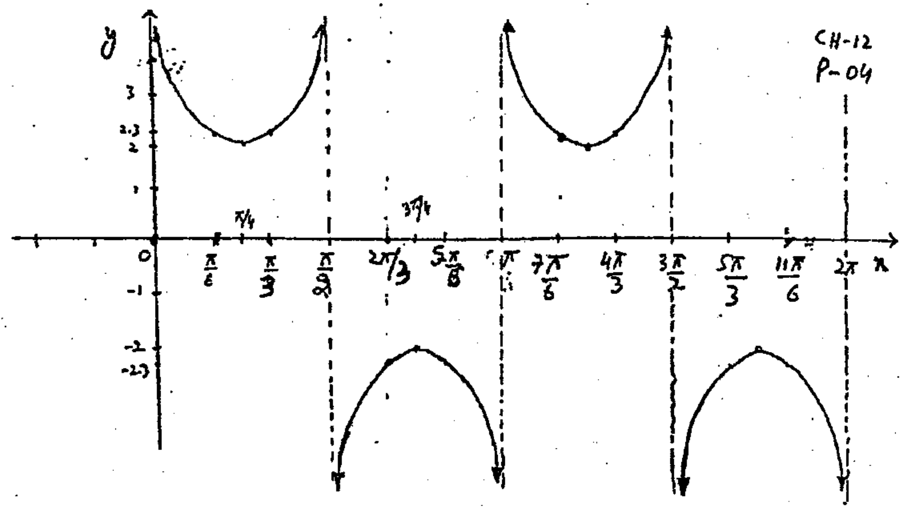
Sol

x	$-\pi$	$-\frac{3\pi}{4}$	$-\pi/2$	$-\frac{\pi}{4}$	0	$\pi/4$	$\pi/2$	$\frac{3\pi}{4}$	π
y	∞	-1	0	1	∞	-1	0	1	∞



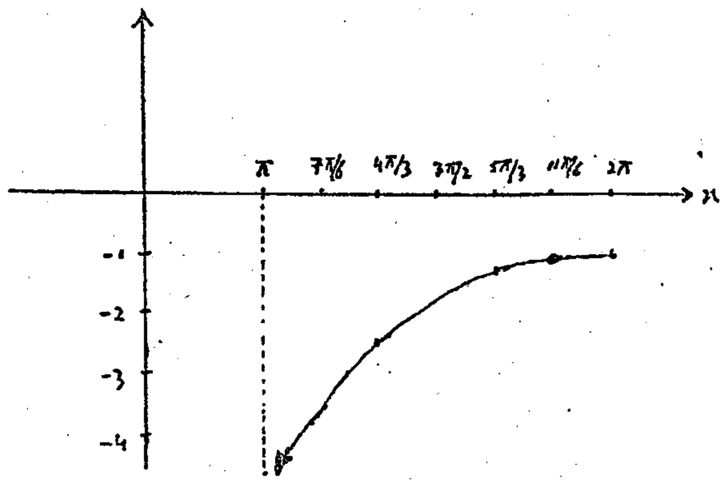
Q:9 $y = 2\operatorname{cosec} 2x$ $\quad 0 \le x \le 2\pi$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$5\pi/3$	$\frac{11\pi}{6}$	2π
y	∞	2.3	2	2.3	∞	-2.3	-2	-2.3	∞	2.3	2.3	∞	-2.3	-2.3	∞



Q:10 $y = \sec \frac{x}{2}$ $\quad \pi \le x \le 2\pi$

x	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$11\pi/6$	2π
y	∞	-3.86	-2.4	-1.41	-1.15	-1.03	-1



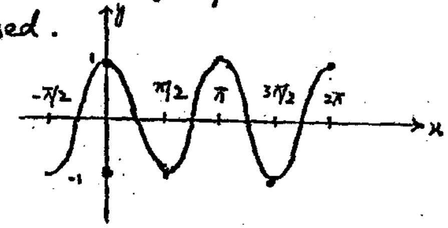
Exercise # 12.4

Q:1 Without drawing, guess the graph of each of the following functions. Also find its period, frequency and amplitude?

(i) $y = \cos 2\theta$

Sol It is $y = \cos A\theta$ form where $A = 2 > 1$

$\Rightarrow$ we will have two cycles in a length of 2π. So the graph will be compressed.



Period $= \frac{2\pi}{A}$

$= \frac{2\pi}{2}$

$= \pi$

Frequency $= \frac{1}{\text{Period}} = \frac{1}{\pi}$ Ans

Amplitude $= \frac{1}{2}$ (Difference b/w maximum and minimum values)

$= \frac{1}{2}\{1 - (-1)\}$

$= \frac{1}{2}(2) = 1$ Ans

(ii) $y = \sin 6\theta \Rightarrow \sin A\theta$ form

Here $A = 6 > 1$ $\Rightarrow$ The graph will be compressed to have six repeatitions in the interval 2π

Period $= \frac{2\pi}{A} = \frac{2\pi}{6} = \frac{\pi}{3}$

Frequency $= \frac{3}{\pi}$ (Frequency = Reciprocal of period)

Amplitude $= \frac{1}{2}\{1 - (-1)\}$

$= \frac{1}{2}(2) = 1$

(iii) $y = \sin \pi\theta$

Sol which is $y = \sin A\theta$ form

Here $A = \pi > 1$

$\Rightarrow$ The graph will be compressed

Period $= \frac{2\pi}{\pi} = 2$ Ans

Frequency $= \frac{1}{\text{Period}} = \frac{1}{2}$

Amplitude $= 1$

(iv) $y = \cos \frac{\pi}{2}\theta$

Sol

Here $A = \frac{\pi}{2} > 1$

$\Rightarrow$ The graph will be compressed

Period $= \frac{2\pi}{A} = \frac{2\pi}{\pi/2} = 4$

Frequency $= \frac{1}{4}$

Amplitude $= 1$ Ans

Golden words.

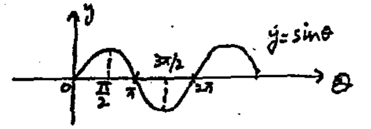
People grow through experience if they meet life honestly and courageously. This is how character is built. (Eleanor Roosevelt 1884-1962)

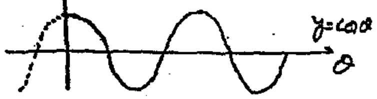
Q.2 Use the symmetric and periodic properties of sine, cosine and tangent functions to establish the following identities.

(i) $\sin(\frac{\pi}{2}+\theta) = \cos\theta$

Sol By translating (moving) forward the graph of $y=\sin\theta$ by $\pi/2$, we get the graph of $\cos\theta$. Hence

$\sin(\frac{\pi}{2}+\theta) = \cos\theta$.

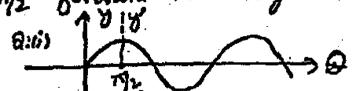




Note:- By translating the graph we mean making new x & y axis.

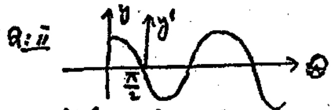
e.g In the above question moving the graph forward by $\pi/2$ means if we make x & y axis $\pi/2$ forward than original.

(ii) $\cos(\frac{\pi}{2}+\theta) = -\sin\theta$



Sol As we know that if we make x-y axis of $y=\cos\theta$ to move forward by $\frac{\pi}{2}$, we get $-$ve $\sin\theta$ i.e $-\sin\theta$ which is called reflection of $\sin\theta$

Hence $\cos(\frac{\pi}{2}+\theta) = -\sin\theta$



(iii) $\sin(\pi-\theta) = \sin\theta$

Sol As $\sin(\theta-\pi) = -\sin\theta$

$\Rightarrow \sin(-1)(-\theta+\pi) = -\sin\theta$

$\Rightarrow -\sin(-\theta+\pi) = -\sin\theta$

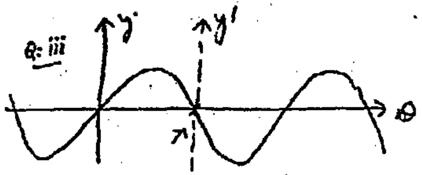
$\Rightarrow \sin(-\theta+\pi) = \sin\theta$

$\Rightarrow \sin(\pi-\theta) = \sin\theta$

$\sin(\pi-\theta) = \sin\theta$

$\sin(\theta-\pi) = -\sin\theta$

If we decrease angle by π unit, we get $-$ve $\sin\theta$



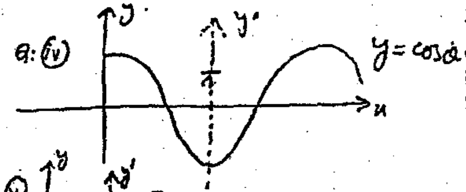
(iv) $\cos(\pi-\theta) = -\cos\theta$

Sol If we increase or decrease the angle of $\cos\theta$ by π unit the sign is changed i.e the graph is reversed.

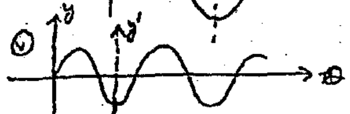
$\cos(\theta-\pi) = -\cos\theta$

$\Rightarrow \cos(-1)(-\theta+\pi) = -\cos\theta$

$\Rightarrow \cos(\pi-\theta) = -\cos\theta$.



(v) $\sin(\pi+\theta) = -\sin\theta$



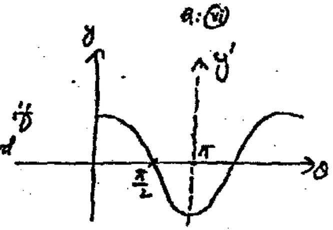
Sol If we make x & y axis forward by π unit, the graph of $\sin\theta$ is reversed.

i.e $\sin(\pi+\theta) = -\sin\theta$

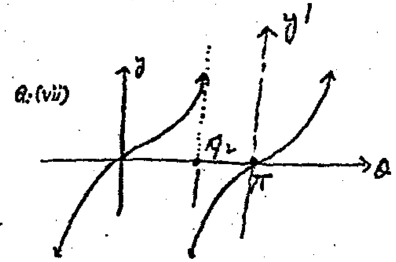
(vi) $\cos(\pi+\theta) = -\cos\theta$

Sol As clear from the figure if we move the axis forward by π unit, we get the graph of $-\cos\theta$

Hence $\cos(\pi+\theta) = -\cos\theta$



(vii) $\tan(\pi-\theta) = -\tan\theta$



Sol As $\tan(\theta-\pi) = \tan\theta$

i.e By going π unit forward or backward the graph does not change.

Now $\tan(\theta-\pi) = \tan\theta$

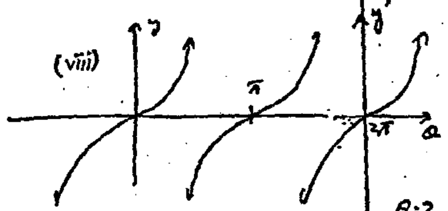
take -1 as common from angle

$\tan(-1)(\theta-\pi) = \tan\theta$

$\Rightarrow \; -\tan(\theta-\pi) = \tan\theta \qquad \because \tan(-\theta) = -\tan\theta$

$\Rightarrow \; \tan(\theta-\pi) = -\tan\theta$.

(viii) $\tan(2\pi-\theta) = -\tan\theta$



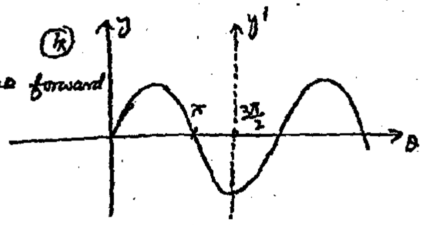
Sol: As $\tan(\theta-2\pi) = \tan\theta$

$\Rightarrow \tan(-1)(-\theta+2\pi) = \tan\theta$

$\Rightarrow \; -\tan(2\pi-\theta) = \tan\theta$

$\Rightarrow \; \tan(2\pi-\theta) = -\tan\theta$

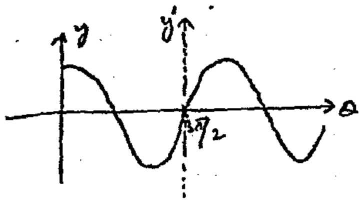
(ix) $\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$



Sol: As we know that moving sine forward by $\frac{3\pi}{2}$ we get reverse $\cos\theta$

Hence $\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$

(x) $\cos\left(3\frac{\pi}{2}+\theta\right) = \sin\theta$



Sol: By moving the graph of $y=\cos\theta$ by $\frac{3\pi}{2}$, we get the graph of $y=\cos\theta$

Hence $\cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta$

(xi) $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

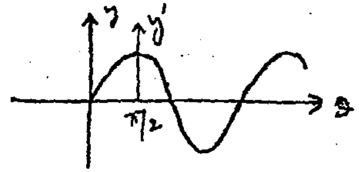
Sol: As $\sin(\theta-\pi/2) = -\cos\theta$

$\Rightarrow \sin(-1)(-\theta+\pi/2) = -\cos\theta$

$\Rightarrow \; -\sin(\pi/2-\theta) = -\cos\theta$

$\Rightarrow \; \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$.

(xii) $\sin\left(-\theta-\frac{\pi}{2}\right) = -\cos\theta$



As $\sin(-\theta-\pi/2)$

$= \sin(-1)\left(\theta+\frac{\pi}{2}\right)$

$= -\sin(\theta+\pi/2)$

$= -\cos\theta \qquad \because \sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta$

Q:3 For any integer, deduce that

(i) $\sin(\theta+2k\pi) = \sin\theta$

L.H.S $\sin(\theta+2k\pi)$ Apply $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \sin\theta\cos(2k\pi) + \cos\theta\sin(2k\pi)$

$= \sin\theta\,(1) + \cos\theta\,(0) \qquad \sin(2k\pi) = 0, \; \cos(2k\pi) = 1$

$= \sin\theta = R.H.S$

(ii) $\cos(\theta+2k\pi) = \cos\theta$

L.H.S $\cos(\theta+2k\pi)$

$= \cos\theta\cos 2k\pi - \sin\theta\sin(2k\pi)$

$= \cos\theta\,(1) - \sin\theta\,(0)$

$= \cos\theta = R.H.S$

(iii) $\tan(\theta+2k\pi) = \tan\theta$

L.H.S $\tan(\theta+2k\pi)$

$= \dfrac{\sin(\theta+2k\pi)}{\cos(\theta+2k\pi)}$

$= \dfrac{\sin\theta\cos(2k\pi) + \cos\theta\sin 2k\pi}{\cos\theta\cos 2k\pi - \sin\theta\sin 2k\pi}$

$= \dfrac{\sin\theta(1) + \cos\theta(0)}{\cos\theta(1) - \sin\theta(0)} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta = R.H.S$

(iv) $\cot(\theta+2k\pi)=\cot\theta$

L.H.S $\cot(\theta+2k\pi)$

$$= \frac{\cos(\theta+2k\pi)}{\sin(\theta+2k\pi)} = \frac{\cos\theta}{\sin\theta} = \cot\theta = R.H.S$$

(v) $\sec(\theta+2k\pi)=\sec\theta$

L.H.S $\sec(\theta+2k\pi)$

$$= \frac{1}{\cos(\theta+2k\pi)}$$

$$= \frac{1}{\cos\theta\cos 2k\pi - \sin\theta\sin 2k\pi}$$

$\sin 2k\pi = 0$

$\cos 2k\pi = 1$

$$= \frac{1}{\cos\theta\,(1) - \sin\theta\,(0)}$$

$$= \frac{1}{\cos\theta - 0} = \frac{1}{\cos\theta} = \sec\theta = R.H.S$$

(vi) $\operatorname{cosec}(\theta+2k\pi)=\operatorname{cosec}\theta$

L.H.S $\operatorname{cosec}(\theta+2k\pi)$

$$= \frac{1}{\sin(\theta+2k\pi)}$$

$$= \frac{1}{\sin\theta\cos 2k\pi + \cos\theta\sin 2k\pi}$$

$$= \frac{1}{\sin\theta\,(1) + \cos\theta\,(0)}$$

$$= \frac{1}{\sin\theta}$$

$$= \operatorname{cosec}\theta = R.H.S$$

Q.4 Find the maximum and minimum of each of the following functions.

(i) $y = -2 + \frac{1}{2}\sin(\frac{1}{3}\theta+2)$

compare with

$y = a + b\sin(c\theta+d)$,

$a=-2$ & $b=1/2$

Maximum value $= a+|b|$

$= -2+|\frac{1}{2}|$

$= -2+\frac{1}{2}$

$= -3/2$

Minimum value $= a-|b|$

$= -2-|\frac{1}{2}|$

$= -2-\frac{1}{2}$

$= -5/2$

(ii) $y = 5-4\sin(\theta+30)$

$a=5$ & $b=-4$

Max $= a+|b|$

$= 5+|-4|$

$= 5+4$

$= 9$

Min $= a-|b|$

$= 5-|-4|$

$= 5-4$

$= 1$

(iii) $y = \dfrac{1}{19-10\sin(3\theta-45)}$ $\Rightarrow$ $y' = 19-10\sin(3\theta-45)$

Here $a=19$ $b=-10$

$\left.\begin{array}{l}\underline{Max} = M = 19+|-10| = 19+10 = 29 \\ Min = m = 19-|-10| = 19-10 = 9\end{array}\right\}$ for y'

Now for reciprocal

i.e $y = \dfrac{1}{19-10\sin(3\theta-45)}$

$\left.\begin{array}{l} MAX = M' = \frac{1}{m} = \frac{1}{9} \\ min = m' = \frac{1}{M} = \frac{1}{29}\end{array}\right\}$

(ii) $y = \dfrac{1}{4\cos 2\pi\theta}$

$\Rightarrow y' = 4\cos 2\pi\theta$

$\Rightarrow y' = 0 + 4\cos 2\pi\theta$ $\quad$ $(y' = a + b\cos n\theta)$ form

$a = 0 \qquad b = 4$

Max for y' $= M = a + |b|$
$= 0 + |4| = 4$

Min for y' $= m = a - |b|$
$= 0 - |4| = -4$

Now for reciprocal i.e $y = \dfrac{1}{4\cos 2\pi\theta}$

Since $m < 0$ & $M > 0$

$\Rightarrow$ Max for $y = M' = \dfrac{1}{M} = \dfrac{1}{4}$

Min for $y = m' = \dfrac{1}{m} = \dfrac{1}{-4}$

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Exercise # 12.5

Q:1 Find all the solutions of the trigonometric functions

(i) $\sin\theta = \dfrac{\sqrt{2}}{2}$

Sol

$\sin\theta = \dfrac{\sqrt{2}}{\sqrt{2}\sqrt{2}}$ $\qquad \because 2 = \sqrt{2}\sqrt{2}$

$\Rightarrow \sin\theta = \dfrac{1}{\sqrt{2}}$

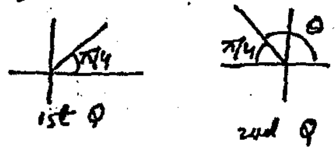
$\Rightarrow$ Either θ is in 1st quadrant or 2nd quadrant.

$\Rightarrow \theta = \pi/4$ or Acute angle is $\pi/4$

$\Rightarrow \theta = \dfrac{\pi}{4}$ for 1st Quadrant

and $\theta = \pi - \dfrac{\pi}{4}$ for 2nd "

$\theta = \dfrac{4\pi - \pi}{4} \Rightarrow \theta = \dfrac{3\pi}{4}$



So the general solution will be

$\theta = \left\{\dfrac{\pi}{4} + 2k\pi\right\} \cup \left\{3\dfrac{\pi}{4} + 2k\pi\right\}$ where $k \in \mathbb{Z}$

(ii) $\cos\theta = -\sqrt{3}/2$

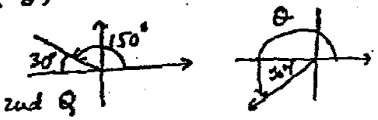
Sol θ will be in 2nd or 3rd quadrant $\qquad \cos 30^\circ = \sqrt{3}/2$

$\Rightarrow$ Since $\cos 30^\circ = \sqrt{3}/2$

$\Rightarrow$ Acute angle will be 30° $\left(\dfrac{\pi}{6}\right)$ in 2nd or 3rd quadrant

So the solution of $\cos\theta = -\sqrt{3}/2$

is $\theta = 180^\circ - 30^\circ = 150^\circ = 5\dfrac{\pi}{6}$ for 2nd Q



& $\theta = 180^\circ + 30^\circ = 210^\circ = 7\dfrac{\pi}{6}$ for 3rd Q

So the general solution will be $\left\{5\dfrac{\pi}{6} + 2k\pi\right\} \cup \left\{7\dfrac{\pi}{6} + 2k\pi\right\}$

Q.3 $\tan\theta = \sqrt{3}$

Sol $\tan\theta = \sqrt{3}$ implies that either θ is in 1st quadrant or 3rd quadrant.

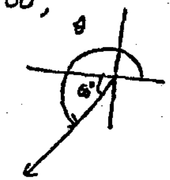
for 1st quadrant

$\tan\theta = \sqrt{3} \Rightarrow \theta = 60^\circ = \pi/3$

& for 3rd quadrant acute angle will be 60°, then θ is

$\theta = \pi + \frac{\pi}{3} \quad (180^\circ + 60^\circ = 240^\circ)$

$\theta = 4\frac{\pi}{3}$



Then the general solution will be

$\left\{\frac{\pi}{3} + k\pi\right\} \cup \left\{4\frac{\pi}{3} + k\pi\right\}$

Note Period of $\tan\theta$ is π

④ $\cos\theta = -1$

Sol $\Rightarrow \theta$ is along $-ve$ x-axis

$\Rightarrow \theta = \pi \quad (180^\circ)$

Then the general solution will be

$\{\pi + 2k\pi\}$

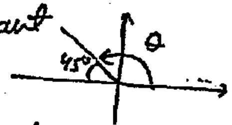
⑤ $\tan\theta = -1$

Sol $\Rightarrow \theta$ is in 2nd quadrant or 4th quadrant

$\tan\theta = -1$ $\quad \tan 45^\circ = 1$

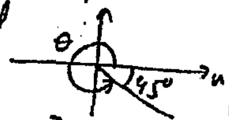
$\Rightarrow \theta = \pi - \frac{\pi}{4} \quad (180^\circ - 45^\circ)$ for 2nd quadrant

$\theta = \frac{3\pi}{4}$



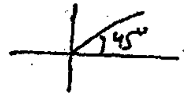
or $\theta = 2\pi - \frac{\pi}{4} \quad (360^\circ - 45^\circ)$ for 4th quadrant

$= 7\frac{\pi}{4}$



Then the general solution is $\left\{3\frac{\pi}{4} + k\pi\right\} \cup \left\{7\frac{\pi}{4} + k\pi\right\}$

⑥ $\cos\theta = \frac{\sqrt{2}}{2}$



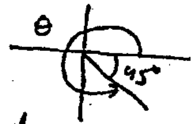
Sol $\cos\theta = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta$ is in 1st or 4th quadrant

$\Rightarrow \theta = 45^\circ = \frac{\pi}{4}$ for 1st quadrant

$\Rightarrow \theta = 360^\circ - 45^\circ = 2\pi - \frac{\pi}{4}$

$= 7\frac{\pi}{4}$ for 4th quadrant



So the general solution will be

$\theta = \left\{\frac{\pi}{4} + 2k\pi\right\} \cup \left\{7\frac{\pi}{4} + 2k\pi\right\}$

⑦ $\tan\theta = 0$

Sol $\tan\theta = 0$

means θ is 0° or 180°

$\Rightarrow \theta = 0$ or π

$\theta = 0 + k\pi$ or $\pi + k\pi$

$\Rightarrow \boxed{\theta = k\pi}$

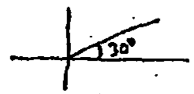
⑧ $\tan\theta = \frac{\sqrt{3}}{3}$

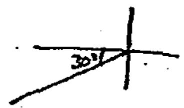
Sol $\tan\theta = \frac{\sqrt{3}}{\sqrt{3}\sqrt{3}}$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$ (θ will be in 1st or 3rd quadrant)

$\Rightarrow \theta = 30^\circ$ or 210°

$\theta = \pi/6$ or $7\pi/6$





S.S $= \left\{\frac{\pi}{6} + k\pi\right\} \cup \left\{7\frac{\pi}{6} + k\pi\right\}$

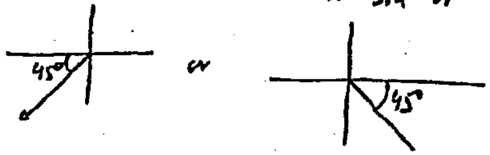
⑨ $\sin\theta = -\frac{\sqrt{2}}{2}$

Sol: $\Rightarrow \sin\theta = -\frac{\sqrt{2}}{\sqrt{2}\sqrt{2}}$

$\sin 45^\circ = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\theta = -\frac{1}{\sqrt{2}}$

$\Rightarrow$ Angle should be 45° in 3rd or 4th quadrant



$\Rightarrow \theta = 180^\circ + 45^\circ \ (\pi + \frac{\pi}{4})$ or $\theta = 360^\circ - 45^\circ \ (2\pi - \frac{\pi}{4})$

$\theta = 5\frac{\pi}{4}$ $\qquad$ $\theta = 7\frac{\pi}{4}$

Hence $S.S = \{5\frac{\pi}{4} + 2k\pi\} \cup \{7\frac{\pi}{4} + 2k\pi\}$

⑩ $\cos\theta = 0$

$\Rightarrow \theta$ is odd multiple of $\pi/2$

$\Rightarrow \boxed{\theta = (2n+1)\frac{\pi}{2}}$

Engr. Majid Amin

Q:- In the following questions use graph to estimate the solution of each question.

⑪ $2\sin\theta - \theta = 0$

$\Rightarrow 2\sin\theta = \theta$

Solution will be those numbers which satisfy both sides. For graphical solution we treat them two separate functions

$y_1 = 2\sin\theta$ & $y_2 = \theta$.

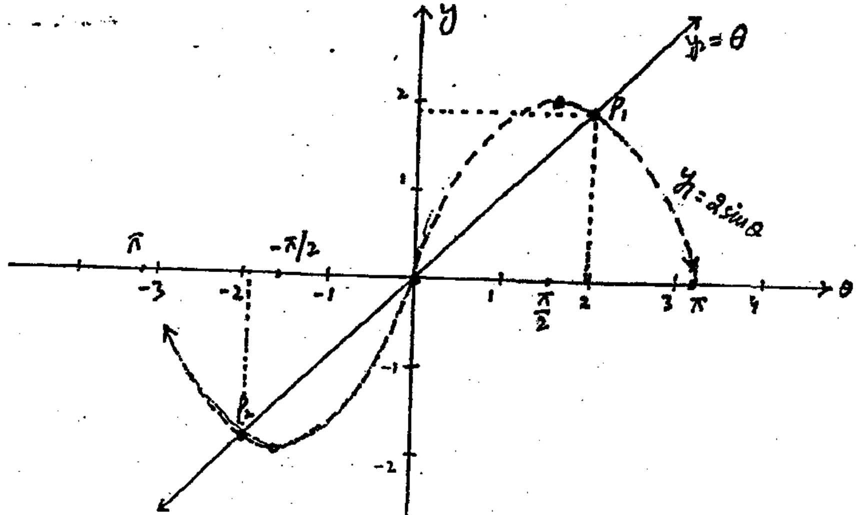
We will draw the graphs of the two functions and the intersection will be the solution

$y_1 = 2\sin\theta$

θ	$-\pi$	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$
y_1	0	-2	0	2	0	-2

$y_2 = \theta$

θ	-2	-1	0	1	2
y_2	-2	-1	0	1	2



Origin, P_1 and P_2 are points of intersection. origin = (0,0)

From diagram $P_1 = (1.9, 1.9)$ & $P_2 = (-1.9, -1.9)$

Hence $\theta = 1.9$ and -1.9 and 0.

(12) $\tan\theta = 2\theta$

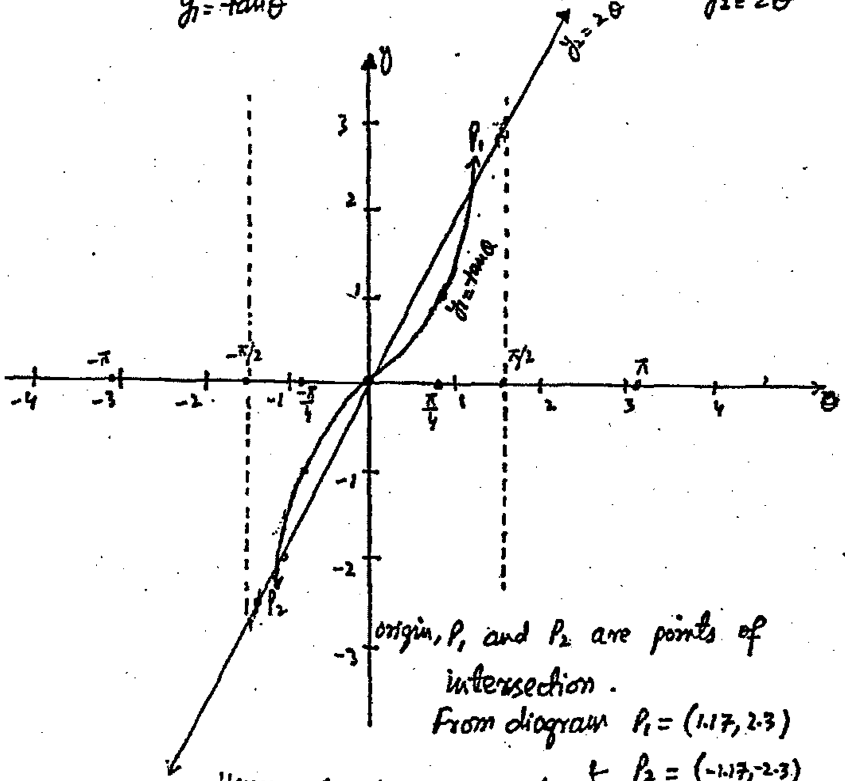
Sol: Let $y_1 = \tan\theta$ & $y_2 = 2\theta$

θ	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
y_1	$-\infty$	-1		1	∞

$y_1 = \tan\theta$

θ	-1	0	1
y_2	-2	0	2

$y_2 = 2\theta$



origin, P_1 and P_2 are points of intersection.

From diagram $P_1 = (1.17, 2.3)$ & $P_2 = (-1.17, -2.3)$

Hence $\theta = 0, 1.17, -1.17$ Ans

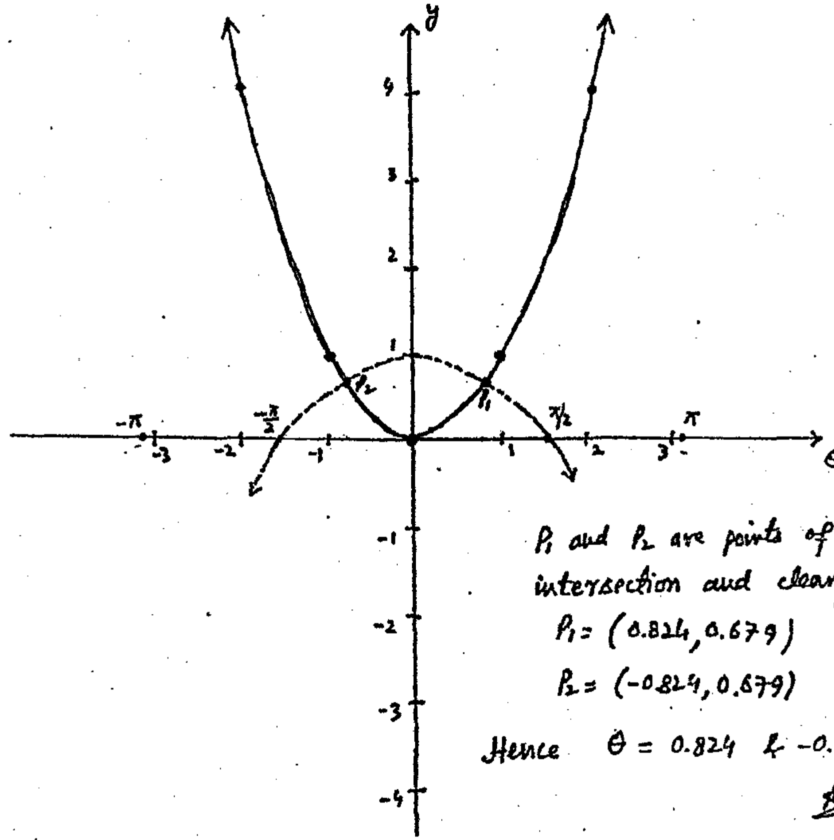
(13) $\cos\theta = \theta^2$

CH-12
P-08

Sol: Let $y_1 = \cos\theta$ and $y_2 = \theta^2$

θ	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
y_1	0	0.707	1	0.707	0

θ	-2	-1	0	1	2
y_2	4	1	0	1	4



P_1 and P_2 are points of intersection and clearly

$P_1 = (0.824, 0.679)$

$P_2 = (-0.824, 0.679)$

Hence $\theta = 0.824$ & -0.824

Ans

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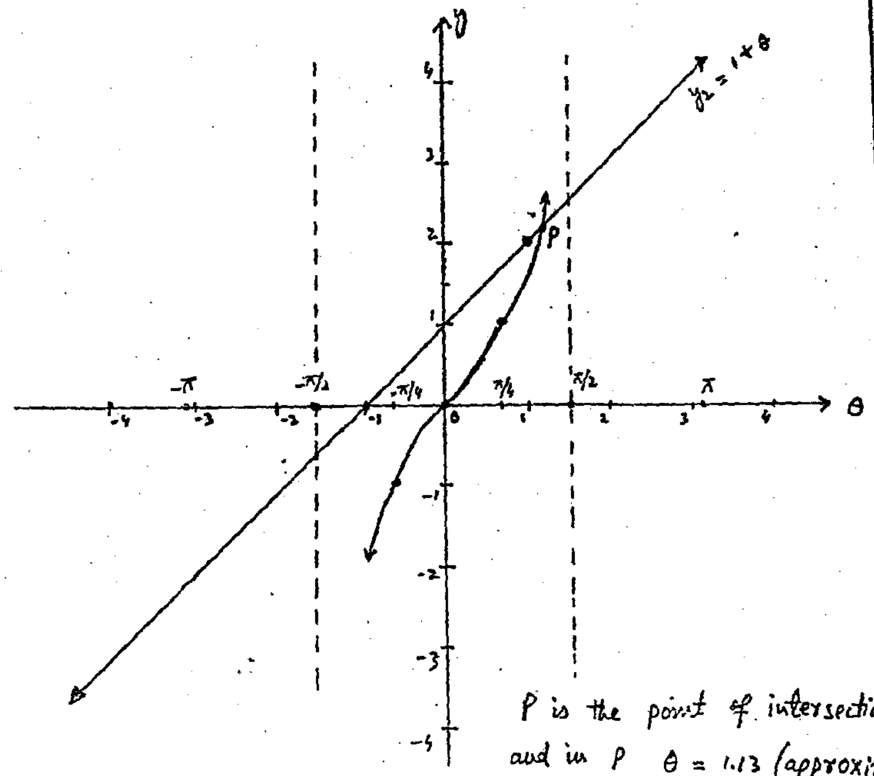
(14) $\tan\theta = 1+\theta$

Sol. let $y_1 = \tan\theta$ & $y_2 = 1+\theta$.

θ	$-\pi/2$	$-\pi/3$	$-\pi/4$	0	$\pi/4$	$\pi/3$	$\pi/2$
y_1	$-\infty$	-1.73	-1	0	1	1.73	∞

θ	-1	0	1
y_2	0	1	2



P is the point of intersection and in P $\theta = 1.13$ (approximately)

Exercise # 12.6

Q: Evaluate the following without using table or calculator.

(i) Arc $\sin(-1)$

Sol. let $x = \sin^{-1}(-1)$

$\Rightarrow \sin x = -1$ where $x \in \left[-\frac{\pi}{2}\ \frac{\pi}{2}\right]$

$\Rightarrow x = -\frac{\pi}{2}$ Ans

Note (i) $\sin x = y$ $\Rightarrow x = \sin^{-1}y$

(ii) Arc $\sin y$ means $\sin^{-1}y$

Arc $\sin(-1)$ is $\sin^{-1}(-1)$

(ii) Arc $\cos(-1)$

Sol. let $x =$ Arc $\cos(-1)$

$\Rightarrow x = \cos^{-1}(-1)$

$\Rightarrow \cos x = -1$ where $x \in [0\ \pi]$

$\Rightarrow x = \pi$ Ans

(iii) Arc $\tan(-1)$

Sol. let $x =$ Arc$\tan(-1)$

$\Rightarrow x = \tan^{-1}(-1)$

$\Rightarrow \tan x = -1$ and $x \in \left(-\frac{\pi}{2}\ \frac{\pi}{2}\right)$

$\Rightarrow x = -\pi/4$

(iv) Arc $\sin\left(\frac{1}{2}\right)$

Sol. let $x =$ Arc $\sin\left(\frac{1}{2}\right)$

$\Rightarrow x = \sin^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \sin x = \frac{1}{2}$ and $x \in \left[-\frac{\pi}{2}\ \frac{\pi}{2}\right]$

$\Rightarrow x = \pi/6$ Ans

(v) $\operatorname{Cosec}^{-1}(-\sqrt{2})$

Sol $\operatorname{cosec}^{-1}(-\sqrt{2}) = \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

Note
$\operatorname{Cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$
$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$
$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

let $x = \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$\Rightarrow \sin x = \frac{-1}{\sqrt{2}}$ and $x \in \left[\frac{-\pi}{2} \;\; \frac{\pi}{2}\right]$

$\Rightarrow x = -\pi/4$ Ans

(vi) $\operatorname{Arc\,Sec}\left(\frac{2}{\sqrt{3}}\right)$

Sol $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \cos^{-1}(\sqrt{3}/2)$

let $x = \cos^{-1}(\sqrt{3}/2)$

$\Rightarrow \cos x = \sqrt{3}/2$ and $x \in [0 \;\; \pi]$

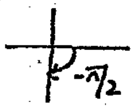
$\Rightarrow x = \pi/6$ Ans

Q:2 Evaluate the following inverse relations of general trigonometric functions.

(i) $\operatorname{arc}\sin(-1)$

Sol $\operatorname{arc}\sin(-1) = \sin^{-1}(-1)$

$= -\frac{\pi}{2}$ or $\frac{3\pi}{2}$



$= \frac{-\pi}{2} - 2k\pi$ or $\frac{3\pi}{2} + 2k\pi$

Hence $S.S = \left\{\frac{3\pi}{2} + 2k\pi\right\} \cup \left\{\frac{-\pi}{2} - 2k\pi\right\}$ where $k \in \mathbb{Z}$

(ii) $\operatorname{arc}\cos(1)$

$= \cos^{-1} 1$

$= 0 + 2k\pi$

Hence $S.Set = \{2k\pi\}$

(iii) $\operatorname{arc}\cos\left(-\frac{\sqrt{2}}{2}\right)$

Sol $\operatorname{arc}\cos\left(-\frac{\sqrt{2}}{2}\right) = \cos^{-1}\left(-\frac{\sqrt{2}}{\sqrt{2}\sqrt{2}}\right)$

$= \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

let $x = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$\Rightarrow \cos x = \frac{-1}{\sqrt{2}} \Rightarrow x$ will be in 2nd or 3rd quadrant

$\Rightarrow x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$

$\Rightarrow$ S.Set $= \left\{\frac{3\pi}{4} + 2k\pi\right\} \cup \left\{\frac{5\pi}{4} + 2k\pi\right\}$ Ans

(iv) $\operatorname{arc}\tan 0$

Sol $\operatorname{arc}\tan 0 = \tan^{-1} 0$

$= n\pi$ Ans

(v) $\operatorname{arc}\tan\left(-\frac{\sqrt{3}}{3}\right)$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

Sol $\operatorname{arc}\tan\left(-\frac{\sqrt{3}}{3}\right) = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

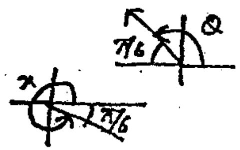
$= \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) \quad \because 3 = \sqrt{3}\sqrt{3}$

let $x = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \tan x = \frac{-1}{\sqrt{3}} \Rightarrow x$ is in 2nd or 4th quadrant

for 2nd quadrant $x = \pi - \frac{\pi}{6} = 5\pi/6$

For 4th quadrant $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$



$S.S = \left\{\frac{5\pi}{6} + 2k\pi\right\} \cup \left\{\frac{11\pi}{6} + 2k\pi\right\}$ A

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(vi) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

Sol $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

let $x = \cos^{-1}(-\sqrt{3}/2)$

$\Rightarrow \cos x = -\sqrt{3}/2 \Rightarrow x$ is in 2nd or 3rd quadrant

for 2nd quadrant $x = \pi - \frac{\pi}{6} = 5\pi/6$

for 3rd quadrant $x = \pi + \frac{\pi}{6} = 7\pi/6$

Hence $S.S = \left\{5\frac{\pi}{6} + 2k\pi\right\} \cup \left\{7\frac{\pi}{6} + 2k\pi\right\}$ Ans

Q:3 Use calculator to find the approximate measure in radians of the following inverse functions.

Sol

(i) $\sin^{-1} 0.1 = 0.10016$

(ii) $\cos^{-1} 0.6 = 0.92729$

(iii) $\tan^{-1} 5 = 1.3734$

(iv) $\tan^{-1} 0.2 = 0.19739$

(v) $\cos^{-1}(7/8) = 0.50536$

(vi) $\cos^{-1}(\sqrt{2}/3) = 1.0799$

Q:4 Find the exact value of each expression

(i) $\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

Sol $\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right) = \cos\left(\sin^{-1}\frac{\sqrt{2}}{\sqrt{2}\sqrt{2}}\right)$

$= \cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)$

$= \cos(\pi/4)$

$= 1/\sqrt{2}$ Ans

(ii) $\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$ $\qquad \cos 30^\circ = \sqrt{3}/2$

Sol $\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) = \tan(\pi/6)$

$= 1/\sqrt{3}$ Ans

(iii) $\sec\left(\cos^{-1}\frac{1}{2}\right)$ $\qquad \cos 60^\circ = 1/2$

Sol $\sec\left(\cos^{-1}\frac{1}{2}\right) = \sec(\pi/3)$

$= \frac{1}{\cos(\frac{\pi}{3})}$

$= \frac{1}{1/2}$

$= 2$ Ans

(iv) $\operatorname{cosec}(\tan^{-1} 1)$

Sol $\operatorname{cosec}(\tan^{-1} 1) = \operatorname{cosec}(\pi/4)$

$= \frac{1}{\sin(\pi/4)}$

$= \sqrt{2}$ Ans

(v) $\sin(\tan^{-1}(-1))$

Sol $\sin(\tan^{-1}(-1)) = \sin(-\pi/4) = -\sin\pi/4 = -\frac{1}{\sqrt{2}}$

(vi) $\sec\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$

Sol $\sec\left(\sin^{-1}\frac{1}{2}\right) = \sec(\pi/6) = \frac{1}{\cos(\pi/6)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$ Ans

Q:5 Compute the following expressions which involve principle as well as general trignometric functions and their inverses.

(i) $\sin(\tan^{-1}(\frac{1}{\sqrt{3}}))$

Sol Let $y = \sin(\tan^{-1}(\frac{1}{\sqrt{3}}))$

$y = \sin(\frac{\pi}{6} \text{ or } 7\frac{\pi}{6})$

$\Rightarrow y = \sin\frac{\pi}{6}$ or $y = \sin 7\frac{\pi}{6}$

$\Rightarrow y = \frac{1}{2}$ or $y = -1/2$

Hence $y = \{\frac{1}{2}, -\frac{1}{2}\}$ Ans

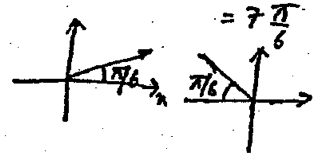
$\tan\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta$ is in 1st or 3rd quadrant

For 1st quadrant $\theta = \pi/6$

For 3rd quadrant $\theta = \pi + \pi/6$

$= 7\frac{\pi}{6}$



(ii) $\sin(\text{Tan}^{-1}\frac{1}{\sqrt{3}})$

Sol $\sin(\text{Tan}^{-1}\frac{1}{\sqrt{3}}) = \sin(\pi/6)$

$= \frac{1}{2}$ Ans

Note

$\text{Tan}^{-1}\frac{1}{\sqrt{3}}$ is principle tangent

$\tan^{-1}\frac{1}{\sqrt{3}}$ is general

(iii) $\sin(\text{arc}\cos(-\frac{\sqrt{3}}{2}))$

Sol Let $y = \sin(\text{arc}\cos(-\frac{\sqrt{3}}{2}))$

$y = \sin(\text{Cos}^{-1}(-\frac{\sqrt{3}}{2}))$

$y = \sin(5\frac{\pi}{6})$

$y = \frac{1}{2}$

(iv) $\text{Arc}\cos(\tan 3\frac{\pi}{4})$

Sol $\cos^{-1}(\tan 3\frac{\pi}{4})$

$= \cos^{-1}(-1)$

$= \pi$

General solution is $\{\pi + 2k\pi\}$

(v) $\text{Tan}^{-1}(\tan 3\frac{\pi}{4})$

Sol $\text{Tan}^{-1}(\tan 3\frac{\pi}{4})$

$= \text{Tan}^{-1}(-1)$

$= -\pi/4$

(vi) $\tan(\text{Arc}\cos(-\frac{4}{5}))$

Sol $\tan(\cos^{-1}(-\frac{4}{5}))$

$= \tan(2.49805)$

$= -0.75$

$= -3/4$ Ans

Engr. Majid Amin

Quote

Education is simply the soul of a society as it passes from one generation to another.

(G-K Chesterton 1874-1936)

Exercise # 12.7

Q:1 Find x, if

(i) $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{2} - x$

Sol $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{2} - x$

Take sin of both sides, we get

$\Rightarrow \sin(\sin^{-1}\frac{1}{2}) = \sin(\frac{\pi}{2} - x)$

$\Rightarrow \frac{1}{2} = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$

$\Rightarrow \frac{1}{2} = (1)\cos x - 0\sin x$

$\Rightarrow \frac{1}{2} = \cos x$

$\Rightarrow x = \cos^{-1}(\frac{1}{2})$

$\Rightarrow x = \pi/3$ Ans

Note

* $\sin^{-1}x$ shows Principle function
* $\sin(\sin^{-1}x) = x$
* $\cos\pi/2 = 0$
* $\sin\pi/2 = 1$

$\sin(\alpha-\beta)$
$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(ii) $\cos^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{2} - \sin^{-1}x$

Sol $\cos^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{2} - \sin^{-1}x$

$\Rightarrow \frac{\sqrt{3}}{2} = \cos(\frac{\pi}{2} - \sin^{-1}x)$

$\cos(\alpha-\beta)$
$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\Rightarrow \frac{\sqrt{3}}{2} = \cos\frac{\pi}{2}\cos(\sin^{-1}x) + \sin\frac{\pi}{2}\cdot\sin(\sin^{-1}x)$

$\Rightarrow \frac{\sqrt{3}}{2} = 0\cdot\cos(\sin^{-1}x) + 1\cdot x$

$\Rightarrow \frac{\sqrt{3}}{2} = 0 + x \Rightarrow \boxed{x = \sqrt{3}/2}$ Ans

(iii) $\sin^{-1}\frac{1}{2} = \frac{\pi}{2} - x$ where $\sin^{-1}\frac{1}{2}$ is inverse relation

Sol $\sin^{-1}\frac{1}{2} = \frac{\pi}{2} - x$

$\Rightarrow \frac{1}{2} = \sin(\frac{\pi}{2} - x)$

$\Rightarrow \frac{1}{2} = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$

$\Rightarrow \frac{1}{2} = 1\cdot\cos x - 0\cdot\sin x$

$\Rightarrow \frac{1}{2} = \cos x$

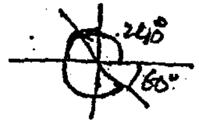
$\Rightarrow x = \cos^{-1}(\frac{1}{2})$

$\Rightarrow x = 60°$ for 1st quadrant

or

$x = 360° - 60° = 300°$ for 4th quadrant

$\cos 60° = 1/2$



$\Rightarrow x = \pi/3$ or $x = 5\frac{\pi}{3}$

$60° = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$

$300° = 300 \times \frac{\pi}{180} = \frac{5\pi}{3}$

$\Rightarrow x = \frac{\pi}{3} + 2k\pi$ or $x = 5\frac{\pi}{3} + 2k\pi$

Hence $x = \{\frac{\pi}{3} + 2k\pi\} \cup \{5\frac{\pi}{3} + 2k\pi\}$ Ans

Q:2 Show that

(i) $\sin^{-1}x + \cos^{-1}x = \pi/2$

Sol Let $\sin^{-1}x + \cos^{-1}x = \pi/2$

$\Rightarrow \sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x$

$\Rightarrow x = \sin(\frac{\pi}{2} - \cos^{-1}x)$

Apply $\sin(\alpha-\beta)$ formula

$\Rightarrow x = \sin\frac{\pi}{2}\cos(\cos^{-1}x) - \cos\frac{\pi}{2}\sin(\cos^{-1}x)$

$\Rightarrow x = 1\cdot x - 0\cdot\sin(\cos^{-1}x)$

$\Rightarrow x = x - 0$

$\Rightarrow x = x$ which is always true.

Hence $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ is also always true.

(ii) $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \pi/2$

Sol L.H.S $\tan^{-1}x + \tan^{-1}\frac{1}{x}$

By formula $\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$= \tan^{-1}\left(\frac{x+\frac{1}{x}}{1-x\cdot\frac{1}{x}}\right)$

$= \tan^{-1}\left(\frac{x+\frac{1}{x}}{1-1}\right)$

$= \tan^{-1}\left(\frac{x+\frac{1}{x}}{0}\right)$

$= \tan^{-1}\infty$

$= \pi/2 = R\cdot H\cdot S$

Q:3 show that $\sec(\text{Arc}\tan x) = \sqrt{1+x^2}$

L.H.S $\sec(\text{Arc}\tan x)$

$= \sec(\tan^{-1}x)$ let $\tan^{-1}x = \theta$

$= \sec\theta$ $\Rightarrow x = \tan\theta$

By formula $1+\tan^2\theta = \sec^2\theta$

$= \sqrt{1+\tan^2\theta}$

$= \sqrt{1+x^2}$ $= R.H.S$

Q:4 Show that $\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$

L.H.S $\tan(\sin^{-1}x)$ Let $\sin^{-1}x = \theta$

$= \tan\theta$ $\Rightarrow x = \sin\theta$

$= \frac{\sin\theta}{\cos\theta}$

$= \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{x}{\sqrt{1-x^2}} = R.H.S$

Q:5 Prove that $\tan(\text{Arc}\sec x) = \sqrt{x^2-1}$, $x \geq 1$ CH-12 P-11

Sol L.H.S $\tan(\text{Arc}\sec x)$

$= \tan(\sec^{-1}x)$ Let $\sec^{-1}x = \theta$

$x = \sec\theta$

$= \tan\theta$

$= \sqrt{\sec^2\theta - 1}$ Formula $1+\tan^2\theta = \sec^2\theta$

$= \sqrt{x^2-1}$

$= R.H.S$

Q:6 Evaluate

(i) $\sin\left(\frac{\pi}{2} - \cos^{-1}\frac{4}{5}\right)$

(ii) $\sin\left(\text{Arc}\cos\frac{\pi}{2} + \pi\right)$

Sol $\sin\left(\frac{\pi}{2} - \cos^{-1}\frac{4}{5}\right)$

Apply $\sin(\alpha-\beta)$ formula

$= \sin\frac{\pi}{2}\cos\left(\cos^{-1}\frac{4}{5}\right) - \cos\frac{\pi}{2}\sin\left(\cos^{-1}\frac{4}{5}\right)$

$= 1\cdot\frac{4}{5} - 0.\sin\left(\cos^{-1}\frac{4}{5}\right)$

$= \frac{4}{5} - 0$

$= 4/5$

(ii) $= \sin\left(\cos^{-1}\frac{\pi}{2} + \pi\right)$

Since $\cos^{-1}\frac{\pi}{2} = \infty$

Hence answer is not possible

Q:7 Show that $\cos(\sin^{-1}x - \sin^{-1}y) = \sqrt{(1-x^2)(1-y^2)} + xy$

L.H.S $\cos(\sin^{-1}x - \sin^{-1}y)$ Let $\sin^{-1}x = \alpha \Rightarrow x = \sin\alpha$

& $\sin^{-1}y = \beta \Rightarrow y = \sin\beta$

$= \cos(\alpha - \beta)$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$ $\cos^2\theta + \sin^2\theta = 1$

$= \sqrt{1-\sin^2\alpha}\sqrt{1-\sin^2\beta} + \sin\alpha\sin\beta$ $\cos\theta = \sqrt{1-\sin^2\theta}$

Putting the value of . . . &

$= \sqrt{1-x^2}\,\sqrt{1-y^2} + xy$

$= \sqrt{(1-x^2)(1-y^2)} + xy$

$=$ R.H.S

Q:8 Show that

(i) $\cos(2\sin^{-1}x) = 1-2x^2 \qquad -1 \le x \le +1$

Sol L.H.S $\cos(2\sin^{-1}x)$ let $\sin^{-1}x = \theta$

$= \cos 2\theta$ $x = \sin\theta$

By double angle identity

$= \cos^2\theta - \sin^2\theta$

$= (1-\sin^2\theta) - \sin^2\theta$

$= 1 - 2\sin^2\theta$

Put the value of $\sin\theta$

$= 1 - 2x^2 =$ R.H.S

(ii) $2\,\text{Arc}\cos x = \text{Arc}\cos(2x^2-1) \qquad 0 \le x \le 1$

L.H.S $2\,\text{Arc}\cos x$ let $\cos^{-1}x = \theta$

$= 2\cos^{-1}x$ $\Rightarrow x = \cos\theta$

$\Rightarrow 2\cos^{-1}x = 2\theta \longrightarrow$ (i)

Given eqn is

$2\cos^{-1}x = \cos^{-1}(2x^2-1)$

$\Rightarrow \cos(2\cos^{-1}x) = 2x^2-1 \qquad (\because 2\cos^{-1}x = 2\theta)$

$\Rightarrow \cos 2\theta = 2x^2-1$

$\Rightarrow \cos^2\theta - \sin^2\theta = 2x^2-1$

By $\sin^2\theta = 1-\cos^2\theta$

$\Rightarrow \cos^2\theta - (1-\cos^2\theta) = 2x^2-1$

$\Rightarrow \cos^2\theta - 1 + \cos^2\theta = 2x^2-1$

$\Rightarrow 2\cos^2\theta - 1 = 2x^2-1$

$\Rightarrow 2x^2 - 1 = 2x^2-1$ which is always true.

Hence proved

i.e $2\cos^{-1}x = \cos^{-1}(2x^2-1)$.

(iii) $\cos(\text{arc}\tan x) = \dfrac{1}{\sqrt{1+x^2}}$ for $x \ge 0$

L.H.S

$\cos(\text{arc}\tan x)$

$= \cos(\tan^{-1}x)$ Let $\tan^{-1}x = \theta$

  $\Rightarrow x = \tan\theta$

$= \cos\theta$

$= \dfrac{1}{\sec\theta}$ But $\sec^2\theta = 1+\tan^2\theta$

  $\Rightarrow \sec\theta = \sqrt{1+\tan^2\theta}$

$= \dfrac{1}{\sqrt{1+\tan^2\theta}}$

$= \dfrac{1}{\sqrt{1+x^2}} =$ R.H.S

Q.9 Evaluate the following expressions without using calculator or table

(i) $\tan[\text{arc}\sec(-3)]$

Sol $\tan(\sec^{-1}(-3))$ Let $\sec^{-1}(-3) = \theta$

$= \tan\theta$ $\Rightarrow -3 = \sec\theta$

$= \dfrac{\sin\theta}{\cos\theta}$ $\Rightarrow -\frac{1}{3} = \cos\theta$

$$= \frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$$

$$= \frac{\sqrt{1-\left(\frac{-1}{3}\right)^2}}{-\frac{1}{3}} = \frac{\sqrt{1-\frac{1}{9}}}{-\frac{1}{3}} = \frac{\sqrt{8/9}}{-1/3} = \frac{2\sqrt{2}/3}{-1/3}$$

$$= -2\sqrt{2} \text{ Ans}$$

(ii) $\cos(\text{arc}\tan(\frac{-3}{4}))$

Sol $\cos(\tan^{-1}\frac{-3}{4})$ Let $\tan^{-1}(\frac{-3}{4}) = \theta$

$$= \cos\theta \qquad \frac{-3}{4} = \tan\theta$$

$$= \frac{1}{\sec\theta}$$

Formula

$$\sec^2\theta = 1+\tan^2\theta$$

$$= \frac{1}{\sqrt{1+\tan^2\theta}}$$

$$= \frac{1}{\sqrt{1+\left(\frac{-3}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{1+\frac{9}{16}}}$$

$$= \frac{1}{\sqrt{\frac{16+9}{16}}} = \frac{1}{\sqrt{25/16}} = \frac{1}{5/4} = 4/5 \text{ Ans}$$

(iii) $\sin(\sin^{-1}\frac{4}{5} - \cos^{-1}\frac{3}{5})$

CH-12
P-12

Sol $\sin(\sin^{-1}\frac{4}{5} - \cos^{-1}\frac{3}{5})$ Let $\sin^{-1}\frac{4}{5} = \alpha \Rightarrow \frac{4}{5} = \sin\alpha$

$= \sin(\alpha - \beta)$ & $\cos^{-1}\frac{3}{5} = \beta \Rightarrow \frac{3}{5} = \cos\beta$

By formula

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \sin\alpha\cos\beta - \sqrt{1-\sin^2\alpha}\sqrt{1-\cos^2\beta}$$

$$= \left(\frac{4}{5}\right)\left(\frac{3}{5}\right) - \sqrt{1-\left(\frac{4}{5}\right)^2}\sqrt{1-\left(\frac{3}{5}\right)^2}$$

$$= \frac{12}{25} - \sqrt{1-\frac{16}{25}}\sqrt{1-\frac{9}{25}}$$

$$= \frac{12}{25} - \sqrt{\frac{25-16}{25}}\sqrt{\frac{25-9}{25}}$$

$$= \frac{12}{25} - \sqrt{\frac{9}{25}}\sqrt{\frac{16}{25}}$$

$$= \frac{12}{25} - \frac{3}{5}\cdot\frac{4}{5}$$

$$= \frac{12}{25} - \frac{12}{25}$$

$$= 0 \text{ Ans}$$

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Q:10 Express the following in terms of $\tan^{-1}x$

(i) $\sin^{-1}x$

Sol Let $\sin^{-1}x = \theta$

$\Rightarrow x = \sin\theta$

Also $\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \cos\theta = \sqrt{1-\sin^2\theta}$

$\cos\theta = \sqrt{1-x^2}$

Now $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow \tan\theta = \dfrac{x}{\sqrt{1-x^2}}$

$\Rightarrow \theta = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

$\Rightarrow \sin^{-1}x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

(ii) $\cos^{-1}x$

Sol Let $\cos^{-1}x = \theta$

$\Rightarrow x = \cos\theta$

Also $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \sin\theta = \sqrt{1-\cos^2\theta}$

$\Rightarrow \sin\theta = \sqrt{1-x^2}$

Now $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow \tan\theta = \dfrac{\sqrt{1-x^2}}{x}$

$\Rightarrow \theta = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

$\Rightarrow \cos^{-1}x = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

(iii) Arc $\cot x$

Sol $\cot^{-1}x$

Let $\cot^{-1}x = \theta \Rightarrow \cot\theta = x$

$\Rightarrow \dfrac{1}{\tan\theta} = x$

$\Rightarrow \tan\theta = \dfrac{1}{x}$

$\Rightarrow \theta = \tan^{-1}\dfrac{1}{x}$

$\Rightarrow \cot^{-1}x = \tan^{-1}\dfrac{1}{x}$

Engr. Majid Amin

Q:11 verify that

(i) $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{7}\right) = \pi/4$

L.H.S $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{7}\right)$

Apply $\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$ formula

$= \tan^{-1}\left(\dfrac{\frac{1}{2}+\frac{1}{2}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}\right) + \tan^{-1}\left(-\frac{1}{7}\right)$

$= \tan^{-1}\left(\dfrac{1}{1-\frac{1}{4}}\right) + \tan^{-1}\left(-\frac{1}{7}\right)$

$= \tan^{-1}\left(\dfrac{1}{3/4}\right) + \tan^{-1}\left(-\frac{1}{7}\right)$

$= \tan^{-1}(4/3) + \tan^{-1}(-1/7)$

Again apply the formula

$= \tan^{-1}\left\{\dfrac{\frac{4}{3}+\frac{-1}{7}}{1-\left(\frac{4}{3}\right)\left(\frac{-1}{7}\right)}\right\}$

$= \tan^{-1}\left\{\dfrac{\frac{28-3}{21}}{\frac{21+4}{21}}\right\} = \tan^{-1}\left(\dfrac{25/21}{25/21}\right) = \tan^{-1} 1 = \pi/4$

$= R.H.S$

(ii) $\sin^{-1}\left(\frac{77}{85}\right) - \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{15}{17}\right)$

Sol $\sin^{-1}\left(\frac{77}{85}\right) - \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{15}{17}\right)$

is actually

$\cos\left\{\sin^{-1}\left(\frac{77}{85}\right) - \sin^{-1}\left(\frac{3}{5}\right)\right\} = 15/17$

L.H.S $\cos\left\{\sin^{-1}\frac{77}{85} - \sin^{-1}\frac{3}{5}\right\}$ let $\sin^{-1}\frac{77}{85} = \alpha \Rightarrow \frac{77}{85} = \sin\alpha$

$= \cos(\alpha - \beta)$ and $\sin^{-1}\frac{3}{5} = \beta \Rightarrow \frac{3}{5} = \sin\beta$

By formula

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \sqrt{1-\sin^2\alpha}\,\sqrt{1-\sin^2\beta} + \sin\alpha\sin\beta$

$= \sqrt{1-\left(\frac{77}{85}\right)^2}\,\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{77}{85}\cdot\frac{3}{5}$

$= \sqrt{1-\frac{(77)^2}{(85)^2}}\,\sqrt{1-\frac{3^2}{5^2}} + \frac{231}{425}$

$= \sqrt{\frac{(85)^2-(77)^2}{(85)^2}}\,\sqrt{\frac{5^2-3^2}{5^2}} + \frac{231}{425}$

$= \frac{\sqrt{7225-5929}}{85}\,\frac{\sqrt{25-9}}{5} + \frac{231}{425}$

$= \frac{\sqrt{1296}}{85}\,\frac{\sqrt{16}}{5} + \frac{231}{425}$

$= \frac{36}{85}\cdot\frac{4}{5} + \frac{231}{425}$

$= \frac{144}{425} + \frac{231}{425} = \frac{144+231}{425} = \frac{375}{425} = \frac{15}{17}$

Hence $\cos(\alpha-\beta) = \frac{15}{17}$.

$\Rightarrow \alpha - \beta = \cos^{-1}\frac{15}{17}$

P.T.V of α & β

$\Rightarrow \sin^{-1}\frac{77}{85} - \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{15}{17}$.

Q:12 Express $\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{11}\right)$ as single inverse tangent.

Sol $\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{11}\right)$

$= \tan^{-1} 1 - \tan^{-1}\frac{1}{11}$ $\quad \because \frac{\pi}{4} = \tan^{-1} 1$

By formula

$= \tan^{-1}\left(\dfrac{1 - \frac{1}{11}}{1 + (1)(\frac{1}{11})}\right)$

Written by Engr. Majid Amin

$= \tan^{-1}\left(\dfrac{\frac{11-1}{11}}{\frac{11+1}{11}}\right)$

$= \tan^{-1}\left(\frac{10}{11} \Big/ \frac{12}{11}\right)$

$= \tan^{-1}\left(\frac{10}{12}\right)$

$= \tan^{-1}\left(5/6\right)$.

which is a single term of $\tan^{-1}$.

Quote:

Life isn't simple. But the beauty of it is, you can always start over. It will get easier.

(Alain Bessette)

Available at
www.mathcity.org

MathCity.org
Merging Man and maths

Exercise # 12.8

Q:1 $\sin 2\theta = \frac{1}{2}$ $\Rightarrow$ 2θ is in 1st or 2nd quadrant.

Sol $\sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \sin^{-1}\frac{1}{2}$

$\Rightarrow 2\theta = \frac{\pi}{6} + 2k\pi$ or $\frac{5\pi}{6} + 2k\pi$
(for 1st Q)

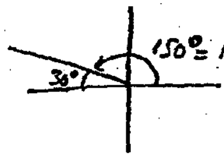
$\Rightarrow \theta = \frac{\pi}{12} + k\pi$ or $\frac{5\pi}{12} + k\pi$

Hence $S.S = \left\{\frac{\pi}{12} + k\pi\right\} \cup \left\{\frac{5\pi}{12} + k\pi\right\}$

Q:2 $\tan\theta = -\frac{1}{\sqrt{3}}$ $\Rightarrow \theta$ is in 2nd or 3rd quadrant

Sol $\theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ $\quad \tan 30^\circ = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta = 5\frac{\pi}{6}$ or $11\frac{\pi}{6}$

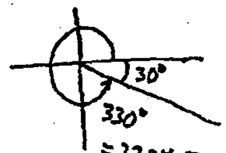


$150^\circ = 150 \times \frac{\pi}{180}$

$= \frac{5\pi}{6}$

Now add period

$\Rightarrow \theta = \left\{5\frac{\pi}{6} + k\pi\right\} \cup \left\{11\frac{\pi}{6} + k\pi\right\}$



$= 330 \times \frac{\pi}{180} = 11\frac{\pi}{6}$

Q:3. $\cos\theta = -\sqrt{3}/2$

Sol θ will be in 2nd or 3rd quadrant

$\theta = \cos^{-1}(-\sqrt{3}/2)$ $\quad \cos 30^\circ = \sqrt{3}/2$

$\theta = 5\frac{\pi}{6}$ or $7\frac{\pi}{6}$

$\theta = 5\frac{\pi}{6} + 2k\pi$ or $7\frac{\pi}{6} + 2k\pi$

$S.S = \left\{5\frac{\pi}{6} + 2k\pi\right\} \cup \left\{7\frac{\pi}{6} + 2k\pi\right\}$

Q.4 $\cos\left(2\theta - \frac{\pi}{2}\right) = -1$

Sol Apply $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ formula

$\Rightarrow \cos 2\theta \cos\frac{\pi}{2} + \sin 2\theta \sin\left(\frac{\pi}{2}\right) = -1$

$\Rightarrow \cos 2\theta\,(0) + \sin 2\theta\,(1) = -1$

$\Rightarrow \sin 2\theta = -1$

$\Rightarrow 2\theta = \sin^{-1}(-1)$

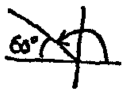
$\Rightarrow 2\theta = \frac{3\pi}{2} + 2k\pi$

$\Rightarrow \boxed{\theta = \frac{3\pi}{4} + k\pi}$ Ans

Q.5 $\sec\frac{3\theta}{2} = -2$

Sol $\frac{3\theta}{2} = \sec^{-1}(-2)$

$\Rightarrow \frac{3\theta}{2} = \cos^{-1}\left(-\frac{1}{2}\right)$ $\cos 60^\circ = 1/2$



$\Rightarrow \frac{3\theta}{2} = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

$\Rightarrow \frac{3\theta}{2} = \frac{2\pi}{3} + 2k\pi$ or $\frac{4\pi}{3} + 2k\pi$

$\Rightarrow 3\theta = \frac{4\pi}{3} + 4k\pi$ or $\frac{8\pi}{3} + 4k\pi$

$\Rightarrow \theta = \frac{4\pi}{9} + \frac{4k\pi}{3}$ or $\frac{8\pi}{9} + \frac{4k\pi}{3}$

Hence $S.S = \left\{\frac{4\pi}{9} + \frac{4}{3}k\pi\right\} \cup \left\{\frac{8\pi}{9} + \frac{4}{3}k\pi\right\}$ Ans

Q.6 $4\cos^2 x - 1 = 0$

Sol $\Rightarrow 4\cos^2 x = 1$

$\cos^2 x = 1/4$

$\Rightarrow \cos x = \pm 1/2$

Either $\cos x = 1/2$ or $\cos x = -1/2$

$\Rightarrow x = \cos^{-1}\left(\frac{1}{2}\right)$ or $x = \cos^{-1}\left(-\frac{1}{2}\right)$

$\Rightarrow x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ or $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

$\Rightarrow x = \frac{\pi}{3} + 2k\pi$ or $\frac{5\pi}{3} + 2k\pi$ or $x = \frac{2\pi}{3} + 2k\pi$ or $\frac{4\pi}{3} + 2k\pi$

$S.S = \left\{\frac{\pi}{3} + 2k\pi\right\} \cup \left\{\frac{5\pi}{3} + 2k\pi\right\} \cup \left\{\frac{2\pi}{3} + 2k\pi\right\} \cup \left\{\frac{4\pi}{3} + 2k\pi\right\}$

Q: Solve each eqn in problem 7–10. Use exact values in the given interval.

⑦ $(\sin x)(\cos x) = 0$ $0 \le x \le 2\pi$

Sol Either $\sin x = 0$ or $\cos x = 0$

$\Rightarrow x = \sin^{-1}(0)$ or $x = \cos^{-1}(0)$

$\Rightarrow x = 0$ or π or $x = \pi/2,\ 3\pi/2$

Hence $S.S = \left\{0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}\right\}$

⑧ $(\sin x)(\cot x) = 0$ $0 \le x \le 2\pi$

Sol $(\sin x)\left(\frac{\cos x}{\sin x}\right) = 0$

$\Rightarrow \cos x = 0$

$\Rightarrow x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ Ans

⑨ $(\sec x - 2)(2\sin x - 1) = 0$ $\quad 0 \le x \le 2\pi$

Sol Either $\sec x - 2 = 0$ or $2\sin x - 1 = 0$

$\Rightarrow \sec x = 2$ $\quad \Rightarrow 2\sin x = 1$

$\Rightarrow \cos x = 1/2$ $\quad \Rightarrow \sin x = 1/2$

$\Rightarrow x = \frac{\pi}{3}, 5\frac{\pi}{3}$ $\quad \Rightarrow x = \frac{\pi}{6}, 2\frac{\pi}{3}$

$S.S = \{\frac{\pi}{3}, 5\frac{\pi}{3}, \frac{\pi}{6}, 2\frac{\pi}{3}\}$ Ans

⑩ $(\operatorname{cosec} x - 2)(2\cos x - 1) = 0$ $\quad 0 \le x \le 2\pi$

Sol Either $\operatorname{cosec} x - 2 = 0$ or $2\cos x - 1 = 0$

$\Rightarrow \operatorname{cosec} x = 2$ $\quad 2\cos x = 1$

$\Rightarrow \sin x = 1/2$ $\quad \cos x = 1/2$

$\Rightarrow x = \frac{\pi}{6}, 5\frac{\pi}{6}$ $\quad x = \frac{\pi}{3}, 5\frac{\pi}{3}$

$S.S = \{\frac{\pi}{6}, 5\frac{\pi}{6}, \frac{\pi}{3}, 5\frac{\pi}{3}\}$ Ans

Q: Use the trignometric identities to solve problem (11-16) giving the general solutions.

⑪ $\cos\theta = \sin\theta$

Sol Divide b.s by $\cos\theta$

$\Rightarrow \frac{\cos\theta}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow 1 = \tan\theta$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \{\pi/4, 5\frac{\pi}{4}\}$ Ans

⑫ $\tan\theta = 2\sin\theta$

Sol $\frac{\sin\theta}{\cos\theta} = 2\sin\theta$

$\Rightarrow \frac{\sin\theta}{\cos\theta} - 2\sin\theta = 0$ take $\sin\theta$ as common

$\Rightarrow \sin\theta\{\frac{1}{\cos\theta} - 2\} = 0$

$\Rightarrow \sin\theta = 0$ or $\frac{1}{\cos\theta} - 2 = 0$

$\Rightarrow \theta = 0$ or π $\quad \frac{1}{\cos\theta} = 2$

$\Rightarrow \cos\theta = 1/2$

$\Rightarrow \theta = \pi/3$ or $5\pi/3$

Hence $\theta = \{0, \pi, \frac{\pi}{3}, 5\frac{\pi}{3}\}$ Ans

⑬ $\sin\theta = \operatorname{cosec}\theta$

Sol $\sin\theta = \operatorname{cosec}\theta$

$\Rightarrow \sin\theta = \frac{1}{\sin\theta}$

$\Rightarrow \sin^2\theta = 1$ take square root of b.S

$\Rightarrow \sin\theta = \pm 1$

$\Rightarrow \sin\theta = 1$ or $\sin\theta = -1$

$\Rightarrow \theta = \pi/2$ or $\theta = 3\pi/2$

$S.S = \{\frac{\pi}{2}, 3\frac{\pi}{2}\}$ Ans

(14) $4\cos^2\frac{\theta}{2} - 3 = 0$

Sol $4\cos^2\frac{\theta}{2} = 3$

$\Rightarrow \cos^2\frac{\theta}{2} = 3/4$; take square root, we get

$\Rightarrow \cos\frac{\theta}{2} = \pm\frac{\sqrt{3}}{2}$

$\Rightarrow \cos\frac{\theta}{2} = \frac{\sqrt{3}}{2}$ or $\cos\frac{\theta}{2} = -\frac{\sqrt{3}}{2}$.

$\Rightarrow \frac{\theta}{2} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ or $\frac{\theta}{2} = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \frac{\theta}{2} = \frac{\pi}{6}$ or $\frac{11\pi}{6}$ or $\frac{\theta}{2} = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$

$\Rightarrow \theta = \frac{\pi}{3}$ or $\frac{11\pi}{3}$ or $\theta = \frac{5\pi}{3}$ or $\frac{7\pi}{3}$

S.S $= \left\{\frac{\pi}{3}, \frac{11\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}\right\}$ Ans

(15) $\cos 2\theta = \cos\theta$

Sol $\cos 2\theta = \cos\theta$

$\Rightarrow \cos 2\theta - \cos\theta = 0$

Double angle identity
$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\Rightarrow (\cos^2\theta - \sin^2\theta) - \cos\theta = 0$

$\Rightarrow \cos^2\theta - (1-\cos^2\theta) - \cos\theta = 0$

$\Rightarrow \cos^2\theta - 1 + \cos^2\theta - \cos\theta = 0$

$\Rightarrow 2\cos^2\theta - 1\cos\theta - 1 = 0$

By quadratic formula

$\cos\theta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-1)}}{2(2)}$

$\Rightarrow \cos\theta = \frac{1 \pm \sqrt{1+8}}{4}$

$\Rightarrow \cos\theta = \frac{1\pm 3}{4}$

$\Rightarrow \cos\theta = \frac{1+3}{4}$ or $\cos\theta = \frac{1-3}{4}$

$\Rightarrow \cos\theta = 1$ or $\cos\theta = -1/2$

$\Rightarrow \theta = 2K\pi$ or $\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

Hence general solution is

$\theta = \{2K\pi\} \cup \left\{\frac{2\pi}{3} + 2K\pi\right\} \cup \left\{\frac{4\pi}{3} + 2K\pi\right\}$ where $K \in \mathbb{Z}$

Q:16 $\sin 2\theta + \sin\theta = 0$

Sol $\sin 2\theta + \sin\theta = 0$

$\Rightarrow 2\sin\theta\cos\theta + \sin\theta = 0$ $\because$ double angle identity $\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow \sin\theta\{2\cos\theta + 1\} = 0$

Either $\sin\theta = 0$ or $2\cos\theta + 1 = 0$

$\Rightarrow \theta = K\pi$ or $\cos\theta = -1/2$

$\Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

So the general solution will be

$\{K\pi\} \cup \left\{\frac{2\pi}{3} + 2K\pi\right\} \cup \left\{\frac{4\pi}{3} + 2K\pi\right\}$ where $K \in \mathbb{Z}$

Q: Use quadratic formula or factorization to solve the problems (17–24).

(17) $2\sin^2 x - 3\sin x + 1 = 0$

Sol Let $\sin x = t$

$\Rightarrow 2t^2 - 3t + 1 = 0$

By factorization

$\Rightarrow 2t^2 - 2t - t + 1 = 0$

$\Rightarrow 2t(t-1) - 1(t-1) = 0$

$\Rightarrow (t-1)(2t-1) = 0$

$\Rightarrow$ Either $(t-1)=0$ or $(2t-1)=0$

$\Rightarrow t=1$ $\qquad$ $2t=1$

$\Rightarrow t = 1/2$

Now $\sin x = t$ (Put the value of t)

$\Rightarrow \sin x = 1$ or $\sin x = 1/2$

$\Rightarrow x = \pi/2$, $x = \frac{\pi}{6}$ or $5\frac{\pi}{6}$

Hence $x = \left\{\frac{\pi}{2}+2k\pi\right\} \cup \left\{\frac{\pi}{6}+2k\pi\right\} \cup \left\{5\frac{\pi}{6}+2k\pi\right\}$ Ans

where $k \in \mathbb{Z}$

$\boxed{Q:18}$ $3\cos x + 3 = 2\sin^2 x$

Sol $3\cos x + 3 = 2(1-\cos^2 x)$ $\quad \because \sin^2 x + \cos^2 x = 1$

$\Rightarrow 3\cos x + 3 = 2 - 2\cos^2 x$

$\Rightarrow 2\cos^2 x + 3\cos x + 1 = 0$

$\Rightarrow 2\cos^2 x + 2\cos x + 1\cos x + 1 = 0$

$\Rightarrow 2\cos x(\cos x + 1) + 1(\cos x + 1) = 0$

$\Rightarrow (\cos x + 1) \times (2\cos x + 1) = 0$

$\Rightarrow$ Either $\cos x + 1 = 0$ or $2\cos x + 1 = 0$

$\Rightarrow \cos x = -1$, $\cos x = -1/2$

$\Rightarrow x = \pi$ $\qquad$ $\Rightarrow x = 2\pi/3,\ 4\frac{\pi}{3}$

$\Rightarrow x = \pi + 2k\pi$ $\qquad$ $\Rightarrow x = 2\frac{\pi}{3}+2k\pi,\ 4\frac{\pi}{3}+2k\pi$

$\Rightarrow x = (2k+1)\pi$

S.S $= \{(2k+1)\pi\} \cup \left\{2\frac{\pi}{3}+2k\pi\right\} \cup \left\{4\frac{\pi}{3}+2k\pi\right\}$ Ans

$\boxed{Q:19}$ $\cos^2 x \cdot \sin x = 2$

Sol $\cos^2 x \cdot \sin x = 2$

$\Rightarrow (1-\sin^2 x)\sin x = 2$

$\Rightarrow \sin x - \sin^3 x = 2$

$\Rightarrow -\sin^3 x + \sin x - 2 = 0$ $\quad$ xing by -1, we get

$\Rightarrow \sin^3 x - \sin x + 2 = 0$

$\Rightarrow \sin^3 x - \sin x = -2$

Now this eqn does not have a real solution Because the minimum value of $\sin x$ is -1 when $x = \frac{3\pi}{2}$

Try $x = 3\pi/2$

$\left(\sin 3\frac{\pi}{2}\right)^3 - \left(\sin 3\frac{\pi}{2}\right) = -2$

$\Rightarrow (-1)^3 - (-1) = -2$

$\Rightarrow -1 + 1 = -2$

$\Rightarrow 0 = -2$ (which is false)

Note
$\sin^3 x + \sin x = -2$
has one solution
$x = 3\pi/2$

Hence S.S of $\sin^3 x - \sin x = -2$ is $\{\ \}$.

$\boxed{Q:20}$ $\cos^2 x - \sin^2 x = \sin x$

Sol As $\cos^2 x = 1 - \sin^2 x$

$\Rightarrow (1-\sin^2 x) - \sin^2 x = \sin x$

$\Rightarrow 1 - 2\sin^2 x = \sin x$

$\Rightarrow -2\sin^2 x - \sin x + 1 = 0$ $\quad$ xing by -1

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$

Let $\sin x = t$

$\Rightarrow 2t^2 + t - 1 = 0$

By factorization

$\Rightarrow 2t^2 + 2t - t - 1 = 0$

$\Rightarrow 2t(t+1) - 1(t+1) = 0$

$\Rightarrow (t+1)(2t-1) = 0$

Either $t+1=0$ or $2t-1=0$

$t=-1$ $\quad t = 1/2$

Now since $\sin x = t$

$\Rightarrow \sin x = -1$ & $\sin x = 1/2$

$x = \frac{3\pi}{2}$, $x = \frac{\pi}{6}$ or $5\frac{\pi}{6}$

General solution is

$\{\frac{3\pi}{2} + 2k\pi\} \cup \{\frac{\pi}{6} + 2k\pi\} \cup \{\frac{5\pi}{6} + 2k\pi\}$ Ans

Q:21 $\cos 2x + \cos x + 1 = 0$

Sol $(\cos^2 x - \sin^2 x) + \cos x + 1 = 0$ $\quad \because \cos 2x = \cos^2 x - \sin^2 x$ by double angle identity

$\Rightarrow \cos^2 x - (1-\cos^2 x) + \cos x + 1 = 0$

$\Rightarrow \cos^2 x - 1 + \cos^2 x + \cos x + 1 = 0$

$\Rightarrow 2\cos^2 x + \cos x = 0$

$\Rightarrow \cos x(2\cos x + 1) = 0$

Either $\cos x = 0$ or $2\cos x + 1 = 0$

$\Rightarrow x = (2k+1)\frac{\pi}{2}$, $\cos x = -1/2$

$\Rightarrow x = 2\frac{\pi}{3}, 4\frac{\pi}{3}$

$S.S = \{(2k+1)\frac{\pi}{2}\} \cup \{2\frac{\pi}{3} + 2k\pi\} \cup \{4\frac{\pi}{3} + 2k\pi\}$ where $k \in Z$

Q:22 $1 + \sin x = 2\cos^2 x$

Sol $1 + \sin x = 2(1 - \sin^2 x)$

$\Rightarrow 1 + \sin x = 2 - 2\sin^2 x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$, Factorize the eqn

$\Rightarrow 2\sin^2 x + 2\sin x - \sin x - 1 = 0$

$\Rightarrow 2\sin x(\sin x + 1) - 1(\sin x + 1) = 0$

$\Rightarrow (\sin x + 1)(2\sin x - 1) = 0$

Either $\sin x + 1 = 0$ or $2\sin x - 1 = 0$

$\sin x = -1$; $\sin x = 1/2$

$\Rightarrow x = 3\frac{\pi}{2} + 2k\pi$ $\quad \Rightarrow x = \frac{\pi}{6}, 5\frac{\pi}{6}$

$S.S = \{3\frac{\pi}{2} + 2k\pi\} \cup \{\frac{\pi}{6} + 2k\pi\} \cup \{5\frac{\pi}{6} + 2k\pi\}$ Ans where $k \in Z$

Q:23 $\tan^2 x = \frac{3}{2}\sec x$

Sol $(\sec^2 x - 1) = \frac{3}{2}\sec x$ $\quad \because \tan^2\theta + 1 = \sec^2\theta$

$\Rightarrow 2\sec^2 x - 2 = 3\sec x$

$\Rightarrow 2\sec^2 x - 3\sec x - 2 = 0$

By factorization

$\Rightarrow 2\sec^2 x - 4\sec x + \sec x - 2 = 0$

$\Rightarrow 2\sec x(\sec x - 2) + (\sec x - 2) = 0$

$\Rightarrow (\sec x - 2)(2\sec x + 1) = 0$

$\sec x - 2 = 0$ or $2\sec x + 1 = 0$

$\sec x = 2$ $\quad \sec x = -1/2$

$\Rightarrow \cos x = 1/2$ $\quad \Rightarrow \cos x = -2$ (not possible)

$\Rightarrow x = \frac{\pi}{3}, 5\frac{\pi}{3}$

Hence $S.S = \{\frac{\pi}{3} + 2k\pi\} \cup \{5\frac{\pi}{3} + 2k\pi\}$

(24) $3 - \sin x = \cos 2x$

Sol
$3 - \sin x = \cos^2 x - \sin^2 x$

$\Rightarrow 3 - \sin x = (1 - \sin^2 x) - \sin^2 x$

$\Rightarrow 3 - \sin x = 1 - 2\sin^2 x$

$\Rightarrow 2\sin^2 x - \sin x + 2 = 0$

Let $\sin x = t$

$\Rightarrow 2t^2 - t + 2 = 0$

By quadratic formula

$$t = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(2)}}{2(2)}$$

$$t = \frac{1 \pm \sqrt{-15}}{4}$$ which is not real

Hence $\sin x = t$

So real solution of $\sin x$ does not exist.

i.e $\sin x = \{ \}$.

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Exercise # 12.9

Q- Use reduction identity to solve the problems

Q.1 $\sin\theta + \cos\theta = 1 \longrightarrow (A)$

Sol compare with $a\sin\theta + b\cos\theta = c$, we have

$a = 1, \quad b = 1, \quad c = 1$

Let $a = r\cos\phi$ and $b = r\sin\phi$

$\Rightarrow 1 = r\cos\phi \longrightarrow (i)$ $\quad \Rightarrow 1 = r\sin\phi \longrightarrow (ii)$

$Eq^2 (i) + Eq^2 (ii)$, we get

$$\begin{aligned} 1^2 &= r^2\cos^2\phi \\ 1^2 &= r^2\sin^2\phi \\ \hline 1^2 + 1^2 &= r^2\cos^2\phi + r^2\sin^2\phi \end{aligned}$$

$\Rightarrow 2 = r^2(\cos^2\phi + \sin^2\phi)$

$\Rightarrow 2 = r^2(1)$

$\Rightarrow 2 = r^2$

$\Rightarrow \sqrt{2} = r$

& Equ (ii) $\div$ Equ (i)

$$\frac{r\sin\phi}{r\cos\phi} = \frac{1}{1}$$

$\Rightarrow \tan\phi = 1$

$\Rightarrow \phi = \tan^{-1}(1)$

$\Rightarrow \phi = \pi/4$

Equ (A) $\Rightarrow \underset{\downarrow}{1}\sin\theta + \underset{\downarrow}{1}\cos\theta = 1$

$\Rightarrow r\cos\phi\sin\theta + r\sin\phi\cos\theta = 1$

$\Rightarrow r\{\cos\phi\sin\theta + \sin\phi\cos\theta\} = 1$

$\Rightarrow r\sin(\theta + \phi) = 1$

$\Rightarrow \sqrt{2}\sin(\theta + \pi/4) = 1$

$\Rightarrow \sin(\theta + \pi/4) = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta + \frac{\pi}{4} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta + \frac{\pi}{4} = \frac{\pi}{4}$ or $3\pi/4$

$\Rightarrow \theta + \frac{\pi}{4} = \frac{\pi}{4}$ OR $\theta + \frac{\pi}{4} = 3\pi/4$

$\Rightarrow \theta = 0$ $\qquad \theta = 3\frac{\pi}{4} - \frac{\pi}{4}$

$\theta = \pi/2$

Hence $S.Sol = \{0 + 2k\pi\} \cup \{\frac{\pi}{2} + 2k\pi\}$ where $k \in Z$

$\boxed{Q:2}$ $\cos\theta + \sin\theta = 0$

Sol $1\sin\theta + 1\cos\theta = 0 \longrightarrow (1)$

compare with $a\sin\theta + b\cos\theta = C$, we get

$a = 1, \quad b = 1, \quad c = 0$

As $a\sin\theta + b\cos\theta = r\sin(\theta + \phi) \longrightarrow (A)$

where $r = \sqrt{a^2 + b^2}$ & $\tan\phi = b/a$

$r = \sqrt{1^2 + 1^2}$ $\qquad \Rightarrow \phi = \tan^{-1}(\frac{b}{a})$

$r = \sqrt{2}$ $\qquad \Rightarrow \phi = \tan^{-1}(\frac{1}{1})$

$\Rightarrow \phi = \pi/4$

Equ (A) $\Rightarrow$ $1\sin\theta + 1\cos\theta = \sqrt{2}\sin(\theta + \pi/4)$

Equ (1) $\Rightarrow$ $\sqrt{2}\sin(\theta + \frac{\pi}{4}) = 0$

$\Rightarrow \sin(\theta + \pi/4) = 0$

$\Rightarrow \theta + \frac{\pi}{4} = \pi$ or 2π

$\Rightarrow \theta + \frac{\pi}{4} = \pi$ OR $\theta + \frac{\pi}{4} = 2\pi$

$\Rightarrow \theta = \pi - \frac{\pi}{4}$ OR $\theta = 2\pi - \frac{\pi}{4}$

$= 3\frac{\pi}{4}$ $\qquad = 7\pi/4$

$S.S = \{3\frac{\pi}{4} + 2k\pi\} \cup \{7\frac{\pi}{4} + 2k\pi\}$

formulae

$r = \sqrt{a^2 + b^2}$

$\tan\phi = b/a$

$\cos\phi = \frac{b}{\sqrt{a^2+b^2}}$

$\sin\phi = \frac{a}{\sqrt{a^2+b^2}}$

$\boxed{Q:3}$ $\sqrt{3}\sin\theta + \cos\theta = 1 \longrightarrow (1)$

Sol compare with $a\sin\theta + b\cos\theta = c$, we get

$a = \sqrt{3}, \quad b = 1, \quad c = 1$

As $a\sin\theta + b\cos\theta = r\sin(\theta + \phi) \longrightarrow (A)$

where $r = \sqrt{a^2 + b^2}$ & $\tan\phi = b/a$

$= \sqrt{(\sqrt{3})^2 + 1^2}$ $\qquad \Rightarrow \tan\phi = 1/\sqrt{3}$

$= \sqrt{3+1}$ $\qquad \Rightarrow \phi = \pi/6$

$= 2$

Equ (A) $\Rightarrow$ $\sqrt{3}\sin\theta + 1\cos\theta = 2\sin(\theta + \frac{\pi}{6})$

Equ (1) $\Rightarrow$ $2\sin(\theta + \frac{\pi}{6}) = 1$

$\Rightarrow \sin(\theta + \frac{\pi}{6}) = \frac{1}{2}$

$\Rightarrow \theta + \frac{\pi}{6} = \sin^{-1}(\frac{1}{2})$

$\Rightarrow \theta + \frac{\pi}{6} = \frac{\pi}{6}$ or $5\frac{\pi}{6}$

$\Rightarrow \theta + \frac{\pi}{6} = \frac{\pi}{6}$ or $\theta + \frac{\pi}{6} = 5\frac{\pi}{6}$

$\Rightarrow \theta = 0$ or $\theta = 5\frac{\pi}{6} - \frac{\pi}{6}$

$\theta = 4\frac{\pi}{6}$

$= 2\pi/3$

$S.S = \{0 + 2k\pi\} \cup \{2\frac{\pi}{3} + 2k\pi\}$

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Q.4 $\sqrt{3}\cos\theta - \sin\theta = 1/2$

Sol $\Rightarrow -1\sin\theta + \sqrt{3}\cos\theta = 1/2 \longrightarrow (i)$

compare with

$a\sin\theta + b\cos\theta = c$, we get

$\Rightarrow a = -1$, $b = \sqrt{3}$ & $c = 1/2$

As $a\sin\theta + b\cos\theta = r\sin(\theta+\phi) \longrightarrow (A)$

where $r = \sqrt{a^2+b^2}$ & $\phi = \frac{b}{\sqrt{a^2+b^2}}$ & $\cos\phi = \frac{a}{\sqrt{a^2+b^2}}$

$= \sqrt{(-1)^2+(\sqrt{3})^2}$ $\quad \sin\phi = \frac{b}{r}$ $\quad \cos\phi = \frac{a}{r}$

$= \sqrt{1+3}$

$= 2$ $\quad \sin\phi = \frac{\sqrt{3}}{2}$ $\quad \cos\phi = \frac{-1}{2}$

$\Rightarrow \phi = \sin^{-1}(\sqrt{3}/2)$ $\quad \phi = \cos^{-1}(\frac{-1}{2})$

$\Rightarrow \phi = 2\pi/3$ $\quad \phi = 2\frac{\pi}{3}$

OR

$\tan\phi = b/a$

$\tan\phi = \frac{\sqrt{3}}{-1}$

$\Rightarrow \phi = 2\pi/3$

Eqn (A) $\Rightarrow a\sin\theta + b\cos\theta = r\sin(\theta+\phi)$

$\Rightarrow -1\sin\theta + \sqrt{3}\cos\theta = 2\sin(\theta + \frac{2\pi}{3})$

Eqn (i) $\Rightarrow 2\sin(\theta + \frac{2\pi}{3}) = 1/2$

$\Rightarrow \sin(\theta + \frac{2\pi}{3}) = \frac{1}{4}$

$\Rightarrow \theta + \frac{2\pi}{3} = \sin^{-1}(\frac{1}{4})$

$\Rightarrow \theta = \sin^{-1}(\frac{1}{4}) - \frac{2\pi}{3}$ Ans

Q.5 $\sqrt{3}\cos\theta + \sin\theta = 1 \longrightarrow (i)$

Sol $1\sin\theta + \sqrt{3}\cos\theta = 1$ $\quad a\sin\theta + b\cos\theta = c$ form

$\Rightarrow a = 1$ $\quad b = \sqrt{3}$ $\quad c = 1$

As $a\sin\theta + b\cos\theta = r\sin(\theta+\phi) \longrightarrow (A)$

where $r = \sqrt{a^2+b^2}$ & $\cos\phi = \frac{a}{r}$; $\sin\phi = \frac{b}{r}$

$= \sqrt{1^2+(\sqrt{3})^2}$ $\Rightarrow \underbrace{\cos\phi = \frac{1}{2} \quad \sin\phi = \frac{\sqrt{3}}{2}}$

$= \sqrt{1+3}$

$= \sqrt{4}$ $\quad \Rightarrow \phi = \pi/3$

$= 2$

Eqn (A) $\Rightarrow a\sin\theta + b\cos\theta = r\sin(\theta+\phi)$

$\Rightarrow 1\sin\theta + \sqrt{3}\cos\theta = 2\sin(\theta + \pi/6)$

Eqn (i) $\Rightarrow 2\sin(\theta + \frac{\pi}{3}) = 1$

$\Rightarrow \sin(\theta + \frac{\pi}{3}) = \frac{1}{2}$

$\Rightarrow \theta + \frac{\pi}{3} = \sin^{-1}(\frac{1}{2})$

$\theta + \frac{\pi}{3} = \frac{\pi}{6}$ or $5\frac{\pi}{6}$

$\Rightarrow \theta + \frac{\pi}{3} = \frac{\pi}{6}$ or $\theta + \frac{\pi}{3} = 5\frac{\pi}{6}$

$\Rightarrow \theta = \frac{\pi}{6} - \frac{\pi}{3}$ $\quad \theta = 5\frac{\pi}{6} - \frac{\pi}{3} = \frac{5\pi - 2\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$

$\theta = \frac{\pi - 2\pi}{6} = \frac{-\pi}{6}$ $\quad \Rightarrow \theta = \pi/2$

$S.S = \{-\frac{\pi}{6} + 2k\pi\} \cup \{\frac{\pi}{2} + 2k\pi\}$

Q:- Solve the following equations containing principle trignometric function giving exact values in their restricted domains.

⑥ $4\sin^2 x = 1$

Sol $\Rightarrow \sin^2 x = \frac{1}{4}$ Take square root, we get

$\sin x = \pm \frac{1}{2}$

Either $\sin x = \frac{1}{2}$ where $x \in [-\frac{\pi}{2}\ \frac{\pi}{2}]$ or $\sin x = -\frac{1}{2}$

$\Rightarrow x = \pi/6$ $\qquad x = -\pi/6$

Hence $S.S = \{\frac{\pi}{6}, -\frac{\pi}{6}\}$

⑦ $2\sqrt{2}\cos^2 x + (2-\sqrt{2})\cos x - 1 = 0$

Sol $2\sqrt{2}\cos^2 x + 2\cos x - \sqrt{2}\cos x - 1 = 0$

$\Rightarrow 2\cos x(\sqrt{2}\cos x + 1) - 1(\sqrt{2}\cos x + 1) = 0$

$\Rightarrow (\sqrt{2}\cos x + 1)(2\cos x - 1) = 0$

Either $\sqrt{2}\cos x + 1 = 0$ OR $2\cos x - 1 = 0$ where $x \in [0\ \pi]$

$\Rightarrow \sqrt{2}\cos x = -1$ $\qquad \Rightarrow 2\cos x = 1$

$\Rightarrow \cos x = \frac{-1}{\sqrt{2}}$ $\qquad \Rightarrow \cos x = 1/2$

$\Rightarrow x = \cos^{-1}(\frac{-1}{\sqrt{2}})$ $\qquad \Rightarrow x = \cos^{-1}(\frac{1}{2})$

$\Rightarrow x = 3\pi/4$ $\qquad \Rightarrow x = \pi/3$

Hence $S.S = \{\frac{3\pi}{4}, \frac{\pi}{3}\}$ Ans

⑧ $\cot^2 x + (\sqrt{3} - 1)\cot x - \sqrt{3} = 0$

Sol $\cot^2 x + \sqrt{3}\cot x - 1\cot x - \sqrt{3} = 0$

$\Rightarrow \cot x(\cot x + \sqrt{3}) - 1(\cot x + \sqrt{3}) = 0$

$\Rightarrow (\cot x + \sqrt{3})(\cot x - 1) = 0$

Either $\cot x + \sqrt{3} = 0$ or $\cot x - 1 = 0$ where $x \in [-\frac{\pi}{2}\ \frac{\pi}{2}] - \{0\}$

$\Rightarrow \cot x = -\sqrt{3}$ $\qquad \cot x = 1$

$\Rightarrow x = \cot^{-1}(-\sqrt{3})$ $\qquad \Rightarrow x = \cot^{-1}(1)$

$\Rightarrow x = \tan^{-1}(-\frac{1}{\sqrt{3}})$ $\qquad \Rightarrow x = \tan^{-1}(\frac{1}{1})$

$\Rightarrow x = -\pi/6$ $\qquad \Rightarrow x = \pi/4$

Hence $S.S = \{\frac{\pi}{4}, -\frac{\pi}{6}\}$ Ans

⑨ $4\cot^2 x + 2(\sqrt{3} - 1)\cot x - \sqrt{3} = 0$

Sol $4\cot^2 x + 2\sqrt{3}\cot x - 2\cot x - \sqrt{3} = 0$

$\Rightarrow 2\cot x(2\cot x + \sqrt{3}) - 1(2\cot x + \sqrt{3}) = 0$

$\Rightarrow (2\cot x + \sqrt{3})(2\cot x - 1) = 0$

Either $2\cot x + \sqrt{3} = 0$ or $2\cot x - 1 = 0$ where $x \in [-\frac{\pi}{2}\ \frac{\pi}{2}] - \{0\}$

$\Rightarrow 2\cot x = -\sqrt{3}$ $\qquad \Rightarrow 2\cot x = 1$

$\Rightarrow \cot x = -\sqrt{3}/2$ $\qquad \Rightarrow \cot x = 1/2$

$\Rightarrow x = \cot^{-1}(-\sqrt{3}/2)$ $\qquad \Rightarrow x = \cot^{-1}(1/2)$

$\qquad \Rightarrow x = \tan^{-1}(2)$

$\Rightarrow x = \tan^{-1}(-\frac{2}{\sqrt{3}})$ $\qquad \Rightarrow x = 63.4^0$

$\Rightarrow x = -49.1^0$

Hence $S.S = \{-49.1^0, 63.4^0\}$

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Q:10 $4\cos^2 x + 2(\sqrt{3}-1)\cos x - \sqrt{3} = 0$

Sol $4\cos^2 x + 2\sqrt{3}\cos x - 2\cos x - \sqrt{3} = 0$

$\Rightarrow 2\cos x(2\cos x + \sqrt{3}) - 1(2\cos x + \sqrt{3}) = 0$

$\Rightarrow (2\cos x + \sqrt{3})(2\cos x - 1) = 0$

Either $2\cos x + \sqrt{3} = 0$ or $2\cos x - 1 = 0$, $x \in [0 \quad \pi]$

$\Rightarrow 2\cos x = -\sqrt{3}$ $\qquad \Rightarrow 2\cos x = 1$

$\Rightarrow \cos x = -\sqrt{3}/2$ $\qquad \Rightarrow \cos x = 1/2$

$\Rightarrow x = \cos^{-1}(-\sqrt{3}/2)$ $\qquad \Rightarrow x = \cos^{-1}(1/2)$

$\Rightarrow x = 5\frac{\pi}{6}$ $\qquad \Rightarrow x = \pi/3$

Hence $S.S = \{5\frac{\pi}{6}, \frac{\pi}{3}\}$

Q:11 $4\cos^2 x - 4\cos x - 3 = 0$

Sol By factorization, we get

$4\cos^2 x - 6\cos x + 2\cos x - 3 = 0$

$\Rightarrow 2\cos x(2\cos x - 3) + 1(2\cos x - 3) = 0$

$\Rightarrow (2\cos x - 3)(2\cos x + 1) = 0$

Either $2\cos x - 3 = 0$ or $2\cos x + 1 = 0$ where $x \in [0 \quad \pi]$

$\Rightarrow 2\cos x = 3$ $\qquad \Rightarrow 2\cos x = -1$

$\Rightarrow \cos x = 3/2 > 1$ $\qquad \Rightarrow \cos x = -1/2$

Since maximum value of $\cos x$ is 1 $\qquad \Rightarrow x = \cos^{-1}(-\frac{1}{2}) = 2\pi/3$

Hence $\cos x = 3/2$ is not possible

$\Rightarrow x = $ Undefined

Hence $S.S = \{2\frac{\pi}{3}\}$ Ans

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(12) $\sin 4x + \sin 2x = 0$

Sol $\sin 4x + \sin 2x = 0$

$\Rightarrow \sin 2(2x) + \sin 2x = 0$

Double angle identity
$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\Rightarrow 2\sin 2x\cos 2x + \sin 2x = 0$

taking $\sin 2x$ as common

$\Rightarrow \sin 2x\{2\cos 2x + 1\} = 0$

Either $\sin 2x = 0$ or $2\cos 2x + 1 = 0$

$\Rightarrow 2x = \sin^{-1}(0)$ $\qquad \cos 2x = -1/2$

$\Rightarrow 2x = 0, \pi$ $\qquad \Rightarrow 2x = \cos^{-1}(-1/2)$

$\Rightarrow x = 0, \pi/2$ $\qquad \Rightarrow 2x = 2\frac{\pi}{3}$

$\Rightarrow x = \pi/3$

Hence $S.S = \{0, \frac{\pi}{2}, \frac{\pi}{3}\}$ Ans

Q:- Use inverse trigonometric function to find the solutions in the given intervals correct to four decimal places?

Q:13 $2\tan^2 x + 9\tan x + 3 = 0$ $\qquad [-\frac{\pi}{2} \quad \frac{\pi}{2}]$.

Sol let $\tan x = y$

$\Rightarrow 2y^2 + 9y + 3 = 0$

By quadratic formula

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-9 \pm \sqrt{9^2 - 4(2)(3)}}{2(2)} = \frac{-9 \pm \sqrt{81 - 24}}{4}$$

$\Rightarrow y = \frac{-9 \pm \sqrt{57}}{4}$ $\qquad$ put $y = \tan x$

$\Rightarrow \tan x = \frac{-9 \pm \sqrt{57}}{4}$

$\Rightarrow \tan x = \frac{-9 + \sqrt{57}}{4}$ & $\tan x = \frac{-9 - \sqrt{57}}{4}$

$\Rightarrow x = \tan^{-1}\left(\frac{-9+\sqrt{57}}{4}\right)$ & $x = \tan^{-1}\left(\frac{-9-\sqrt{57}}{4}\right)$

$\Rightarrow x = \tan^{-1}(-0.3625)$ & $x = \tan^{-1}(-4.1374)$

$\Rightarrow x = -0.3477$ rad & $x = -1.3336$

Hence $S.S = \{-0.3477, -1.3336\}$ Ans

Q:14 $3\sin^2 x + 7\sin x + 3 = 0$ $\left[-\frac{\pi}{2} \quad \frac{\pi}{2}\right]$

Sol By quadratic formula

$\sin x = \frac{-7 \pm \sqrt{7^2 - 4(3)(3)}}{2(3)}$

$\Rightarrow \sin x = \frac{-7 \pm \sqrt{49-36}}{6} = \frac{-7 \pm \sqrt{13}}{6}$

$\Rightarrow \sin x = \frac{-7+\sqrt{13}}{6}$, $\sin x = \frac{-7-\sqrt{13}}{6}$

$\Rightarrow x = \sin^{-1}\left(\frac{-7+\sqrt{13}}{6}\right)$; $x = \sin^{-1}\left(\frac{-7-\sqrt{13}}{6}\right)$

$\Rightarrow x = \sin^{-1}(-0.5657)$, $x = \sin^{-1}(-1.7676)$

$\Rightarrow x = -0.6013$ rad $x = \infty$

$S.S = \{-0.6013\}$ Ans

Q:15 $15\cos^4 x - 14\cos^2 x + 3 = 0$ $[0 \quad \pi]$

Sol $15\cos^4 x - 14\cos^2 x + 3 = 0$

By factorization

$\Rightarrow 15\cos^4 x - 9\cos^2 x - 5\cos^2 x + 3 = 0$

$\Rightarrow 3\cos^2 x\,(5\cos^2 x - 3) - 1\,(5\cos^2 x - 3) = 0$

$\Rightarrow (5\cos^2 x - 3)(3\cos^2 x - 1) = 0$

Either $5\cos^2 x - 3 = 0$ or $3\cos^2 x - 1 = 0$

$\Rightarrow 5\cos^2 x = 3$ $\quad$ $3\cos^2 x = 1$

$\Rightarrow \cos^2 x = 3/5$ $\quad$ $\Rightarrow \cos^2 x = 1/3$

$\Rightarrow \cos x = \pm\sqrt{3/5}$ $\quad$ $\Rightarrow \cos x = \pm\sqrt{1/3}$

$\Rightarrow \cos x = \pm\ 0.7746$ $\quad$ $\Rightarrow \cos x = \pm\ 0.5773$

$\Rightarrow \cos x = 0.7746$ or $\cos x = -0.7746$ $\quad$ $\Rightarrow \cos x = 0.5773$ or $\cos x = -0.5773$

$\Rightarrow x = \cos^{-1}(0.7746)$ or $x = \cos^{-1}(-0.7746)$ $\quad$ $\Rightarrow x = \cos^{-1}(0.5773)$ or $x = \cos^{-1}(-0.5773)$

$\Rightarrow x = 0.6847$ rad or $x = 2.4568$ rad $\quad$ $x = 0.955$ rad or $x = 2.1809$

Hence $S.S = \{0.6847, 2.4568, 0.955, 2.1809\}$

Q:16 $\sin^2 t - 4\sin t + 1 = 0$

Sol By quadratic formula

$\sin t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$

$\sin t = \frac{4 \pm \sqrt{12}}{2}$

$\Rightarrow \sin t = \frac{4 \pm \sqrt{4 \times 3}}{2}$

$\Rightarrow \sin t = \frac{4 \pm 2\sqrt{3}}{2}$

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$$\Rightarrow \sin t = \frac{2(2\pm\sqrt{3})}{2}$$

$$\Rightarrow \sin t = 2\pm\sqrt{3}$$

$\Rightarrow \sin t = 2+\sqrt{3}$ or $\sin t = 2-\sqrt{3}$

$\Rightarrow t = \sin^{-1}(2+\sqrt{3})$ or $t = \sin^{-1}(2-\sqrt{3})$

$\Rightarrow t = \sin^{-1}(3.73)$ $\quad t = \sin^{-1}(0.268)$

$\Rightarrow t =$ Not possible $\quad t = 0.2713$ rad

$S.S = \{0.2713\}$ Ans

$\boxed{Q.17}$ $5\sin^2\alpha + 3\cos\alpha - 2 = 0$

Sol $5\sin^2\alpha + 3\cos\alpha - 2 = 0$

$\Rightarrow 5(1-\cos^2\alpha) + 3\cos\alpha - 2 = 0$

$\Rightarrow 5 - 5\cos^2\alpha + 3\cos\alpha - 2 = 0$

$\Rightarrow -5\cos^2\alpha + 3\cos\alpha + 3 = 0$

xing by -1

$\Rightarrow 5\cos^2\alpha - 3\cos\alpha - 3 = 0$

By quadratic formula

$$\Rightarrow \cos\alpha = \frac{-(-3)\pm\sqrt{(-3)^2-4(5)(-3)}}{2(5)}$$

$$\Rightarrow \cos\alpha = \frac{3\pm\sqrt{9+60}}{10}$$

$$\Rightarrow \cos\alpha = \frac{3\pm\sqrt{69}}{10}$$

$\Rightarrow \cos\alpha = \frac{3+\sqrt{69}}{10}$ & $\cos\alpha = \frac{3-\sqrt{69}}{10}$

$\Rightarrow \cos\alpha = 1.13$ & $\cos\alpha = -0.5306$

Not possible $\quad \Rightarrow \alpha = \cos^{-1}(-0.5306)$

$\Rightarrow \alpha = 2.13$ rad

Hence $\alpha = \{2.13\}$ Ans

$\boxed{Q.18}$ $2\sin^3\beta + \sin^2\beta - 2\sin\beta - 1 = 0$

Sol take $\sin^2\beta$ as common

$\Rightarrow \sin^2\beta(2\sin\beta+1) - 1(2\sin\beta+1) = 0$

$\Rightarrow (2\sin\beta+1)(\sin^2\beta-1) = 0$

Either $2\sin\beta+1=0$ or $\sin^2\beta - 1 = 0$

$\Rightarrow \sin\beta = -\frac{1}{2}$ $\quad \sin^2\beta = 1$

$\Rightarrow \beta = \sin^{-1}(-\frac{1}{2})$ $\quad \Rightarrow \sin\beta = \pm 1$

$\Rightarrow \beta = 210^\circ$ or 330° $\quad \Rightarrow \sin\beta = 1$ or $\sin\beta = -1$

$\Rightarrow \beta = 90^\circ$ or $\beta = 270^\circ$

Hence $S.S = \{90^\circ, 210^\circ, 270^\circ, 330^\circ\}$.

End of chapter # 12.

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